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# Quasipositivity and homogeneity

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*Abstract*

We prove the following characterization of positive knots: a knot is positive if and only if it is homogeneous and strongly quasipositive.

## 1. *Introduction and main result*

A knot is called quasipositive if it has a special braid diagram, namely a product of conjugates of the positive standard generators of the braid group. If this product contains words of the form

$$\sigma_{i,j} = (\sigma_i \cdots \sigma_{j-2})\sigma_{j-1}(\sigma_i \cdots \sigma_{j-2})^{-1}$$

only, then we call the knot strongly quasipositive. Here $\sigma_i$ is the $i$th positive standard generator of the braid group. The word $\sigma_{i,j}$ is often called a positive band, see Figure 1, where the corresponding section of its Seifert diagram is drawn, too.

The concept of quasipositive knots is due to L. Rudolph ([**11**]). In [**13**], Rudolph proved that positive knots are strongly quasipositive and asked whether positive knots could be characterized as strongly quasipositive knots which satisfy some extra geometric conditions. It turns out that the appropriate extra condition is homogeneity. Homogeneous links were introduced by P. R. Cromwell ([**4**]) as a generalization of alternating links and positive links. A link diagram is homogeneous around a Seifert circle $S$ if the following two conditions are met:

(1) all the crossings inside $S$, which are adjacent to $S$, have the same sign;
(2) all the crossings outside $S$, which are adjacent to $S$, have the same sign.

A link is homogeneous if it has a diagram which is homogeneous around each Seifert circle. A homogeneous link diagram is shown in Figure 2, together with its Seifert diagram.

We observe that a braid diagram is strongly quasipositive and homogeneous if and only if it is a positive braid diagram. This suggests that the two classes of strongly quasipositive knots and homogeneous knots have a small intersection. We shall make this statement more precise.

Theorem 1. *A knot is positive if and only if it is homogeneous and strongly quasipositive.*

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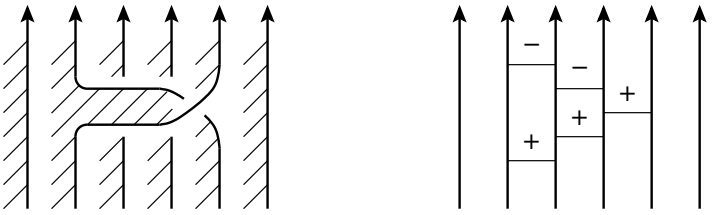



Fig. 1. A positive band: $\sigma_{2,5}$.

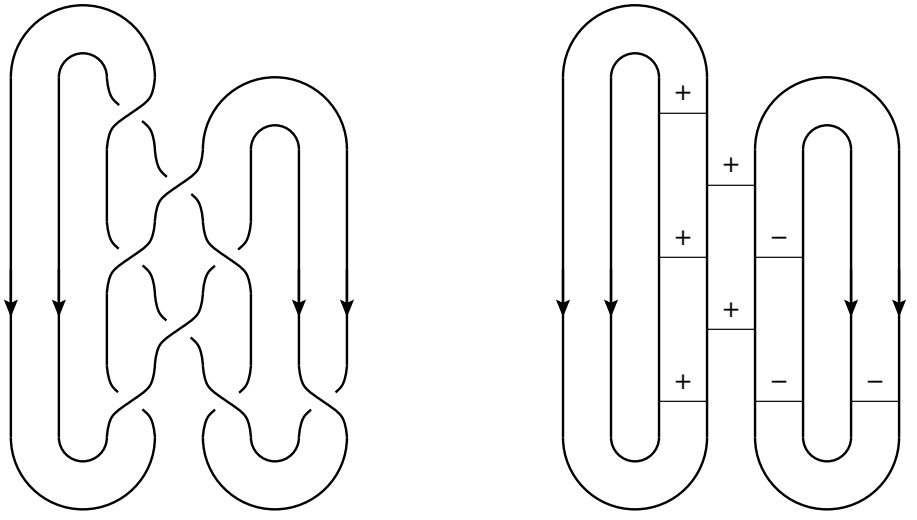



Fig. 2. A homogeneous diagram.

