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# NONLINEAR DYNAMIC SYSTEMS AND OPTIMAL CONTROL PROBLEMS ON TIME SCALES *

Yunfei Peng[1], Xiaoling Xiang[1] and Yang Jiang[1]

**Abstract.** This paper is mainly concerned with a class of optimal control problems of systems governed by the nonlinear dynamic systems on time scales. Introducing the reasonable weak solution of nonlinear dynamic systems, the existence of the weak solution for the nonlinear dynamic systems on time scales and its properties are presented. Discussing $L^1$-strong-weak lower semicontinuity of integral functional, we give sufficient conditions for the existence of optimal controls. Using integration by parts formula and Hamiltonian function on time scales, the necessary conditions of optimality are derived respectively. Some examples on continuous optimal control problems, discrete optimal control problems, mathematical programming and variational problems are also presented for demonstration.

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## 1. Introduction

Important advancements in all the physical, life and social sciences heavily on the existence of a mathematical framework to describe, to solve and to better understand the problems from these fields. Historically, two separate approaches have dominated mathematical modelling: the field of differential equations and the area of difference equation. In order to create a theory that can unify discrete and continuous analysis, the calculus of time scales was initiated by Hilger in his Ph.D. thesis in 1988. The time scales calculus has a tremendous potential for applications in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics, neural networks, social sciences, as is pointed out in the monographs of Benchohra *et al.* [1] and Lakshmikantham *et al.* [7].

In recent years dynamic systems on time scales are considered for both initial value problems and boundary value problems. Some results on the existence, uniqueness and properties of classical solution were obtained [1,7,8,14]. In addition to, the theory of the calculus of variations on time scales is already well developed [2,9].

The optimal control problems on time sales is a very important topic for both theory and application. In 2006, we put forward the optimal control problem on time scales and present the existence of optimal controls, derived the necessary conditions of optimality for LQ problem on time scales in [3,10]. Then Zhan and Wei considered

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[1] College of Science, Guizhou University, Guiyang, Guizhou 550025, P.R. China. pengyf0803@163.com; xxl3621070@yahoo.com.cn; jiangyang415@163.com

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Hamilton-Jacobi-Bellman equations on time scales and presented the existence of optimal controls, derived the necessary conditions of optimality for the optimal control problems of system governed by a linear dynamic systems (see [16–18]). In 2009, using Hamiltonian function and classical variation, Hilscher and Zeidan [5] derived a weak maximum principle for some special optimal control problems of systems governed by the semilinear dynamic equation on time scales. The optimal condition was first proved in [5] in the context of the calculus of variations on time scales and the optimal inequality is not given. We prove it here in a more general setting of optimal control.

In this paper, under quite weak conditions, we study the optimal systematically control problem $(P^\sigma)$ below:

$$\min J(u), \quad J(u) = \int_{[a,b)} g\left(x(t), x^\sigma(t)\right) \Delta t + \int_{[a,b)} h(u(t)) \Delta t, \tag{1.1}$$

on all pairs $(x, u) \in C_{rd}(\mathbb{T}, R) \times L^1(\mathbb{T}, R)$ such that

$$\begin{cases} x^\Delta(t) + p(t)x^\sigma(t) = f\left(t, x(t), x^\sigma(t)\right) + u(t), & a \le t < b, \\ x(a) = x_0, \end{cases} \tag{1.2}$$

where $p \in L^1(\mathbb{T}, R)$, $\mathbb{T}$ is a bounded time scale, $a = \inf \mathbb{T}$, $b = \sup \mathbb{T}$. The problem $(P^\sigma)$ is very difficult since $p \in L^1(\mathbb{T}, R)$ and $f$ is dependent on $x^\sigma$. Firstly, all of Gronwall inequality on time scales given before cannot be used to obtain the *a priori* estimate of solutions for the nonlinear dynamic system (1.2). In addition, the backward problems on time scales cannot be turned into the Cauchy problem by simple transformation $s = b - t$.

First reasonably extending the exponential function $e_p$ with $p \in C_{rd}(\mathbb{T}, R)$ to the case that $p \in L^1_{\mathbb{T}}(\mathbb{T}, R)$ we introduce the reasonable weak solution of (1.2). Deriving a generalized Gronwall inequality associated $x$ and $x^\sigma$ on time scales to obtain the *a priori* estimate of solutions and fully considering the structure character of time scales we first use the Leray-Schauder fixed theorem to present the existence of the weak solution. Then introducing new norm $\|\cdot\|_\beta$, we use the contraction mapping principle to give the uniqueness of weak solution. The conditions are quite weak (see Thm. 3.A). Before, the classical solution of (1.2) were considered. Under the very strong condition (such as $f$ satisfies uniform Lipschitz condition), some authors gave the existence and uniqueness of classical solution by the contracting mapping principle [13].

Next, the existence of optimal controls for problem $(P^\sigma)$ is presented (see Thm. 4.A). Finally utilizing the integration by parts on time scales, we derive the necessary conditions of optimality containing optimal controlled system, adjoint equation and optimal inequality (see Thms. 5.A and 6.C). In order to discover relation both the optimal control problem and the variational problem on time scales we give Hamiltonian formulations to Theorems 5.A and 6.C further (see Thms. 5.B and 6.D). Moreover, as contrast, we compare the problem $(P^\sigma)$ with problem (P) which are two typical optimal control problems on time scales (see Tab. 1). Some examples on continuous optimal control problems, discrete optimal control problems and mathematical programming are also presented for demonstration. The results obtained generalize and improve the corresponding results [3,5,9,10,16–18].

The paper is organized as follows. In Section 2, we give some basic notations and some basic results on time scales. In Section 3, the generalized exponential function $e_p$ with $p \in L^1(\mathbb{T}, R)$ and the weak solutions of the semilinear dynamic systems are presented. Discussing $L^1$-strong-weak lower semicontinuity of integral functional, sufficient conditions for the existence of optimal controls is given in Section 4. Necessary conditions of optimality are derived in Section 5. In Section 6, we compare the problem $(P^\sigma)$ with problem (P) which are two typical optimal control problems on time scales. Finally, some typical examples are given for demonstration.

## 2. Terminology and preliminaries

In this section we collect some important concepts and results on time scales which are very useful in sequel.

A time scale $\mathbb{T}$ is a nonempty closed subset of $R$. The two most popular examples are $\mathbb{T} = R$ and $\mathbb{T} = Z$. Define the forward and backward jump operators $\sigma, \rho : \mathbb{T} \longrightarrow \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} \mid s > t\}, \ \rho(t) = \sup\{s \in T \mid s < t\}, \ t \in \mathbb{T},$$

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where, in this definition, we write $\sup\emptyset = \inf\mathbb{T} \equiv a$ and $\inf\emptyset = \sup\mathbb{T} \equiv b$. A point $t \in \mathbb{T}$ is said to be left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. The forward graininess $\mu : \mathbb{T} \longrightarrow [0, +\infty)$ and the backward graininess $\nu : \mathbb{T} \longrightarrow [0, +\infty)$ are defined by $\mu(t) = \sigma(t) - t$ and $\nu(t) = t - \rho(t)$, respectively.

**Definition 2.1.** A function $f : \mathbb{T} \longrightarrow R$ is $\Delta$-differentiable at $t \in \mathbb{T}$, if there exists a number $f^{\Delta}(t)$, with the following property: for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \text{ for } s \in \mathbb{T} \text{ and } |s - t| < \delta.$$

If $f$ is $\Delta$-differentiable for every $t \in \mathbb{T}$, we say that $f$ is $\Delta$-differentiable on $\mathbb{T}$.

In the definition, if $\mathbb{T} = R$ this definition coincides with classical derivative definition and if $\mathbb{T} = Z$ it coincides with forward difference.

The Lebesgue $\Delta$-measure $\mu_{\Delta}$ was defined as the Caratheodory extension of a set function on time scale $\mathbb{T}$ in [4,11]. In particular, for each $t_0$ in $\mathbb{T}$ the single-point set $\{t_0\}$ is $\Delta$-measurable and its $\Delta$-measure is given by

$$\mu_{\Delta}(\{t_0\}) = \sigma(t_0) - t_0. \tag{2.1}$$

It is clear that the Lebesgue $\Delta$-measure of single-point set on $\mathbb{T}$ may not be equal to zero. We note that, if $t_0$ is right-scattered,

$$\mu_{\Delta}\left([t_0, c)\bigcap\mathbb{T}\right) = \mu(t_0) = \sigma(t_0) - t_0 > 0 \text{ for any } c \in (t_0, \sigma(t_0)]$$

(see proof of Thm. 3.A). In addition, the Lebesgue $\Delta$-measure $\mu_{\Delta}$ has closed association with Lebesgue measure $\mu$. Let $E$ be a subset of $\mathbb{T}$, define $\tilde{E} = E \bigcup\limits_{t_i \in E} (t_i, \sigma(t_i))$, where $\sigma(t_i) - t_i > 0$. $\tilde{E}$ is called the extension of $E$. $E$ is Lebesgue $\Delta$-measurable if and only if $\tilde{E}$ is Lebesgue measurable and

$$\mu_{\Delta}(E) = \mu(\tilde{E}) = \mu(E) + \sum_{t_i \in E} (\sigma(t_i) - t_i). \tag{2.2}$$

It is obvious that $\mu_{\Delta}(E) = \mu(E)$ if and only if $E$ has no right-scattered points.

As a straightforward consequence of equality (2.2), one can deduce the simple formula to calculate the Lebesgue $\Delta$-integral. For a function $f : \mathbb{T} \longrightarrow R$, define the step function interpolation $\tilde{f} : [a, b] \longrightarrow R$ as

$$\tilde{f}(t) = \begin{cases} f(t_i), & t \in (t_i, \sigma(t_i)), \\ f(t), & t \in \mathbb{T}. \end{cases}$$

Let $E \subset \mathbb{T} \subseteq [a, b]$ be $\Delta$-measurable set such that $b \notin E$. Now we say that $f$ is Lebesgue $\Delta$-integrable on $E$ if and only if $\tilde{f}$ is Lebesgue integrable on $\tilde{E}$, in which case the equality holds:

$$\int_E f(s)\Delta s = \int_{\tilde{E}} \tilde{f}(s)\mathrm{d}s.$$

The right side of the equality above is the classical Lebesgue integral over the real interval $\tilde{E} \subset [a, b]$. We can obtain a formula for calculating the Lebesgue $\Delta$-integral. For all $s$, $t \in \mathbb{T}$ with $s \leq t$, the following expression holds:

$$\int_{[s,t)} f(\tau)\Delta\tau = \int_{[s,t)\bigcap\mathbb{T}} f(\tau)\mathrm{d}\tau + \sum_{s \leq t_i < t, t_i \in \mathbb{T}} \mu(t_i)f(t_i),$$

$$\int_{[t,\sigma(t))} f(\tau)\Delta\tau = \mu(t)f(t), \int_{[t,t)} f(\tau)\Delta\tau = 0.$$

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Now define

$$L^1(\mathbb{T},R)=\{f:\mathbb{T}\longrightarrow R\,|\,f \text{ is Lebesgue } \Delta\text{-integrable on } \mathbb{T}\},$$
$$L^r(\mathbb{T},R)=\left\{f\in L^1(\mathbb{T},R)\mid |f|^r\in L^1(\mathbb{T},R)\right\}(r\geq 1).$$

Endowed with norm

$$\|f\|_{L^r}=\left(\int_{\mathbb{T}}|f(\tau)|^r\Delta\tau\right)^{\frac{1}{r}},$$

the spaces $L^r(\mathbb{T},R)(r\geq 1)$ is a Banach space (see [11]).

Let $e\subset\mathbb{T}$, $e$ is called $\Delta$-null set if $\mu_\Delta(e)=0$. Say that a property $Q$ holds $\Delta$-a.e. on $E$ if there is a $\Delta$-null set $e\subset E$ such that $Q$ holds for all $t\in E\setminus e$.

Guseinov [4] and Rynne [11] given the Newton-Leibniz formula and integration by parts formula in $C^1_{rd}(\mathbb{T},R)$, $H^1(\mathbb{T},R)$, respectively. For our purpose, we extend their results to the weaker case.

**Theorem 2.1.** (1) *Let $f\in L^1(\mathbb{T},R)$, define*

$$F(t)=\int_{[a,t)}f(\tau)\Delta\tau \text{ for } t\in\mathbb{T}.$$

*Then $F\in C_{rd}(\mathbb{T},R)$ is differentiable $\Delta$-a.e. on $\mathbb{T}$ and*

$$F(t)-F(s)=\int_{[s,t)}f(\tau)\Delta\tau \text{ for } s,\ t\in\mathbb{T}.$$

(2) *If $f$ and $g$ are differentiable $\Delta$-a.e. on $\mathbb{T}$, then*

$$\int_{[s,t)}\left[f^\Delta(\tau)g(\tau)+f^\sigma(\tau)g^\Delta(\tau)\right]\Delta\tau=f(t)g(t)-f(s)g(s) \text{ for } s,\ t\in\mathbb{T}. \tag{2.3}$$

*Proof.* (1) Obviously, we have $F\in C_{rd}(\mathbb{T},R)$ (see Lem. 3.2 of [11]).

If $t$ is right-scattered *i.e.* $t<\sigma(t)$, then we have

$$\frac{F(\sigma(t))-F(t)}{\sigma(t)-t}=\frac{1}{\sigma(t)-t}\int_{[t,\sigma(t))}f(\tau)\Delta\tau=f(t).$$

Note that, if $t$ is right-dense *i.e.* $t=\sigma(t)$, in addition to $s\neq t$, then

$$\frac{F(\sigma(t))-F(s)}{\sigma(t)-s}=\frac{1}{t-s}\int_{[s,t)}f(\tau)\Delta\tau.$$

By Lebesgue integral theory, we have

$$\lim_{s\to t}\frac{F(\sigma(t))-F(s)}{\sigma(t)-s}=f(t) \text{ a.e. on } \mathbb{T}\setminus\{t\in\mathbb{T}\,|\,t<\sigma(t)\}.$$

Hence, $F$ is differentiable $\Delta$-a.e. on $\mathbb{T}$ and

$$F^\Delta(t)=f(t)\ \Delta\text{-a.e. on } \mathbb{T}.$$

Further, we obtain that Newton-Leibniz formula is hold, that is,

$$F(t)-F(s)=\int_{[s,t)}f(\tau)\Delta\tau \text{ for } s,\ t\in\mathbb{T}.$$

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(2) By conclusion (1), since $f$ and $g$ are differentiable $\Delta$-a.e. on $\mathbb{T}$, $f,\ g \in C_{rd}(\mathbb{T}, R)$. If $t$ is right-scattered *i.e.* $t < \sigma(t)$, then we have

$$\frac{f(\sigma(t))g(\sigma(t)) - f(t)g(t)}{\sigma(t) - t} = f^{\Delta}(t)g(t) + f^{\sigma}(t)g^{\Delta}(t).$$

If $t$ is right-dense *i.e.* $t = \sigma(t)$, in addition to $s \neq t$, then

$$\frac{f(\sigma(t))g(\sigma(t)) - f(s)g(s)}{\sigma(t) - s} = g(s)\frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} + f^{\sigma}(t)\frac{g(\sigma(t)) - g(s)}{\sigma(t) - s}.$$

By assumption on functions $f$ and $g$, $fg$ is differentiable $\Delta$-a.e. on $\mathbb{T}$, and

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f^{\sigma}(t)g^{\Delta}(t).$$

Integrating from $s$ to $t$, we obtain the integration by parts formula (2.3). □

## 3. Weak solution of dynamic systems

In this section, we extend the exponential function $e_p$ with $p \in C_{rd}(\mathbb{T}, R)$ on time scales to the case that $p \in L^1(\mathbb{T}, R)$ and discuss the weak solution of the nonlinear dynamic equation (1.2).

The exponential function on time scales plays a very important role for discussing dynamic equations on time scales. The exponential function $e_p$ on time scales is defined as the unique solution $y(t) = e_p(t, a)$ of the Cauchy problem $y^{\Delta}(t) = p(t)y(t)$, $y(a) = 1$, where $p \in \Gamma_C(\mathbb{T}) = \{p \in C_{rd}(\mathbb{T}, R) \,|\, 1 + \mu(t)p(t) \neq 0\}$. An explicit formula for $e_p(t, a)$ is given by

$$e_p(t, s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right\} \text{ with } \xi_{\mu(t)}(p(t)) = \begin{cases} \frac{\ln(1+\mu(t)p(t))}{\mu(t)}, & \text{if } \mu(t) > 0, \\ p(t), & \text{if } \mu(t) = 0. \end{cases}$$

Define $\Gamma_1(\mathbb{T}) = \left\{p \in L^1_{\text{loc}}(\mathbb{T}, R) | 1 + \mu(t)p(t) \neq 0\right\}$. Using the proof by contradiction and Lebesgue $\Delta$-integral we can show that for any $p \in \Gamma_1(\mathbb{T})$ and $a,\ b \in R$ but fixed, there are positive number $m$ and $M$ such that

$$m \leq |1 + \mu(t)p(t)| \leq M \text{ for all } t \in [a, b] \bigcap \mathbb{T} \tag{3.1}$$

and the set $\{t \in \mathbb{T} \,|\, 1 + \mu(t)p(t) < 0\} \subseteq \mathbb{T}$ is a finite set. Starting with the explicit formula of exponential function $e_p$ on time scales, we can prove that the so-called cylinder transformation given by

$$\xi_{\mu(t)}(p(t)) = \begin{cases} \frac{\ln(1+\mu(t)p(t))}{\mu(t)} & \text{if } \mu(t) \neq 0, \\ p(t) & \text{if } \mu(t) = 0, \end{cases} \text{ with } p \in \Gamma_1(\mathbb{T}),$$

is meaningful and $\xi_{\mu(\cdot)}(p(\cdot)) \in L^1_{\text{loc}}(\mathbb{T}, R)$.

**Definition 3.1.** For $p \in \Gamma_1(\mathbb{T})$, define the generalized exponential function as follows:

$$e_p(t, s) = \exp\left\{\int_{[s,t)} \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right\}.$$

For $p,\ q \in \Gamma_1(\mathbb{T})$, we still define

$$p \oplus q = p + q + \mu pq,\ \ominus p = -\frac{p}{1 + \mu p},\ p \ominus q = \frac{p - q}{1 + \mu q}.$$

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Further, we can also show that $p \oplus q,\ p \ominus q,\ \ominus p \in \Gamma_1(\mathbb{T})$. Furthermore, we can show the following some fundamental properties (operation and analytic properties) of the generalized exponential function on time scales.

**Theorem 3.1.** *Assume that $p,\ q \in \Gamma_1(\mathbb{T})$, then the following hold:*

(1) $e_0(t,s) \equiv 1, \quad e_p(t,t) \equiv 1, \quad e_p(t,s)e_p(s,r) = e_p(t,r);$

(2) $e_p(\sigma(t),s) = [1+\mu(t)p(t)]e_p(t,s), \quad e_p(t,s) = \dfrac{1}{e_p(s,t)} = e_{\ominus p}(s,t);$

(3) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s), \quad \dfrac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s);$

(4) $e_p(\cdot,s) \in C_{rd}(\mathbb{T},R);$

(5) $(e_p(\cdot,s))^{\Delta} = p(\cdot)e_p(\cdot,s),\ (e_p(s,\cdot))^{\Delta} = -p(\cdot)e_p(s,\sigma(\cdot))\ \Delta\text{-}a.e.\ on\ \mathbb{T}.$

In order to derive *a priori* estimates on solutions of the nonlinear dynamic system (1.2), we need the following generalized Gronwall inequality on time scales.

**Proposition 3.1.** *Let $x \in C_{rd}(\mathbb{T},R)$, $p \in L^1(\mathbb{T},R^+)$ with $R^+ = [0,+\infty)$, $f \in L^1(\mathbb{T},R)$. Then*

$$x^{\Delta}(t) \le p(t)x(t) + f(t),\ \Delta\text{-}a.e.\ on\ \mathbb{T}, \tag{3.2}$$

*implies*

$$x(t) \le e_p(t,a)x(a) + \int_{[a,t)} e_p(t,\sigma(\tau))f(\tau)\Delta\tau \ for\ all\ t \in \mathbb{T}.$$

*Proof.* Note that $p \in L^1(\mathbb{T},R^+)$ implies $p \in \Gamma_1(\mathbb{T})$ and $1+\mu(t)p(t) > 0$ for all $t \in \mathbb{T}$. Further, for any $t,\ s \in \mathbb{T}$, we have $e_{\ominus p}(t,s) > 0$. Now

$$\begin{aligned}
[x(t)e_{\ominus p}(t,a)]^{\Delta} &= x^{\Delta}(t)e_{\ominus p}(\sigma(t),a) + x(t)(\ominus p)(t)e_{\ominus p}(t,a) \quad \Delta\text{-a.e. on } \mathbb{T} \\
&= \frac{x^{\Delta}(t)}{1+\mu(t)p(t)}e_{\ominus p}(t,a) - \frac{p(t)x(t)}{1+\mu(t)p(t)}e_{\ominus p}(t,a) \\
&= \left[x^{\Delta}(t) - p(t)x(t)\right]e_{\ominus p}(\sigma(t),a) \quad \Delta\text{-a.e. on } \mathbb{T}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
x(t)e_{\ominus p}(t,a) - x(a) &= \int_{[a,t)} \left[x^{\Delta}(\tau) - p(\tau)x(\tau)\right]e_{\ominus p}(\sigma(\tau),a)\Delta\tau \\
&\le \int_{[a,t)} f(\tau)e_{\ominus p}(\sigma(\tau),a)\Delta\tau,
\end{aligned}$$

that is,

$$x(t) \le e_p(t,a)x(a) + \int_{[a,t)} e_p(t,\sigma(\tau))f(\tau)\Delta\tau \text{ for all } t \in \mathbb{T}.$$

The proof is completed. □

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**Theorem 3.2.** *Let $x \in C_{rd}(\mathbb{T}, R^+)$ satisfies the following inequality*

$$x(t) \leq \alpha + \int_{[a,t)} p(\tau)x(\tau)\Delta\tau + \int_{[a,t)} g(\tau)x^{\lambda}(\sigma(\tau))\Delta\tau \text{ for all } t \in \mathbb{T}, \tag{3.3}$$

*where $p$, $g \in L^1(\mathbb{T}, R^+)$ and $\alpha \geq 0$, $0 < \lambda < 1$, then there exists a constant $M > 0$ such that*

$$x(t) \leq M \text{ for all } t \in \mathbb{T}.$$

*Proof.* Define

$$y(t) = \alpha + \int_{[a,t)} p(\tau)x(\tau)\Delta\tau + \int_{[a,t)} g(\tau)x^{\lambda}(\sigma(\tau))\Delta\tau, \text{ for all } t \in \mathbb{T}.$$

By Theorem 2.1, $y$ is differentiable $\Delta$-a.e. on $\mathbb{T}$ and $y(a) = \alpha$,

$$y^{\Delta}(t) = p(t)x(t) + g(t)x^{\lambda}(\sigma(t)) \leq p(t)y(t) + g(t)y^{\lambda}(\sigma(t)) \ \Delta\text{-a.e. on } \mathbb{T}.$$

It follows from Proposition 3.1 that

$$y(t) \leq \alpha e_p(b,a) + e_p(b,a)\int_{[a,b)} g(\tau)y^{\lambda}(\sigma(\tau))\Delta\tau \text{ for all } t \in \mathbb{T}.$$

Define

$$q(t) = \alpha e_p(b,a) + e_p(b,a)\int_{[a,t)} g(\tau)y^{\lambda}(\sigma(\tau))\Delta\tau + e_p(b,a)\int_{[a,b)} g(\tau)y^{\lambda}(\sigma(\tau))\Delta\tau$$

for all $t \in \mathbb{T}$, then $q$ is monotone increasing function and $q(b) = 2q(a) - \alpha e_p(b,a)$,

$$q^{\Delta}(t) = e_p(b,a)g(t)y^{\lambda}(\sigma(t)) \leq e_p(b,a)g(t)q^{\lambda}(t) \ \Delta\text{-a.e. on } \mathbb{T}.$$

$\Delta$-integrating from $a$ to $t$, we obtain

$$q^{1-\lambda}(t) - q^{1-\lambda}(a) \leq (1-\lambda)e_p(b,a)\int_{[a,t)} g(\tau)\Delta\tau \text{ for all } t \in \mathbb{T}.$$

Now, we observe that

$$(2q(a) - \alpha e_p(b,a))^{1-\lambda} - q^{1-\lambda}(a) \leq (1-\lambda)e_p(b,a)\int_{[a,b)} g(\tau)\Delta\tau.$$

Let

$$\Gamma(z) = (2z - \alpha e_p(b,a))^{1-\lambda} - z^{1-\lambda},$$

we have $\Gamma \in C\left(\left[\frac{\alpha e_p(b,a)}{2}, +\infty\right), R\right)$ and

$$\lim_{z\to+\infty}\Gamma(z) = \lim_{z\to+\infty}\frac{\Gamma(z)}{z^{1-\lambda}}z^{1-\lambda} = \lim_{z\to+\infty}\left[\left(2 - \frac{\alpha e_p(b,a)}{z}\right)^{1-\lambda} - 1\right]z^{1-\lambda} = +\infty.$$

Using the proof by contraction, we can show that there exists a constant $M > 0$ such that $q(a) < M$. Thus

$$x(t) \leq M \text{ for all } t \in \mathbb{T}.$$

The proof is completed. □

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First consider the following Cauchy problem

$$\begin{cases} x^{\Delta}(t) + p(t)x^{\sigma}(t) = f(t), \\ x(a) = x_0, \end{cases} \tag{3.4}$$

where $p \in \Gamma_1(\mathbb{T})$. For $f \in L^1(\mathbb{T}, R)$, the integral function

$$\int_{[a,\cdot)} e_{\ominus p}(\cdot, \tau) f(\tau) \Delta\tau$$

is well-defined. The function $x \in C_{rd}(\mathbb{T}, R)$ given by

$$x(t) = e_{\ominus p}(t, a)x_0 + \int_{[a,t)} e_{\ominus p}(t, \tau) f(\tau) \Delta\tau, \quad t \in \mathbb{T}$$

is said to be the weak solution of (3.4). By Theorem 2.1, we have the following result.