The proof of Theorem 1 is based upon three famous inequalities. Two of them involve the HOMFLY polynomial, which is normalized to one for the trivial knot and satisfies the following relation (see [**5**]):

$$\frac{1}{v}P_{\times}(v,z) - vP_{\times}(v,z) = zP_{)(}(v,z).$$

We denote the minimal degree in the variable $v$ of $P_K(v,z)$ by $e(K)$, the maximal degree by $E(K)$. The minimal genus among all orientable surfaces spanning a knot $K$ will be denoted by $g(K)$.

Theorem 2 (D. Bennequin [**3**]). *Let $D$ be a closed braid diagram with $s(D)$ strings of a knot $K$. Let $w(D)$ be the writhe of $D$, i.e. the number of positive minus the number of negative crossings of $D$. Then*

$$g(K) \geqslant \frac{1}{2}(1 + w(D) - s(D)).$$

Theorem 3 (H. R. Morton [**7**]). *For any diagram $D$ of an oriented knot $K$,*

$$w(D) - (s(D) - 1) \leqslant e(K) \leqslant E(K) \leqslant w(D) + (s(D) - 1).$$

Here $s(D)$ is the number of Seifert circles of $D$.

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THEOREM 4 (P. R. Cromwell [**4**]). *For any homogeneous knot $K$,*

$$e(K) \leqslant 2g(K).$$

*Moreover, equality holds if and only if $K$ is positive.*

*Proof of Theorem* 1. First we observe that a positive knot is homogeneous and strongly quasipositive (see Rudolph [**13**] or T. Nakamura [**8**]). Conversely, let $K$ be a homogeneous strongly quasipositive knot. $K$ has a closed braid diagram $D$ which is a product of $\sigma_{i,j}$'s. From $D$ we can construct a natural Seifert surface of $K$. It is a union of $s(D)$ disks and $w(D)$ bands with a positive half-twist. Such a band is depicted on the left side of Figure 1. The Euler characteristic of this Seifert surface equals $s(D) - w(D)$; its genus is $\frac{1}{2}(1 + w(D) - s(D))$. By Bennequin's inequality, it is a minimal genus Seifert surface of $K$, i.e.

$$g(K) = \frac{1}{2}(1 + w(D) - s(D)).$$

Now Morton's inequality tells us that

$$2g(K) \leqslant e(K),$$

whereas Cromwell's inequality tells us that

$$e(K) \leqslant 2g(K).$$

We conclude that equality holds, and hence, by Cromwell's Theorem, that $K$ is positive.

## 2. *Construction of non-homogeneous knots*

Theorem 1 enables us to construct many non-homogeneous knots. For instance, it has been shown by Rudolph that any Alexander polynomial of a knot can be realized as the Alexander polynomial of a strongly quasipositive knot (see [**10**]). On the other hand, the coefficients of the Alexander polynomial of a positive knot satisfy strong conditions. In this way, we can find many non-homogeneous knots.

We discuss one application, concerning positive Hopf plumbings. Let $K$ be a knot which can be obtained from the unknot by iterated plumbing of positive Hopf bands. By a result of Rudolph ([**12**]), $K$ is strongly quasipositive. Hence $K$ is either positive or non-homogeneous.

*Example*. The knot $10_{145}$ (in Rolfsen's notation [**9**]) is a non-positive divide knot (see N. A'Campo [**1**] for a definition of divide knots). Divide knots bound a unique Seifert surface which is a plumbing of positive Hopf bands (see M. Ishikawa [**6**]). Hence the knot $10_{145}$ is non-homogeneous.

## 3. *Problems*

(i) Does a homogeneous quasipositive knot have a positive diagram? In particular, does an alternating quasipositive knot have a positive diagram?

(ii) Is it true that a knot is strongly quasipositive if and only if it is quasipositive and its 4-genus equals its ordinary genus? (Up to ten crossings, this is true, see S. Baader [**2**].)

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