**Lemma 3.1.** *Let $p \in \Gamma_1(\mathbb{T})$, $f \in L^1(\mathbb{T}, R)$, the weak solution $x$ of (3.4) satisfies*

$$x^{\Delta}(t) + p(t)x^{\sigma}(t) = f(t) \quad \Delta\text{-a.e. on } \mathbb{T}.$$

*Now, consider the following semilinear dynamic system on time scale*

$$\begin{cases} x^{\Delta}(t) + p(t)x^{\sigma}(t) = f\left(t, x(t), x^{\sigma}(t)\right), \quad a \le t < b, \\ x(a) = x_0. \end{cases} \tag{3.5}$$

**Definition 3.2.** A function $x \in C_{rd}(\mathbb{T}, R)$ is said to be a weak solution of the dynamic system (3.5), if $x$ satisfies the following integral equation

$$x(t) = e_{\ominus p}(t, a)x_0 + \int_{[a,t)} e_{\ominus p}(t, \tau) f\left(\tau, x(\tau), x^{\sigma}(\tau)\right) \Delta\tau, \quad t \in \mathbb{T}. \tag{3.6}$$

Suppose that:

[F] (1) $f : \mathbb{T} \times R \times R \longrightarrow R$ is $\Delta$-measurable in $t \in \mathbb{T}$ and locally Lipschitz continuous, *i.e.* for all $x_1$, $y_1$, $x_2$, $y_2 \in R$, satisfying $|x_1|$, $|y_1|$, $|x_2|$, $|y_2| \le \rho$, we have

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \le L(\rho)\left(|x_1 - x_{12}| + |y_1 - y_2|\right) \text{ for all } t \in \mathbb{T}.$$

(2) There exist a constant $0 < \lambda < 1$ and a function $q \in L^1(\mathbb{T}, R^+)$ such that

$$|f(t, x, y)| \le q(t)\left(1 + |x| + |y|^{\lambda}\right) \text{ for all } x,\ y \in R,\ t \in \mathbb{T}.$$

**Theorem 3.3.** *Let $p \in \Gamma_1(\mathbb{T})$. Under the assumption* [F]*, the dynamic system (3.5) has a unique weak solution $x \in C_{rd}(\mathbb{T}, R)$ and*

$$x^{\Delta}(t) + p(t)x^{\sigma}(t) = f\left(t, x(t), x^{\sigma}(t)\right) \ \Delta\text{-a.e. on } \mathbb{T}.$$

*Proof.* **Step I**. Existence of weak solution. Define the operator $Q$ on $C_{rd}(\mathbb{T}, R)$ given by

$$(Qx)(t) = e_{\ominus p}(t, a)x_0 + \int_{[a,t)} e_{\ominus p}(t, \tau) f\left(\tau, x(\tau), x^{\sigma}(\tau)\right) \Delta\tau. \tag{3.7}$$

For any $x \in C_{rd}(\mathbb{T}, R)$ but fixed, it can be seen from assumption [F] that $f\left(\cdot, x(\cdot), x^{\sigma}(\cdot)\right) \in L^1(\mathbb{T}, R)$. By the properties of generalized exponential function on time scales, we can verify that $Qx \in C_{rd}(\mathbb{T}, R)$.

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For $x,\ y \in C_{rd}(\mathbb{T},R)$ and $\|x\|_{C_{rd}},\ \|y\|_{C_{rd}} \leq \rho$, where $\rho$ is a constant, using assumption [F](1), we have

$$\|Qx - Qy\|_{C_{rd}} \leq 2ML(\rho)(b-a)\|x-y\|_{C_{rd}},$$

where $M = \sup\{|e_{\ominus p}(t,s)|\,|t,\ s \in \mathbb{T}\}$. Hence $Q : C_{rd}(\mathbb{T},R) \longrightarrow C_{rd}(\mathbb{T},R)$ is a continuous operator.

Next, we show that $Q$ is a compact operator. Let $\rho > 0$, set $\mathbb{W} = \{x \in C_{rd}(\mathbb{T},R) | \|x\|_{C_{rd}} \leq \rho\}$. For $x \in \mathbb{W}$, we have

$$|f\left(t,x(t),x^{\sigma}(t)\right)| \leq q(t)\left[1+\rho+\rho^{\lambda}\right] = \omega q(t),$$

where $\omega = 1+\rho+\rho^{\lambda}$. It is easy to see that $Q\mathbb{W} \subseteq C_{rd}(\mathbb{T},R)$ is bounded. Let $t_1,\ t_2 \in T$ with $t_1 \leq t_2$, we have

$$|(Qx)(t_2) - (Qx)(t_1)| \leq \beta(M+1)^2\left|e_{\ominus p}(t_2,t_1) - 1\right| + M\omega\int_{[t_1,t_2)} q(\tau)\Delta\tau,$$

where $\beta = |x_0| + \omega\|q\|_{L^1}$. This implies that $Q\mathbb{W}$ is rd-equicontinuous. By Arzela-Ascoli theorem on time scales (see [3]), $Q$ is a compact operator in $C_{rd}(\mathbb{T},R)$.

Define $\Upsilon = \{x \in C_{rd}(\mathbb{T},R) | x = \delta Qx,\ \delta \in [0,1]\}$. Let $y = \frac{1}{\delta}x$ for $\delta \neq 0$, otherwise $y = 0$ for $x \in \Upsilon$. Note that

$$\begin{aligned}
|y(t)| &= |(Q(\delta y))(t)| \\
&\leq |e_{\ominus p}(t,a)|\,|x_0| + \int_{[a,t)} |e_{\ominus p}(t,\tau)|\,|f\left(\tau,\delta y(\tau),\delta y^{\sigma}(\tau)\right)|\,\Delta\tau \\
&\leq M|x_0| + M\int_{[a,t)} q(\tau)\Delta\tau + \delta\int_{[a,t)} q(\tau)|y(\tau)|\Delta\tau + M\delta^{\lambda}\int_{[a,t)} q(\tau)\,|y^{\sigma}(\tau)|^{\lambda}\,\Delta\tau,
\end{aligned}$$

by virtue of generalized Gronwall inequality on time scales (see Thm. 3.2), there is a constant $r > 0$ such that

$$|y(t)| \leq r \text{ for all } t \in \mathbb{T}.$$

Thus $\Upsilon$ is a bounded set. According to Leray-Schauder fixed point theorem, $Q$ has a fixed point in $C_{rd}(\mathbb{T},R)$. That is, the dynamic system (3.5) has a weak solution $x \in C_{rd}(\mathbb{T},R)$, given by

$$x(t) = e_{\ominus p}(t,a)x_0 + \int_{[a,t)} e_{\ominus p}(t,\tau)f\left(\tau,x(\tau),x^{\sigma}(\tau)\right)\Delta\tau, \quad t \in \mathbb{T}.$$

**Step II**. Uniqueness of weak solution. Define

$$\|x\|_{\beta} = \sup_{t\in\mathbb{T}} \frac{|x(t)|}{e_{\beta}(t,a)},$$

where $\beta > 0$ is a constant, $x \in C_{rd}(\mathbb{T},R)$. Then we have the following results (see Lem. 3.3 of [13]):

(i) $\|\cdot\|_{\beta}$ is a norm and is equivalent to the sup-norm $\|\cdot\|_{C_{rd}}$;

(ii) $(C_{rd}(\mathbb{T},R),\|\cdot\|_{\beta})$ is a Banach space.

If $x \in C_{rd}(\mathbb{T},R)$ is a weak solution of the dynamic system (3.5), there is a constant $r > 0$ which is depend only on $x_0$, $p$, $q$ and $\lambda$ such that

$$\|x\|_{C_{rd}} \leq r.$$

In addition to we can show that there exists a constant $\beta > 0$ such that

$$\int_{[a,b)} \frac{\Delta\tau}{e_{\beta}(b,\tau)} < \frac{1}{4L(r)M}. \tag{3.8}$$

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Define

$$B = \{x \in C_{rd}(\mathbb{T}, R) \,|\, \|x\|_{C_{rd}} \leq r\},$$

$(B, \|\cdot\|_\beta)$ is a Banach space. Define a map $H$ on $B$ which is given by

$$(Hx)(t) = e_{\ominus p}(t,a)x_0 + \int_{[a,t)} e_{\ominus p}(t,\tau) f(\tau, x(\tau), x^\sigma(\tau))\,\Delta\tau.$$

Obviously, $HB \subseteq B$. By assumption [F](1) and (3.8), for any $x$, $y \in B$, we have

$$\begin{aligned}
\|Hx - Hy\|_\beta &= \sup_{t\in\mathbb{T}} \frac{1}{e_\beta(t,a)} \left| \int_{[a,t)} e_{\ominus p}(t,\tau) \left[ f(\tau, x(\tau), x^\sigma(\tau)) - f(\tau, y(\tau), y^\sigma(\tau)) \right] \Delta\tau \right| \\
&\leq \sup_{t\in\mathbb{T}} \frac{ML(r)}{e_\beta(t,a)} \int_{[a,t)} \left[ |x(\tau) - y(\tau)| + |x^\sigma(\tau) - y^\sigma(\tau)| \right] \Delta\tau \\
&\leq \|x-y\|_\beta \left[ ML(r) \sup_{t\in\mathbb{T}} \frac{1}{e_\beta(t,a)} \int_{[a,t)} \left( e_\beta(\tau,a) + e_\beta(\sigma(\tau),a) \right) \Delta\tau \right] \\
&= \|x-y\|_\beta \left[ ML(r) \sup_{t\in\mathbb{T}} \int_{[a,t)} \left( \frac{1}{e_\beta(t,\tau)} + \frac{1}{e_\beta(t,\sigma(\tau))} \right) \Delta\tau \right] \\
&= \|x-y\|_\beta \left[ ML(r) \sup_{t\in\mathbb{T}} \int_{[a,t)} \frac{1}{e_\beta(t,\sigma(\tau))} \left( 1 + \frac{1}{1+\beta\mu(\tau)} \right) \Delta\tau \right] \\
&\leq \|x-y\|_\beta \left[ 2ML(r) \sup_{t\in\mathbb{T}} \int_{[a,t)} \frac{\Delta\tau}{e_\beta(b,\tau)} \right] \\
&\leq \frac{1}{2}\|x-y\|_\beta.
\end{aligned}$$

Thus $H$ is a contractive map on $(B, \|\cdot\|_\beta)$, that is, $H$ has a unique fixed point $x$ on $(B, \|\cdot\|_\beta)$. This mean that the dynamic system (3.5) has a weak solution $x \in C_{rd}(\mathbb{T}, R)$, given by

$$x(t) = e_{\ominus p}(t,a)x_0 + \int_{[a,t)} e_{\ominus p}(t,\tau) f(\tau, x(\tau), x^\sigma(\tau))\,\Delta\tau, \quad t \in \mathbb{T}.$$

**Step III**. Property of weak solution. By Lemma 3.1, we can verify that $x$ is $\Delta$-differential $\Delta$-a.e. on $\mathbb{T}$ and

$$x^\Delta(t) + p(t)x^\sigma(t) = f(t, x(t), x^\sigma(t)) \ \ \Delta\text{-a.e. on } \mathbb{T}.$$

The proof is completed. □

## 4. Existence of optimal controls

In order to study the existence of optimal control, we discus $L^1$-strong-weak lower semicontinuity of integral functional first. In the following, we say that the function sequence $\{f_n | f_n : \mathbb{T} \longrightarrow R\}$ converges to $f$ in Lebesgue $\Delta$-measure, if for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu_\Delta \left( \{t \in \mathbb{T} \,|\, |f_n(t) - f(t)| > \varepsilon\} \right) = 0.$$

**Theorem 4.1.** *Suppose that function $f : R \times R \longrightarrow R \bigcup \{+\infty\}$ satisfies:*
(1) *$f(\cdot,\cdot)$ is lower semicontinuous on $R \times R$;*
(2) *$f \geq 0$ and $f(\xi,\cdot)$ is convex on $R$ for every $\xi \in R$.*

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*Set*

$$J(x,u)=\int_{[a,b)} f(x(t),u(t))\Delta t,$$

*if* $\{x_n\}$, $\{u_n\}\subseteq L^1(\mathbb{T},R)$ *and* $x_n \xrightarrow{s} x$, $u_n \xrightarrow{w} u$ *in* $L^1(\mathbb{T},R)$, *then*

$$J(x,u)\le \varliminf_{n\to\infty} J(x_n,u_n).$$

*Proof.* Suppose that $x_n \xrightarrow{s} x$, $u_n \xrightarrow{w} u$ in $L^1(\mathbb{T},R)$ and there exists a number $c\ge 0$ such that $J(x_n,u_n)\le c$ for all $n\in N$. Set $\alpha_n(t)=f(x_n(t),u_n(t))$ for all $t\in\mathbb{T}$, the function sequence $\{\alpha_n\}$ is bounded in $L^1(\mathbb{T},R)$. By Dunford-Pettis theorem and Mazur theorem, there are $\lambda_i^n\ge 0$, $i=1,2,\ldots,k_n$, $n\in N$, and $\alpha\in L^1(\mathbb{T},R)$ such that

$$\sum_{i=1}^{k_n}\lambda_i^n=1,\quad \alpha_n'(t)\equiv\sum_{i=1}^{k_n}\lambda_i^n\alpha_{n+i}(t)\longrightarrow\alpha(t)\ \Delta\text{-a.e. on }\mathbb{T}. \tag{4.1}$$

If the following inequality

$$f(x(t),u(t))\le\alpha(t)\ \Delta\text{-a.e. on }\mathbb{T} \tag{4.2}$$

holds, by Fatou lemma, it is easy to get

$$\begin{aligned} J(x,u) &\le \int_{[a,b)}\alpha(t)\Delta t=\int_{[a,b)}\varliminf_{n\to\infty}\alpha_n'(t)\Delta t\\ &\le \varliminf_{n\to\infty}\int_{[a,b)}\alpha_n'(t)\Delta t=\varliminf_{n\to\infty}\int_{[a,b)}\alpha_n(t)\Delta t=\varliminf_{n\to\infty}J(x_n,u_n).\end{aligned}$$

This implies that the theorem is true.

By the ideal of Theorem 2 in [12] and Lebesgue $\Delta$-measure theory, we can show that the inequality (4.2) holds. Here we only give an outline.

From $x_n\longrightarrow x$ in $L^1(\mathbb{T},R)$ as $n\to\infty$, it follows that there exists a subsequence, relabeled as $\{x_n\}$ such that $x_n(t)\longrightarrow x(t)$ as $n\to\infty$ $\Delta$-a.e. on $\mathbb{T}$. For $y\in R$ but fixed, define the measurable function $\beta_n$, $d_n^y(t):\mathbb{T}\longrightarrow R$ as follows:

$$\begin{aligned}\beta_n(t) &= \max\{0,\min\{f(x(t),v)\,|\,v\in R\}-\alpha_n(t)\},\ t\in\mathbb{T},\\ d_n^y(t) &= \max\{0,yu_n(t)-\sup\{yv\,|\,v\in W(f,x,\alpha_n+\beta_n)(t)\}\},\ t\in\mathbb{T},\end{aligned}$$

where $W(f,x,\alpha_n+\beta_n)(t)=\{v\in R\,|\,f(x(t),v)\le\alpha_n(t)+\beta_n(t)\}$, $t\in\mathbb{T}$. Using Lebesgue $\Delta$-integral, Lebesgue $\Delta$-measure, Banach-Steinhaus theorem and the convexity and lower semicontinuity of $f(x,\cdot)$ on $R$, one can show that the sequences $\{\beta_n\}$ and $\{d_n^y\}$ converges to 0 in Lebesgue $\Delta$-measure. By Riese theorem, there are subsequences, relabeled as $\{\beta_n\}$ and $\{d_n^y\}$, respectively, such that

$$\beta_n(t)\longrightarrow 0,\ d_n^y(t)\longrightarrow 0\ \Delta\text{-a.e. on }\mathbb{T}. \tag{4.3}$$

By the definition of the functions $\alpha_n$, $\beta_n$, $d_n^y$, $n\in N$, we have

$$u_n(t)\in B^y(W(f,x,\alpha_n+\beta_n)(t),d_n^y(t))\ \Delta\text{-a.e. on }\mathbb{T}, \tag{4.4}$$

where

$$B^y(W(f,x,\alpha_n+\beta_n)(t),d_n^y(t))=\{v\in R\,|\,d^y(v,W(f,x,\alpha_n+\beta_n)(t))\le d_n^y(t)\}.$$

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Therefore, (4.3) and (4.4) and convexity of the function $f(\xi,\cdot)$ yield that

$$u_n'(t) \in B^y\left(W\left(f,x,\alpha_n'+\beta_n'\right)(t), d_n'^y(t)\right) \ \Delta\text{-a.e. on } \mathbb{T}, \tag{4.5}$$

$$u_n' \longrightarrow \bar{u}, \quad \beta_n'(t) \longrightarrow 0, \quad d_n'^y(t) \longrightarrow 0 \ \Delta\text{-a.e. on } \mathbb{T}, \tag{4.6}$$

where

$$u_n' = \sum_{i=1}^{k_n} \lambda_i^n u_{n+i}, \quad \beta_n' = \sum_{i=1}^{k_n} \lambda_i^n \beta_{n+i}, \quad d_n'^y = \sum_{i=1}^{k_n} \lambda_i^n d_{n+i}^y.$$

In view of (4.1), (4.6), Egorov theorem and Lebesgue $\Delta$-measure theory, we can infer for (4.6) that

$$u(t) \in B^y\left(W\left(f,x,\alpha\right)(t),0\right) \ \Delta\text{-a.e. on } \mathbb{T}. \tag{4.7}$$

Suppose that a sequence $\{y_n\}$ is dense in $R$. Then, from assumption (2) we can obtain

$$W(f,x,\alpha)(t) = \bigcap_{n\in N} B^{y_n}\left(W(f,x,\alpha)(t),0\right) \ \Delta\text{-a.e. on } \mathbb{T}.$$

In view of (4.7) and arbitrariness of $y \in R$, this implies $u(t) \in W(f,x,\alpha)(t)$ $\Delta$-a.e. on $\mathbb{T}$. Moreover, the inequality (4.2) holds. □

In fact, it is easy to see that the condition $f \geq 0$ is not essential in Theorem 4.1.

Let $U_{ad}$ be a nonempty closed convex subset of $L^1(\mathbb{T},R)$, consider the controlled system

$$\begin{cases} x^{\Delta}(t) + p(t)x^{\sigma}(t) = f\left(t,x(t),x^{\sigma}(t)\right) + u(t), \quad t > a, \\ x(a) = x_0, \\ u \in U_{ad}, \end{cases} \tag{4.8}$$

where $p \in \Gamma_1(\mathbb{T})$. By Theorem 3.A, we have the following theorem.

**Theorem 4.2.** *Assume that $p \in \Gamma_1(\mathbb{T})$. Under the assumption of Theorem* 3.A*, the controlled system* (4.8) *has a unique weak solution $x \in C_{rd}(\mathbb{T},R)$ corresponding to the control $u \in U_{ad}$.*

We consider the Lagrange problem $(P^{\sigma})$: find $u^0 \in U_{ad}$ such that

$$J\left(u^0\right) \leq J(u) \text{ for all } u \in U_{ad},$$

where

$$J(u) = \int_{[a,b)} g\left(x(\tau),x^{\sigma}(\tau)\right)\Delta\tau + \int_{[a,b)} h(u(\tau))\Delta\tau,$$

$x$ is a weak solution of controlled system (4.8) corresponding to the control $u \in U_{ad}$.

We introduce the following assumptions on $g$ and $h$.

[G] (1) The function $g : R \times R \longrightarrow R$ is lower semicontinuous.

(2) There is a constant $c \in R$ such that

$$g(x,y) \geq c \text{ for all } x,\ y \in R.$$

[H] (1) The function $h : R \longrightarrow R$ is convex.

(2)

$$\lim_{|u|\to+\infty} \frac{h(u)}{|u|} = +\infty. \tag{4.9}$$

We study the Lagrange problems $(P^{\sigma})$.

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**Theorem 4.3.** *Let $p \in \Gamma_1(\mathbb{T})$. Under the assumptions* [F], [G] *and* [H], *the problem $(P^\sigma)$ has at least one solution.*

*Proof.* If $\inf\limits_{u\in U_{ad}} J(u) = +\infty$, there is nothing to prove.

Assume that $\inf\limits_{u\in U_{ad}} J(u) = c < +\infty$. By the assumptions [G](2) and [H](2), we have $c > -\infty$. There exists a minimizing sequence $\{u_n\} \subseteq U_{ad}$ such that

$$c \le J(u_n) = \int_{[a,b)} g(x_n(\tau), x_n^\sigma(\tau))\,\Delta\tau + \int_{[a,b)} h(u_n(\tau))\,\Delta\tau \le c + \frac{1}{n}, \tag{4.10}$$

for $n \ge N$, where $N$ is a natural number, $x_n$ is the weak solution of the controlled system (4.8) corresponding to $u_n$.

Next, we show that $\{u_n\}$ is weakly compact in $L^1(\mathbb{T}, R)$. By assumption (4.9), for any $\delta > 0$, there exists $\theta = \theta(\delta)$ such that

$$h(u) \ge \theta(\delta)|u| \text{ for all } |u| \ge \delta,$$

where $\lim\limits_{\delta\to+\infty} \theta(\delta) \longrightarrow +\infty$. Then, for every measurable subset $E \subseteq \mathbb{T}$, we have

$$\begin{aligned}
\int_E |u_n(\tau)|\,\Delta\tau &= \int_{\tilde{E}\bigcap\{s\in\mathbb{T}||\tilde{u}_n(s)|<\delta\}} |\tilde{u}_n(\tau)|\,\mathrm{d}\tau + \int_{\tilde{E}\bigcap\{s\in\mathbb{T}||\tilde{u}_n(s)|\ge\delta\}} |\tilde{u}_n(\tau)|\,\mathrm{d}\tau \\
&\le \delta\mu_\Delta(E) + \frac{1}{\theta(\delta)}\int_E |h(\tilde{u}_n(\tau))|\,\Delta\tau \\
&\le \delta\mu_\Delta(E) + \frac{C}{\theta(\delta)},
\end{aligned}$$

where $C > 0$ is independent of $\delta$. Note that

$$\lim_{\delta\to+\infty} \theta(\delta) = +\infty,$$

we infer that

$$\lim_{\delta\to+\infty} \sup_{u_n} \int_{\{s\in\mathbb{T}||u_n(s)|\ge\delta\}} |u_n(\tau)|\,\Delta\tau \le \lim_{\delta\to+\infty} \frac{C}{\theta(\delta)} = 0.$$

This means that $\{u_n\} \subseteq L^1(\mathbb{T}, R)$ is uniformly integrable (see p. 907, Prop. A.2.52 of [6]). By the Dunford-Pettis theorem (see p. 918, Thm. A.3.102 of [6]), $\{u_n\}$ is weakly compact in $L^1(\mathbb{T}, R)$. Since $U_{ad} \subseteq L^1(\mathbb{T}, R)$ is closed and convex, from the Mazur lemma, there is a subsequence, relabeled as $\{u_n\}$, $\bar{u} \in U_{ad}$ such that

$$u_n \xrightarrow{w} \bar{u} \text{ in } L^1(\mathbb{T}, R).$$

Since $x_n$ is the weak solution of the controlled system (4.8) corresponding to control $u_n$. Then there is a constant $r > 0$ such that

$$\|x_n\|_{C_{rd}} \le r. \tag{4.11}$$

Let

$$F_n(t) = f(t, x_n(t), x_n^\sigma(t)) \text{ for all } t \in \mathbb{T},$$

by assumption [F](2), we have

$$|F_n(t)| \le (1 + 2r)q(t) \text{ for all } t \in \mathbb{T}.$$

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Moreover, using the Dunford-Pettis theorem, there is a subsequence, relabeled as $\{F_n\}$, and $\overline{F} \in L^1(\mathbb{T}, R)$ such that

$$F_n \xrightarrow{w} \overline{F} \text{ in } L^1(\mathbb{T}, R).$$

Define

$$\eta_n(t) = \int_{[a,t)} e_{\ominus p}(t,\tau)\left[F_n(\tau) + u_n(\tau)\right]\Delta\tau, \quad \bar{\eta}(t) = \int_{[a,t)} e_{\ominus p}(t,\tau)\left[\overline{F}(\tau) + \bar{u}(\tau)\right]\Delta\tau$$

for all $t \in \mathbb{T}$. By Ascoli-Arzela theorem on time scales, we can show that

$$\|\eta_n - \bar{\eta}\|_{C_{rd}(\mathbb{T},R)} \longrightarrow 0 \text{ as } n \to +\infty.$$

Consider the following dynamic equation

$$\begin{cases} y^{\Delta}(t) + p(t)y^{\sigma}(t) = \overline{F}(t) + \bar{u}(t), \quad a \le t < b, \\ y(a) = x_0. \end{cases} \tag{4.12}$$

By Lemma 3.1, the dynamic equation (4.12) has a unique weak solution $\bar{x} \in C_{rd}(\mathbb{T}, R)$ given by

$$\bar{x}(t) = e_{\ominus p}(t,a)x_0 + \int_{[a,t)} e_{\ominus p}(t,\tau)\left[\overline{F}(\tau) + \bar{u}(\tau)\right]\Delta\tau.$$

By Proposition 3.1

$$x_n \longrightarrow \bar{x}, \quad x_n^{\sigma} \longrightarrow \bar{x}^{\sigma} \text{ in } C_{rd}(\mathbb{T}, R).$$

Using assumption [F](1), we have

$$\left|f\left(t, x_n(t), x_n^{\sigma}(t)\right) - f\left(t, \bar{x}(t), \bar{x}^{\sigma}(t)\right)\right| \le L(\rho)\left[\left|x_n(t) - \bar{x}(t)\right| + \left|x_n^{\sigma}(t) - \bar{x}^{\sigma}(t)(t)\right|\right], \ \forall t \in \mathbb{T},$$

for some constant $\rho > 0$. So

$$F_n \longrightarrow f\left(\cdot, \bar{x}(\cdot), \bar{x}^{\sigma}(\cdot)\right) \text{ in } L^1(\mathbb{T}, R).$$

By the uniqueness of limit, we have

$$\overline{F}(\cdot) = f\left(\cdot, \bar{x}(\cdot), \bar{x}^{\sigma}(\cdot)\right).$$

Furthermore,

$$\bar{x}(t) = e_{\ominus p}(t,a)x_0 + \int_{[a,t)} e_{\ominus p}(t,\tau)\left[f\left(\tau, \bar{x}(\tau), \bar{x}^{\sigma}(\tau)\right) + \bar{u}(\tau)\right]\Delta\tau.$$

Thus $\bar{x}$ is a weak solution of the controlled system (4.8) corresponding to control $\bar{u}$. By Theorem 4.1, we have

$$c \le J(\bar{u}) \le \lim_{n \to +\infty} J(u_n) = c.$$

This implies that $\bar{u}$ is an optimal control of the problem $(P^{\sigma})$. □

## 5. Necessary conditions of optimality

In this section, we derive the necessary conditions of optimality containing optimal controlled system, adjoint equation and optimal inequality.

For this purpose, we must study the backward problem for dynamic equations on time scales which can not be directly obtained from the Cauchy problem by simple transformation $s = T - t$. Particularly, we need to deal with the following backward problem of dynamic equations with term $\varphi^{\sigma}$ and term $\varphi$ in the right side

$$\begin{cases} \varphi^{\Delta}(t) = p(t)\varphi(t) + \omega(t)\varphi(t) + q(t)\varphi^{\sigma}(t) + \nu(t), \quad a \le t < b, \\ \varphi(b) = 0. \end{cases}$$

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Here we consider more general backward problem

$$\begin{cases} \varphi^{\Delta}(t) + p(t)\varphi^{\sigma}(t) = w\left(t, \varphi(t), \varphi^{\sigma}(t)\right), \quad a \le t < b, \\ \varphi(b) = \varphi_1. \end{cases} \tag{5.1}$$

First, in order to obtain the existence of weak solution of (5.1), we give a backward generalized Gronwall inequality on time scales which can not be directly obtained from Gronwall inequality.

**Theorem 5.1.** *Suppose that the function $\varphi \in C_{rd}(\mathbb{T}, R^+)$ satisfies the following inequality*

$$\varphi(t) \le \alpha + \int_{[t,b)} q(\tau)\varphi^{\lambda}(\tau)\Delta\tau + \int_{[t,b)} g(\tau)\varphi^{\sigma}(\tau)\Delta\tau \tag{5.2}$$

*where $q,\ g \in L^1(\mathbb{T}, R^+)$, $\alpha \ge 0$, $0 < \lambda < 1$. There exists a constant $M > 0$ such that*

$$\varphi(t) \le M \text{ for } t \in \mathbb{T}.$$

*Proof.* **Step I**. We show that if the function $x \in C_{rd}(\mathbb{T}, R^+)$ satisfies the following inequality

$$\varphi(t) \le f(t) + \int_{[t,b)} q(\tau)\varphi^{\sigma}(\tau)\Delta\tau, \tag{5.3}$$

where $q \in L^1(\mathbb{T}, R^+)$, $f \in C_{rd}(\mathbb{T}, R)$, then

$$\varphi(t) \le f(t) + \int_{[t,b)} e_q(\tau, t)q(\tau)f^{\sigma}(\tau)\Delta\tau \text{ for } t \in \mathbb{T}. \tag{5.4}$$

Define

$$\psi(t) = \int_{[t,b)} q(\tau)\varphi^{\sigma}(\tau)\Delta\tau \text{ for } t \in \mathbb{T}.$$

Then $\psi(b) = 0$ and

$$\psi^{\Delta}(t) = -q(t)\varphi^{\sigma}(t) \ge -q(t)\psi^{\sigma}(t) - q(t)f^{\sigma}(t) \ \Delta - \text{a.e. on } \mathbb{T}.$$

Note that,

$$[\psi(t)e_q(t,b)]^{\Delta} = \left[\psi^{\Delta}(t) + q(t)\psi^{\sigma}(t)\right] e_q(t,b) \ \Delta - \text{a.e. on } \mathbb{T},$$

therefore

$$-\psi(t)e_q(t,b) \ge -\int_{[t,b)} q(\tau)f^{\sigma}(\tau)e_q(\tau,b)\Delta\tau.$$

Moreover, we obtain

$$\varphi(t) \le f(t) + \int_{[t,b)} e_q(\tau,t)q(\tau)f^{\sigma}(\tau)\Delta\tau \text{ for } t \in \mathbb{T}.$$

**Step II**. By the inequality (5.2) and Step I, we have

$$\begin{aligned} \varphi(t) &\le \alpha + \int_{[t,b)} q(\tau)\varphi^{\lambda}(\tau)\Delta\tau + \alpha\int_{[t,b)} e_g(\tau,t)g(\tau)\Delta\tau \\ &\quad + \int_{[t,b)} e_g(\tau,t)g(\tau)\int_{[\tau,b)} q(\nu)\varphi^{\lambda}(\nu)\Delta\nu\Delta\tau \\ &\le \gamma + \gamma\int_{[a,b)} q(\tau)\varphi^{\lambda}(\tau)\Delta\tau, \end{aligned} \tag{5.5}$$

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where

$$\gamma = (\alpha + 1)(\beta + 1), \quad \beta = \int_{[a,b)} e_g(\tau, a) g(\tau) \Delta\tau.$$

Define

$$h(t) = \gamma + \gamma \int_{[t,b)} q(\tau)\varphi^{\lambda}(\tau)\Delta\tau + \gamma \int_{[a,b)} q(\tau)\varphi^{\lambda}(\tau)\Delta\tau,$$

where $t \in \mathbb{T}$, then $h$ is monotone decreasing function and

$$h^{\Delta}(t) \geq -\gamma q(t) h^{\lambda}(t).$$

$\Delta$-integrating from $t$ to $b$, we obtain

$$h^{1-\lambda}(t) - h^{1-\lambda}(b) \leq (1-\lambda)\gamma \int_{[t,b)} q(\tau)\Delta\tau.$$

Now, we observe that

$$(2h(b) - \gamma)^{1-\lambda} - h^{1-\lambda}(b) \leq (1-\lambda)\gamma \int_{[a,b)} q(\tau)\Delta\tau.$$

Furthermore, one can show that there exists a constant $M > 0$ such that $h(b) \leq M$. Thus

$$\varphi(t) \leq h(b) \leq M$$

for all $t \in \mathbb{T}$. The proof is completed. □

A function $\varphi \in C_{rd}(\mathbb{T}, R)$ is said to be a weak solution of the backward problem for nonlinear dynamical equation (5.1), if $\varphi$ satisfies the following integral equation

$$\varphi(t) = e_p(b,t)\varphi_1 - \int_{[t,b)} e_p(\tau, t) w\left(\tau, \varphi(\tau), \varphi^{\sigma}(\tau)\right) \Delta\tau, \quad t \in T. \tag{5.6}$$

Our assumption on function $w$ is as follows:

[W] (1) The function $w : \mathbb{T} \times R \times R \longrightarrow R$ is $\Delta$-measurable in $t \in \mathbb{T}$ and locally Lipschitz continuous, *i.e.* for all $\varphi_1$, $\varphi_2$, $\psi_1$, $\psi_2 \in R$, satisfying $|\varphi_1|$, $|\varphi_2|$, $|\psi_1|$, $|\psi_2| \leq \rho$, we have

$$|w(t, \varphi_1, \psi_1) - w(t, \varphi_2, \psi_2)| \leq L(\rho)\left(|\varphi_1 - \varphi_2| + |\psi_1 - \psi_2|\right) \text{ for all } t \in \mathbb{T}.$$

(2) There exist a constant $0 < \lambda < 1$ and a function $q \in L^1(\mathbb{T}, [0, +\infty))$ such that

$$|w(t, \varphi, \psi)| \leq q(t)\left(1 + |\varphi|^{\lambda} + |\psi|\right) \text{ for all } \varphi, \ \psi \in R.$$

By the idea of Theorem 3.A, we can prove the following result.

**Theorem 5.2.** *Let $p \in \Gamma_1(\mathbb{T})$. Under the assumption* [W]*, the backward problem of the nonlinear dynamical equation* (5.1) *has a unique weak solution $\varphi \in C_{rd}(\mathbb{T}, R)$.*

For the other nonlinear backward problem on time scales

$$\begin{cases} \varphi^{\Delta}(t) = p(t)\varphi(t) + w\left(t, \varphi(t), \varphi^{\sigma}(t)\right), & a \leq t < b, \\ \varphi(b) = \varphi_1, \end{cases} \tag{5.7}$$

we can also prove the following result.

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**Theorem 5.3.** *Let $p \in \Gamma_1(\mathbb{T})$, $\varphi_1 \in R$. Under the assumption* [W], (5.7) *has unique weak solution $\varphi \in C_{rd}(\mathbb{T}, R)$ given by*

$$\varphi(t) = e_{\ominus p}(b,t)\varphi_1 - \int_{[t,b)} e_{\ominus p}(\sigma(\tau),t) w\left(\tau, \varphi(\tau), \varphi^{\sigma}(\tau)\right) \Delta\tau, \quad t \in \mathbb{T}.$$

Now, we present the necessary conditions of optimality for the problem $(P^{\sigma})$.

Let $(\bar{x}, \bar{u})$ be optimal pair, assume that the following conditions are satisfied:

[F](3) Assume that $f(t,\cdot,\cdot) : R \times R \longrightarrow R$ is partial differentiable and $f_{\bar{x}}(\cdot) = f_{\bar{x}}(\cdot, \bar{x}(\cdot), \bar{x}^{\sigma}(\cdot)) \in L^1(\mathbb{T}, R)$, $f_{\bar{x}^{\sigma}}(\cdot) = f_{\bar{x}^{\sigma}}(\cdot, \bar{x}(\cdot), \bar{x}^{\sigma}(\cdot)) \in L^1(\mathbb{T}, R)$.

(4) $P(\cdot) = p(\cdot) - f_{\bar{x}^{\sigma}}(\cdot) \in \Gamma_1(\mathbb{T})$.

[G](3) $g : R \times R \longrightarrow R$ is convex.

**Theorem 5.4.** *Assume that $p \in \Gamma_1(\mathbb{T})$. Under the assumptions* [F](1)-(2)-(3)-(4), [G](2)-(3) *and* [H], *then, in order that the pair $(\bar{x}, \bar{u})$ be optimal pair of the problem $(P^{\sigma})$, it is necessary that there is $\varphi \in C_{rd}(\mathbb{T}, R)$ such that the following equations and inequality hold:*

$$\begin{cases} \bar{x}^{\Delta}(t) + p(t)\bar{x}^{\sigma}(t) = f\left(t, \bar{x}(t), \bar{x}^{\sigma}(t)\right) + \bar{u}(t), \quad a \le t < b, \\ \bar{x}(a) = x_0, \end{cases} \tag{5.8}$$

$$\begin{cases} \varphi^{\Delta}(t) = P(t)\varphi(t) - f_{\bar{x}}(t)\varphi^{\sigma}(t) - \left[1 + \mu(t)P(t)\right]\eta(t), \quad a \le t < b, \\ \eta \in \partial G\left(\bar{x}^{\sigma}\right), \\ \varphi(b) = 0, \end{cases} \tag{5.9}$$

$$\int_{[a,b)} \left[\frac{\varphi^{\sigma}(t)}{1 + \mu(t)P(t)} + \xi(t)\right] \left[u(t) - \bar{u}(t)\right] \Delta t \ge 0, \quad \forall u \in U_{ad},\ \forall \xi \in \partial H\left(\bar{u}\right), \tag{5.10}$$

*where*

$$\partial G\left(\bar{x}\right) = \left\{ \xi \in L^1(\mathbb{T}, R) \,\middle|\, \int_{[a,b)} \xi(t)\left[x(t) - \bar{x}(t)\right]\Delta t \le \int_{[a,b)} \left[g\left(x(t), x^{\sigma}(t)\right) - g\left(x(t), \bar{x}^{\sigma}(t)\right)\right]\Delta t \right\},$$

$$\partial H(\bar{u}) = \left\{ \xi \in L^{\infty}(\mathbb{T}, R) \,\middle|\, \int_{[a,b)} \xi(t)\left[u(t) - \bar{u}(t)\right]\Delta t \le \int_{[a,b)} \left[h(u(t)) - h(\bar{u}(t))\right]\Delta t \right\}.$$

*Proof.* Since $(\bar{x}, \bar{u}) \in C_{rd}(\mathbb{T}, R) \times U_{ad}$ is an optimal pair, it must satisfies the dynamic equation (5.8).

Since $U_{ad}$ is convex, it is clear that $u_{\epsilon} = \bar{u} + \epsilon(u - \bar{u}) \in U_{ad}$ for all $\epsilon \in [0,1]$ and $u \in U_{ad}$. Let $x_{\epsilon}$ be the weak solution of the following dynamic equation

$$\begin{cases} x_{\epsilon}^{\Delta}(t) + p(t)x_{\epsilon}^{\sigma}(t) = f\left(t, x_{\epsilon}(t), x_{\epsilon}^{\sigma}(t)\right) + u_{\epsilon}(t), \quad t > a, \\ x_{\epsilon}(a) = x_0, \end{cases}$$

then $x_{\epsilon}$ can be expressed by

$$x_{\epsilon}(t) = e_{\ominus p}(t,a)x_0 + \int_{[a,t)} e_{\ominus p}(t,\tau)\left[f\left(\tau, x_{\epsilon}(\tau), x_{\epsilon}^{\sigma}(\tau)\right) + u_{\epsilon}(\tau)\right]\Delta\tau, \quad t \in \mathbb{T}.$$

Considering

$$\begin{aligned} x_{\epsilon}(t) - \bar{x}(t) \quad &= \quad \int_{[a,t)} e_{\ominus p}(t,\tau)\left[f\left(\tau, x_{\epsilon}(\tau), x_{\epsilon}^{\sigma}(\tau)\right) - f\left(\tau, \bar{x}(\tau), \bar{x}^{\sigma}(\tau)\right)\right]\Delta\tau \\ &\quad + \epsilon \int_{[a,t)} e_{\ominus p}(t,\tau)\left[u(\tau) - \bar{u}(\tau)\right]\Delta\tau, \end{aligned} \tag{5.11}$$

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and set $y = \lim_{\epsilon \to 0} \frac{x_\epsilon - \bar{x}}{\epsilon}$. By the assumption [F](3), $u \longrightarrow x(u)$ is continuously Gateaux differentiable at $\bar{u}$ in the direction $u - \bar{u}$. Its Gateaux derivative $y$ satisfies the following dynamic equation

$$\begin{cases} y^{\Delta}(t) + p(t)y^{\sigma}(t) = f_{\bar{x}}(t)y(t) + f_{\bar{x}^{\sigma}}(t)y^{\sigma}(t) + u(t) - \bar{u}(t), \quad a \le t < b, \\ y(a) = 0. \end{cases} \tag{5.12}$$

This is usually known as the variational equation. By Theorem 3.A, the variational equation (5.12) has a unique weak $y \in C_{rd}(\mathbb{T}, R)$ given by

$$y(t) = \int_{[a,t)} e_{\ominus(p - f_{\bar{x}^{\sigma}})}(t, \tau) \left[ f_{\bar{x}}(\tau)y(\tau) + u(\tau) - \bar{u}(\tau) \right] \Delta\tau.$$

Define

$$G(x) = \int_{[a,b)} g\left(x(t), x^{\sigma}(t)\right) \Delta t \text{ for } x \in L^1(\mathbb{T}, R).$$

Since $g : R \times R \longrightarrow R$ is convex, $g : R \times R \longrightarrow R$ is continuous. By Theorem 4.1, $G$ is a lower semicontinuous functional on the real locally convex space $C_{rd}(\mathbb{T}, R)$. For any $x_1$, $x_2 \in C_{rd}(\mathbb{T}, R)$ and $\lambda \in [0, 1]$, by assumption [G](3), we have

$$\begin{aligned} G\left(\lambda x_1 + (1-\lambda)x_2\right) &= \int_{[a,b)} g\left(\lambda x_1(t) + (1-\lambda)x_2(t), \lambda x_1^{\sigma}(t) + (1-\lambda)x_2^{\sigma}(t)\right) \Delta t \\ &\le \lambda \int_{[a,b)} g\left(x_1(t), x_1^{\sigma}(t)\right) \Delta t + (1-\lambda) \int_{[a,b)} g\left(x_2(t), x_2^{\sigma}(t)\right) \Delta t \\ &= \lambda G(x_1) + (1-\lambda) G(x_2). \end{aligned}$$

Hence, $G$ is convex on $C_{rd}(\mathbb{T}, R)$. Note that $\bar{x}$ is a weak solution of the controlled system (5.8) corresponding to the optimal control $\bar{u}$, by Corollary 47.7 of [15], we know that $G$ is finite and continuous at $\bar{x}$. Moreover, one can see from Theorem 47.A of [15] that $G$ is subdifferentiable at $\bar{x} \in C_{rd}(\mathbb{T}, R)$ and the subdifferential $\partial G(\bar{x})$ of $G$ at $\bar{x}^{\sigma}$ is given by

$$\partial G(\bar{x}) = \left\{ \eta \in L^1(\mathbb{T}, R) \,\middle|\, \int_{[a,b)} \eta(t) \left[x(t) - \bar{x}(t)\right] \Delta t \le \int_{[a,b)} \left[g\left(x(t), x^{\sigma}(t)\right) - g\left(\bar{x}(t), \bar{x}^{\sigma}(t)\right)\right] \Delta t \right\}. \tag{5.13}$$

That is, $\partial G(\bar{x})$ is nonempty.

Define

$$H(u) = \int_{[a,b)} h(u(t)) \Delta t \text{ for } u \in L^1(\mathbb{T}, R).$$

Similarly as the functional $G$, the functional $H$ is subdifferentiable at $\bar{u} \in U_{ad} \subseteq L^1(\mathbb{T}, R)$ and the subdifferential $\partial H(\bar{u})$ of $H$ at $\bar{u}$ is given by

$$\partial H(\bar{u}) = \left\{ \xi \in L^{\infty}(\mathbb{T}, R) \,\middle|\, \int_{[a,b)} \xi(t) \left[u(t) - \bar{u}(t)\right] \Delta t \le \int_{[a,b)} \left[h(u(t)) - h(\bar{u}(t))\right] \Delta t \right\}, \tag{5.14}$$

and $\partial H(\bar{u})$ is nonempty.

Note that, $J = G + H$, we conclude that $J$ is subdifferentiable at $\bar{u} \in U_{ad}$. Computing the subgradient of $J$ at $\bar{u}$ in the direction $u - \bar{u}$, we find that

$$\partial J(\bar{u}, u - \bar{u}) = \int_{[a,b)} y(t)\eta(t) \Delta t + \int_{[a,b)} \xi(t) \left[u(t) - \bar{u}(t)\right] \Delta t, \tag{5.15}$$

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for all $\eta \in \partial G(\bar{x})$, $\xi \in \partial H(\bar{u})$. Since $\bar{u}$ is the optimal control, so

$$J(x_\epsilon, u_\epsilon) - J(\bar{x}, \bar{u}) \geq 0, \quad \forall \varepsilon \in [0,1], \quad \forall u \in U_{ad}.$$

Hence, for $\bar{u}$ to be optimal it is necessary that

$$\int_{[a,b)} y(t)\eta(t)\Delta t + \int_{[a,b)} \xi(t)\left[u(t) - \bar{u}(t)\right]\Delta t \geq 0 \tag{5.16}$$

for all $\eta \in \partial G(\bar{x})$, $\xi \in \partial H(\bar{u})$.

Let $\eta \in \partial G(\bar{x})$, consider the following adjoint equation

$$\begin{cases} \varphi^{\Delta}(t) = P(t)\varphi(t) - f_{\bar{x}}(t)\varphi^{\sigma}(t) - (1+\mu(t)P(t))\,\eta(t), \quad a \leq t < b, \\ \varphi(b) = 0. \end{cases} \tag{5.17}$$

We note that $P \in \Gamma_1$, by virtue of Theorem 5.3, the adjoint equation (5.17) has a unique weak solution $\varphi \in C_{rd}(\mathbb{T}, R)$ given by

$$\varphi(t) = \int_{[t,b)} e_{\ominus P}(\sigma(\tau), t)\left[f_{\bar{x}}(\tau)\varphi^{\sigma}(\tau) + (1+\mu(\tau)P(\tau))\,\eta(\tau)\right]\Delta\tau.$$

Using the integration by parts (see Thm. 2.1) and (5.17), we have

$$\begin{aligned} \int_{[a,b)} y(t)\eta(t)\Delta t &= \int_{[a,b)} y(t)\frac{P(t)\left(\varphi^{\sigma}(t) - \mu(t)\varphi^{\Delta}(t)\right) - \varphi^{\Delta}(t) - f_{\bar{x}}(t)\varphi^{\sigma}(t)}{1+\mu(t)P(t)}\Delta t \\ &= \int_{[a,b)} y(t)\left[-\varphi^{\Delta}(t) + \frac{P(t) - f_{\bar{x}}(t)}{1+\mu(t)P(t)}\varphi^{\sigma}(t)\right]\Delta t \\ &= \int_{[a,b)} \varphi^{\sigma}(t)\left[y^{\Delta}(t) + \frac{P(t) - f_{\bar{x}}(t)}{1+\mu(t)P(t)}y(t)\right]\Delta t \\ &= \int_{[a,b)} \frac{\varphi^{\sigma}(t)}{1+\mu(t)\left[p(t) - f_{\bar{x}^{\sigma}}(t)\right]}\left[u(t) - \bar{u}(t)\right]\Delta t. \end{aligned} \tag{5.18}$$

Further, substituting (5.18) into (5.16), we have the following inequality

$$\int_{[a,b)} \left[\frac{\varphi^{\sigma}(t)}{1+\mu(t)\left[p(t) - f_{\bar{x}^{\sigma}}(t)\right]} + \xi(t)\right]\left[u(t) - \bar{u}(t)\right]\Delta t \geq 0 \tag{5.19}$$

for $\forall u \in U_{ad}$ and $\forall \xi \in \partial H(\bar{u})$. This completes the proof of all the necessary conditions as stated in the theorem. □

For the special case of the problem $(P^{\sigma})$, we obtain the following results immediately.

**Remark 5.1.** Assume that $p \in \Gamma_1(\mathbb{T})$ and
(1) $f(t, x(t), x^{\sigma}(t)) = f(t, x^{\sigma}(t))$ and $f(t,\cdot) : R \longrightarrow R$ is differentiable, $f_{\bar{x}^{\sigma}}(\cdot) = f_{\bar{x}^{\sigma}}(\cdot, \bar{x}^{\sigma}(\cdot)) \in L^1(\mathbb{T}, R)$;
(2) $g(x(t), x^{\sigma}(t)) = g(x^{\sigma}(t))$ and $g(\cdot) : R \longrightarrow R$ is differentiable, $g_{\bar{x}^{\sigma}}(\cdot) \in L^1(\mathbb{T}, R)$;
(3) $h(\cdot) : R \longrightarrow R$ is differentiable and $h_{\bar{u}}(\cdot) \in L^{\infty}(\mathbb{T}, R)$.
Under the assumptions [F](1)-(2)-(4), then, in order that the pair $(\bar{x}, \bar{u})$ be optimal pair of the problem $(P^{\sigma})$, it is necessary that there is $\varphi \in C_{rd}(\mathbb{T}, R)$ such that the following equations and inequality hold:
(i) the controlled equation

$$\begin{cases} \bar{x}^{\Delta}(t) + p(t)\bar{x}^{\sigma}(t) = f\left(t, \bar{x}^{\sigma}(t)\right) + \bar{u}(t), \quad a \leq t < b, \\ \bar{x}(a) = x_0, \end{cases} \tag{5.20}$$

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(ii) the adjoint equation (see Thm. 6.1 of [5])

$$\begin{cases} \varphi^{\Delta}(t) = p(t)\varphi(t) - f_{\bar{x}^{\sigma}}(t)\varphi(t) - g_{\bar{x}^{\sigma}}(t), \quad a \le t < b, \\ \varphi(b) = 0, \end{cases} \tag{5.21}$$

(iii) the optimal condition (see Thm. 3.3 of [9])

$$\int_{[a,b)} [\varphi(t) + h_{\bar{u}}(t)]\,[u(t) - \bar{u}(t)]\,\Delta t \ge 0, \quad \forall u \in U_{ad}. \tag{5.22}$$

In order to discover relation both the optimal control problem and the variational problem on time scales we give Hamiltonian formulation to Theorem 5.A further.

Define

$$\mathbf{g}(x(t)) = g\left(x(t), x^{\sigma}(t)\right), \ \mathbf{f}(t, x(t)) = f\left(t, x(t), x^{\sigma}(t)\right) - p(t)x^{\sigma}(t),$$
$$H(t, x(t), u(t), \varphi(t)) = \mathbf{g}\left(x(t)\right) + h(u(t)) + \varphi^{\sigma}(t)\left[\mathbf{f}(t, x(t)) + u(t)\right],$$

we have

$$J(u) = \int_{[a,b)} H(t, x(t), u(t), \varphi(t))\Delta t - \int_{[a,b)} \varphi^{\sigma}(t)x^{\Delta}(t)\Delta t.$$

Hence, the integration by parts formula (2.3) yields

$$J(u) = \int_{[a,b)} H(t, x(t), u(t), \varphi(t))\Delta t + \int_{[a,b)} \varphi^{\Delta}(t)x(t)\Delta t - \varphi(b)x(b) + \varphi(a)x_0.$$

If we suppose that $\mathbf{g} : R \longrightarrow R$, $h : R \longrightarrow R$ $\mathbf{f}(t, \cdot) : R \longrightarrow R$ are differential and $\mathbf{g}_{\bar{x}}(\cdot) = \mathbf{g}_{\bar{x}}(\bar{x}(\cdot)) \in L^1(\mathbb{T}, R)$, $\mathbf{f}_{\bar{x}}(\cdot) = \mathbf{f}_{\bar{x}}(\cdot, \bar{x}(\cdot)) \in L^1(\mathbb{T}, R)$, $h_{\bar{u}}(\cdot) = h_{\bar{u}}(\bar{u}(\cdot)) \in L^{\infty}(\mathbb{T}, R)$. Hence $J$ is Gateaux differentiable, and the $G$-derivative of $J$ at $\bar{u}$ in the direction $u - \bar{u}$ can be given by

$$\begin{aligned} \lim_{\varepsilon \to 0} \frac{J(\bar{u} + \varepsilon(u - \bar{u})) - J(\bar{u})}{\varepsilon} &= \int_{[a,b)} H_{\bar{x}}(t)y(t)\Delta t + \int_{[a,b)} \varphi^{\Delta}(t)y(t)\Delta t - \varphi(b)y(b) \\ &\quad + \int_{[a,b)} H_{\bar{u}}(t)\,[u(t) - \bar{u}(t)]\,\Delta t, \end{aligned} \tag{5.23}$$

where $H_{\bar{x}}(t) = H_{\bar{x}}(t, \bar{x}(t), \bar{u}(t), \varphi(t))$, $H_{\bar{u}}(t) = H_{\bar{u}}(t, \bar{x}(t), \bar{u}(t), \varphi(t))$. Note that $\bar{u}$ is the optimal control, it follows that $J$ attains its minimum at $\bar{u}$. Hence

$$\int_{[a,b)} H_{\bar{x}}(t)y(t)\Delta t + \int_{[a,b)} \varphi^{\Delta}(t)y(t)\Delta t - \varphi(b)y(b) + \int_{[a,b)} H_{\bar{u}}(t)\,[u(t) - \bar{u}(t)]\,\Delta t \ge 0.$$

Let $\varphi$ be any solution of

$$\begin{cases} \varphi^{\Delta}(t) = -H_{\bar{x}}(t), \quad a \le t < b, \\ \varphi(b) = 0. \end{cases} \tag{5.24}$$

By our assumption, this equation has a unique solution $\varphi \in C_{rd}(\mathbb{T}, R)$. Further

$$\int_{[a,b)} H_{\bar{u}}(t)\,[u(t) - \bar{u}(t)]\,\Delta t \ge 0 \text{ for all } u \in U_{ad},$$

that is,

$$\int_{[a,b)} [h_{\bar{u}}(\bar{u}(t)) + \varphi^{\sigma}(t)]\,[u(t) - \bar{u}(t)]\,\Delta t \ge 0 \text{ for all } u \in U_{ad}.$$

This means that the adjoint equation and the optimal inequality are derived.

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Next we derive the Hamiltonian formulations to Theorem 5.A. Set

$$\mathbf{J}(x,u)=\int_{[a,b)}H(t,x(t),u(t),\varphi(t))\Delta t,$$

$\mathcal{J}$ is subdifferentiable at $(\bar{x},\bar{u})$ and the subdifferential $\partial\mathbf{J}(\bar{x},\bar{u})$ of $\mathbf{J}$ at $(\bar{x},\bar{u})$ is given by

$$\partial\mathbf{J}(\bar{x},\bar{u})=\left\{(\nu,\omega)\in L^1(\mathbb{T},R)\times L^\infty(\mathbb{T},R)\;\middle|\;\begin{array}{l}\int_{[a,b)}\nu(t)\left[x(t)-\bar{x}(t)\right]\Delta t+\int_{[a,b)}\omega(t)\left[u(t)-\bar{u}(t)\right]\Delta t\\ \\ \leq\int_{[a,b)}\left[H(t,x(t),u(t),\varphi(t))-H(t,\bar{x}(t),\bar{u}(t),\varphi(t))\right]\Delta t\end{array}\right\},$$

and $\partial\mathbf{J}(\bar{x},\bar{u})$ is nonempty. Computing the subgradient of $J$ at $\bar{u}$ in the direction $u-\bar{u}$, we find that

$$\partial J(\bar{u},u-\bar{u})=\int_{[a,b)}y(t)\nu(t)\Delta t+\int_{[a,b)}\omega(t)\left[u(t)-\bar{u}(t)\right]\Delta t+\int_{[a,b)}\varphi^{\Delta}(t)y(t)\Delta t-\varphi(b)y(b),\tag{5.25}$$

for all $(\nu,\omega)\in\partial\mathbf{J}(\bar{x},\bar{u})$, where $y$ satisfies the variational equation (5.12). Since $\bar{u}$ is the optimal control, it follows that $J$ attains its minimum at $\bar{u}$. This implies

$$\int_{[a,b)}y(t)\nu(t)\Delta t+\int_{[a,b)}\omega(t)\left[u(t)-\bar{u}(t)\right]\Delta t+\int_{[a,b)}\varphi^{\Delta}(t)y(t)\Delta t-\varphi(b)y(b)\geq 0,\tag{5.26}$$

for all $(\nu,\omega)\in\partial\mathbf{J}(\bar{x},\bar{u})$. Consider the following adjoint equation

$$\left\{\begin{array}{l}\varphi^{\Delta}(t)=-\nu(t),\quad a\leq t<b,\\ \varphi(b)=0.\end{array}\right.\tag{5.27}$$

This equation (5.27) has a unique weak solution $\varphi\in C_{rd}(\mathbb{T},R)$ given by

$$\varphi(t)\;=\;\int_{[t,b)}\nu(\tau)\Delta\tau.$$

Further, we have

$$\int_{[a,b)}\omega(t)\left[u(t)-\bar{u}(t)\right]\Delta t\geq 0.$$

We obtain the Hamiltonian formulations to Theorem 5.A.

**Theorem 5.5.** *Assume that $p\in\Gamma_1(\mathbb{T})$. Under the assumptions* [F](1)-(2)-(3)-(4), [G](2)-(3) *and* [H], *then, in order that the pair $(\bar{x},\bar{u})$ be optimal pair of the problem $(P^\sigma)$, it is necessary that there is $\varphi\in C_{rd}(\mathbb{T},R)$ such that the following equations and inequality hold:*

$$\left\{\begin{array}{l}\bar{x}^{\Delta}(t)+p(t)\bar{x}^{\sigma}(t)=f\left(t,\bar{x}(t),\bar{x}^{\sigma}(t)\right)+\bar{u}(t),\quad a\leq t<b,\\ \bar{x}(a)=x_0,\end{array}\right.\tag{5.28}$$

$$\left\{\begin{array}{l}\varphi^{\Delta}(t)=-\nu(t),\quad a\leq t<b,\\ \varphi(b)=0,\end{array}\right.\tag{5.29}$$

$$\int_{[a,b)}\omega\left[u(t)-\bar{u}(t)\right]\Delta t\geq 0,\quad\forall u\in U_{ad},\tag{5.30}$$

*where $(\nu,\omega)\in\partial\boldsymbol{J}(\bar{x},\bar{u})$.*

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## 6. Another class of control problems

In this section, as contrast, we consider another class of optimal control problem (P), *i.e.*, consider optimal control problem (P): find $\bar{u} \in U_{ad}$ such that

$$J(\bar{u}) \leq J(u) \text{ for all } u \in U_{ad},$$

where the pairs $(x, u) \in C_{rd}(\mathbb{T}, R) \times U_{ad}$ satisfies the following dynamic equation on time scales

$$\begin{cases} x^{\Delta}(t) = p(t)x(t) + f\left(t, x(t), x^{\sigma}(t)\right) + u(t), \quad a \leq t < b, \\ x(a) = x_0, \\ u \in U_{ad}. \end{cases} \tag{6.1}$$

**Definition 6.1.** A function $x \in C_{rd}(\mathbb{T}, R)$ is said to be a weak solution of the dynamic system (6.1), if $x$ satisfies the following integral equation

$$x(t) = e_p(t, a)x_0 + \int_{[a,t)} e_p(t, \sigma(\tau)) \left[f\left(\tau, x(\tau), x^{\sigma}(\tau)\right) + u(\tau)\right] \Delta\tau, \quad t \in \mathbb{T}. \tag{6.2}$$

For the optimal control problem (P), by similar procedures we can show the following results.

**Theorem 6.1.** *Let $p \in \Gamma_1(\mathbb{T})$. Under the assumption* [F]*, the dynamic system* (6.1) *has a unique weak solution $x \in C_{rd}(\mathbb{T}, R)$.*

**Theorem 6.2.** *Suppose that $p \in \Gamma_1(\mathbb{T})$. Under the assumptions* [F], [G] *and* [H], *the problem* (P) *has at least one solution.*

**Theorem 6.3.** *Assume that $p \in \Gamma_1(\mathbb{T})$ and $-f_{\bar{x}^{\sigma}}(\cdot) \in \Gamma_1(\mathbb{T})$. Under the assumptions* [F](1)-(2)-(3), [G](2)-(3) *and* [H], *then, in order that the pair $(\bar{x}, \bar{u})$ be optimal pair of the problem* (P), *it is necessary that there exists a function $\varphi \in C_{rd}(\mathbb{T}, R)$ such that the following equations and inequality hold:*

$$\begin{cases} \bar{x}^{\Delta}(t) = p(t)\bar{x}(t) + f\left(t, \bar{x}(t), \bar{x}^{\sigma}(t)\right) + \bar{u}(t), \quad a \leq t < b, \\ \bar{x}(a) = x_0, \end{cases} \tag{6.3}$$

$$\begin{cases} \varphi^{\Delta}(t) = -f_{\bar{x}^{\sigma}}(t)\varphi(t) - \left[p(t) + f_{\bar{x}}(t)\right]\varphi^{\sigma}(t) - \left[1 - \mu(t)f_{\bar{x}^{\sigma}}(t)\right]\eta(t), \quad a \leq t < b, \\ \eta \in \partial G\left(\bar{x}\right), \\ \varphi(b) = 0. \end{cases} \tag{6.4}$$

$$\int_{[a,b)} \left[\frac{\varphi^{\sigma}(t)}{1 - \mu(t)f_{\bar{x}^{\sigma}}(t)} + \xi(t)\right] \left[u(t) - \bar{u}(t)\right] \Delta t \geq 0 \text{ for } \forall u \in U_{ad}, \quad \forall \xi \in \partial H\left(\bar{u}\right). \tag{6.5}$$

**Remark 6.1.** Assume that $p \in \Gamma_1(\mathbb{T})$ and
(1) $f(t, x(t), x^{\sigma}(t)) = f(t, x(t))$ and $f(t, \cdot) : R \longrightarrow R$ is differentiable, $f_{\bar{x}}(\cdot) = f_{\bar{x}}(\cdot, \bar{x}(\cdot)) \in L^1(\mathbb{T}, R)$;
(2) $g(x(t), x^{\sigma}(t)) = g(x(t))$ and $g(\cdot) : R \longrightarrow R$ is differentiable, $g_{\bar{x}}(\cdot) \in L^1(\mathbb{T}, R)$;
(3) $h(\cdot) : R \longrightarrow R$ is differentiable and $h_{\bar{u}}(\cdot) \in L^{\infty}(\mathbb{T}, R)$.
Under the assumptions [F](1)-(2), then, in order that the pair $(\bar{x}, \bar{u})$ be optimal pair of the problem (P), it is necessary that there is $\varphi \in C_{rd}(\mathbb{T}, R)$ such that the following equations and inequality hold:

$$\begin{cases} \bar{x}^{\Delta}(t) = p(t)\bar{x}(t) + f\left(t, \bar{x}(t)\right) + \bar{u}(t), \quad a \leq t < b, \\ \bar{x}(a) = x_0, \end{cases} \tag{6.6}$$

$$\begin{cases} \varphi^{\Delta}(t) + p(t)\varphi^{\sigma}(t) = -f_{\bar{x}}(t)\varphi^{\sigma}(t) - g_{\bar{x}}(t), \quad a \leq t < b, \\ \varphi(b) = 0, \end{cases} \tag{6.7}$$

$$\int_{[a,b)} \left[\varphi^{\sigma}(t) + h_{\bar{u}}(t)\right] \left[u(t) - \bar{u}(t)\right] \Delta t \geq 0 \quad \text{for } \forall u \in U_{ad}. \tag{6.8}$$

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TABLE 1. Difference between the problem $(P^\sigma)$ and problem (P).

| Item | Problem $(P^\sigma)$ | Problem (P) |
|---|---|---|
| Cost functional | $J(u)=\int_{[a,b)} g\left(x(t),x^\sigma(t)\right)\Delta t+\int_{[a,b)} h(u(t))\Delta t$ | |
| Controlled system | $\begin{cases} x^\Delta(t)+p(t)x^\sigma(t)=f\left(t,x(t),x^\sigma(t)\right)+u(t) \\ x(a)=x_0 \end{cases}$ | $\begin{cases} x^\Delta(t)=p(t)x(t)+f\left(t,x(t),x^\sigma(t)\right)+u(t) \\ x(a)=x_0 \end{cases}$ |
| Weak solution of controlled system | $x(t)=e_{\ominus p}(t,a)x_0$ $+\int_{[a,t)} e_{\ominus p}(t,\tau)[f\left(\tau,x(\tau),x^\sigma(\tau)\right)+u(\tau)]\Delta\tau$ | $x(t)=e_p(t,a)x_0$ $+\int_{[a,t)} e_p(t,\sigma(\tau))[f\left(\tau,x(\tau),x^\sigma(\tau)\right)+u(\tau)]\Delta\tau$ |
| Adjoint equation | $\begin{cases} \varphi^\Delta(t)=P(t)\varphi(t)-f_{\bar{x}}(t)\varphi^\sigma(t) \\ \qquad -[1+\mu(t)P(t)]\eta(t) \\ \eta\in\partial G\left(\bar{x}\right) \\ \varphi(b)=0 \end{cases}$ | $\begin{cases} \varphi^\Delta(t)=q(t)\varphi(t)-[p(t)+f_{\bar{x}}(t)]\,\varphi^\sigma(t) \\ \qquad -\left[1-\mu(t)f_{\bar{x}^\sigma}(t)\right]\eta(t) \\ \eta\in\partial G(\bar{x}) \\ \varphi(b)=0 \end{cases}$ |
| Weak solution of adjoint equation | $\varphi(t)=\int_{[t,b)} e_{\ominus P}(\sigma(\tau),t)[\beta(\tau)+\eta(\tau)]\Delta\tau$ $+\int_{[t,b)} e_{\ominus P}(\sigma(\tau),t)\omega(\tau)\Delta\tau$ | $\varphi(t)=\int_{[t,b)} e_{\ominus q}(\tau,t)[\beta(\tau)+\eta(\tau)]\Delta\tau$ $+\int_{[t,b)} e_{\ominus q}(\tau,t)\psi(\tau)\Delta\tau$ |
| Optimal inequality | $\int_{[a,b)}\left[\frac{\varphi^\sigma(t)}{1+\mu(t)P(t)}+\xi(t)\right][u(t)-\bar{u}(t)]\,\Delta t\geq 0$ | $\int_{[a,b)}\left[\frac{\varphi^\sigma(t)}{1-\mu(t)f_{\bar{x}^\sigma}(t)}+\xi(t)\right][u(t)-\bar{u}(t)]\,\Delta t\geq 0$ |

**Theorem 6.4.** *Assume that $p\in\Gamma_1(\mathbb{T})$ and $-f_{\bar{x}^\sigma}(\cdot)\in\Gamma_1(\mathbb{T})$. Under the assumptions* [F](1)-(2)-(3), [G](2)-(3) *and* [H], *then, in order that the pair $(\bar{x},\bar{u})$ be optimal pair of the problem* (P), *it is necessary that there is a function $\varphi\in C_{rd}(\mathbb{T},R)$ such that the following equations and inequality hold:*

$$\begin{cases} \bar{x}^\Delta(t)=p(t)\bar{x}(t)+f\left(t,\bar{x}(t)\right)+\bar{u}(t), \quad a\leq t<b, \\ \bar{x}(a)=x_0, \end{cases} \tag{6.9}$$

$$\begin{cases} \varphi^\Delta(t)=-\nu(t), \quad a\leq t<b, \\ \varphi(b)=0, \end{cases} \tag{6.10}$$

$$\int_{[a,b)}\omega(t)\left[u(t)-\bar{u}(t)\right]\Delta t\geq 0 \text{ for } \forall u\in U_{ad}, \tag{6.11}$$

*where $(\nu,\omega)\in\partial\mathcal{J}\left(\bar{x},\bar{u}\right)$ and $\partial\mathcal{J}\left(\bar{x},\bar{u}\right)$ given by*

$$\partial\mathcal{J}\left(\bar{x},\bar{u}\right)=\left\{(\nu,\omega)\in L^1(\mathbb{T},R)\times L^\infty(\mathbb{T},R)\;\middle|\;\begin{array}{l}\int_{[a,b)}\nu(t)\left[x(t)-\bar{x}(t)\right]\Delta t+\int_{[a,b)}\omega(t)\left[u(t)-\bar{u}(t)\right]\Delta t \\ \leq\int_{[a,b)}\left[\mathcal{H}(t,x(t),u(t),\varphi(t))-\mathcal{H}(t,\bar{x}(t),\bar{u}(t),\varphi(t))\right]\Delta t\end{array}\right\},$$

$$\mathcal{H}(t,x(t),u(t),\varphi(t))=\boldsymbol{g}(x(t))+h(u(t))+\varphi^\sigma(t)\left[f\left(t,x(t),x^\sigma(t)\right)+p(t)x(t)+u(t)\right].$$

Define

$$F(t)=f\left(t,x(t),x^\sigma(t)\right), \quad q(t)=-f_{\bar{x}^\sigma}(t), \quad \beta(t)=f_{\bar{x}}(t)\varphi^\sigma(t), \quad \psi(t)=\mu(t)q(t)\eta(t), \quad \omega(t)=\mu(t)P(t)\eta(t).$$

Now, we can show the relationship and difference between the problem (P) and problem $(P^\sigma)$ in Tables 1 and 2.

## 7. EXAMPLE

In this section, some examples are given to illustrate our theory.

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TABLE 2. Equivalence of different formulations of control problems.

| Item | Problem $(P^\sigma)$ | Problem $(P)$ |
|---|---|---|
| Cost functional | $J(u)=\int_{[a,b)} g\left(x^\sigma(t)\right)\Delta t+\int_{[a,b)} h(u(t))\Delta t$ | $J(u)=\int_{[a,b)} g\left(x(t)\right)\Delta t+\int_{[a,b)} h(u(t))\Delta t$ |
| Controlled system | $\begin{cases} x^\Delta(t)+p(t)x^\sigma(t)=f\left(t,x^\sigma(t)\right)+u(t) \\ x(a)=x_0 \end{cases}$ | $\begin{cases} x^\Delta(t)=p(t)x(t)+f\left(t,x(t)\right)+u(t) \\ x(a)=x_0 \end{cases}$ |
| Weak solution of controlled system | $x(t)=e_{\ominus p}(t,a)x_0$ $+\int_{[a,t)} e_{\ominus p}(t,\tau)[f\left(\tau,x^\sigma(\tau)\right)+u(\tau)]\Delta\tau$ | $x(t)=e_p(t,a)x_0$ $+\int_{[a,t)} e_p(t,\sigma(\tau))[f\left(\tau,x(\tau)\right)+u(\tau)]\Delta\tau$ |
| Adjoint equation | $\begin{cases} \varphi^\Delta(t)=p(t)\varphi(t)-f_{\bar{x}^\sigma}(t)\varphi(t)-g'\left(\bar{x}^\sigma(t)\right) \\ \varphi(b)=0 \end{cases}$ | $\begin{cases} \varphi^\Delta(t)+p(t)\varphi^\sigma(t)=-f_{\bar{x}}(t)\varphi^\sigma(t)-g'(\bar{x}(t)) \\ \varphi(b)=0 \end{cases}$ |
| Weak solution of adjoint equation | $\varphi(t)=\int_{[t,b)} e_{\ominus P}(\sigma(\tau),t)g'\left(\bar{x}^\sigma(\tau)\right)\Delta\tau$ | $\varphi(t)=\int_{[t,b)} e_{\ominus p}(\tau,t)[f_{\bar{x}}(\tau)\varphi^\sigma(\tau)+g'(\bar{x}(\tau))]\Delta\tau$ |
| Optimal inequality | $\int_{[a,b)}\left[\varphi(t)+h'(\bar{u}(t))\right]\left[u(t)-\bar{u}(t)\right]\Delta t\geq 0$ | $\int_{[a,b)}\left[\varphi^\sigma(t)+h'(\bar{u}(t))\right]\left[u(t)-\bar{u}(t)\right]\Delta t\geq 0$ |

**Example 7.1.** Let $\mathbb{T}=[0,3]\subset R$ be real interval, consider the following dynamic equation

$$\begin{cases} \dot{x}(t)+\ln\frac{1}{t}\ x(t)=u(t),\ t>0, \\ x(0)=x_0, \\ u\in L^r([0,3],R)\ (r>1) \end{cases} \tag{7.1}$$

with the cost functional

$$J(u)=\int_0^3\left(|x(t)|^2+|u(t)|^r\right)\mathrm{d}t.$$

Since $r>1$, the space $L^{\frac{r}{r-1}}([0,3],R)$ is locally uniformly convex. Moreover, the functional $u\longrightarrow\|u\|_{L^r}$ is Frechet differentiable. Using the necessary conditions of optimality given by Theorem 5, one can show the following theorem.

**Theorem 7.1.** *In order that the pair $(\bar{x},\bar{u})$ is optimal, it is necessary that there exists a function $\varphi\in C([0,3],R)$ such that the following equations and inequality hold:*

$$\begin{cases} \dot{\bar{x}}(t)+\ln\frac{1}{t}\ \bar{x}(t)=\bar{u}(t), \quad t\in(0,3] \\ \bar{x}(0)=x_0; \end{cases} \tag{7.2}$$

$$\begin{cases} \dot{\varphi}(t)=\ln\frac{1}{t}\ \varphi(t)-2\bar{x}(t), \quad t\in[0,3) \\ \varphi(3)=0; \end{cases} \tag{7.3}$$

$$\int_0^3\left[r\bar{u}^{r-1}(t)+\varphi(t)\right]\left[u(t)-\bar{u}(t)\right]\mathrm{d}t\geq 0. \tag{7.4}$$

**Example 7.2.** (see Ex. 2 of [10]). We consider the quantum time scale $\mathbb{T}=\left\{t=q^k|q>1,\ 0\leq k\leq N_0\right\}$. Obviously, we have $\sigma(t)=qt$, $\mu(t)=(q-1)t$. Let

$$U_{ad}=\left\{u:\mathbb{T}\longrightarrow R|\sum_{k=0}^{N_0}\mu\left(q^k\right)\left|u\left(q^k\right)\right|<+\infty\right\},\ \|u\|=\sum_{k=0}^{N_0}\mu\left(q^k\right)\left|u\left(q^k\right)\right|,$$

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one can verify that $U_{ad}$ is a nonempty closed convex set. Study the following dynamic equation

$$\begin{cases} x\left(q^{k+1}\right) = \frac{1}{1+\mu(q^k)p(q^k)}x\left(q^k\right) + \frac{\mu(q^k)}{1+\mu(q^k)p(q^k)}u\left(q^k\right), \quad 0 \le k \le N_0 - 1, \\ x(1) = x_0, \\ u \in U_{ad}, \end{cases} \tag{7.5}$$

with the cost functional

$$J(u) = \sum_{k=0}^{N_0-1} \left|x\left(q^{k+1}\right)\right|^2 + \sum_{k=0}^{N_0} \left|u\left(q^k\right)\right|^2.$$

For this example, our results can be used to (7.5), that is, we have the following theorem.

**Theorem 7.2.** *In order that the pair* $(\bar{x}, \bar{u})$ *is optimal, it is necessary that there exists a sequence* $\{\varphi(q^k)\}$ *such that*

$$\begin{cases} \bar{x}\left(q^{k+1}\right) = \frac{1}{1+\mu(q^k)p(q^k)}\bar{x}\left(q^k\right) + \frac{\mu(q^k)}{1+\mu(q^k)p(q^k)}\bar{u}\left(q^k\right), \quad 0 \le k \le N_0 - 1, \\ \bar{x}(1) = x_0; \end{cases} \tag{7.6}$$

$$\begin{cases} \varphi\left(q^k\right) = \frac{1}{1+\mu(q^k)p(q^k)}\left[\varphi\left(q^{k+1}\right) + \mu\left(q^k\right)\bar{x}\left(q^{k+1}\right)\right], \quad 0 \le k \le N_0 - 1, \\ \varphi\left(q^{N_0}\right) = 0; \end{cases} \tag{7.7}$$

$$\sum_{k=0}^{N_0} \left[\bar{u}\left(q^k\right) + \varphi\left(q^k\right)\right]\left[u\left(q^k\right) - \bar{u}\left(q^k\right)\right] \ge 0. \tag{7.8}$$

**Example 7.3.** (see Ex. 3 of [10]). Consider mathematical programming problem

$$\min \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}\left(x_{ij}\right)$$
$$s.t. \begin{cases} \sum\limits_{j=1}^{m} x_{ij} \le a_i, \quad i = 1, 2, \ldots, n, \\ \sum\limits_{i=1}^{n} x_{ij} \ge b_j, \quad j = 1, 2, \ldots, m, \\ x_{ij} \ge 0, \quad i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m, \end{cases} \tag{7.9}$$

where $\sum\limits_{j=1}^{m} b_j \le \sum\limits_{i=1}^{n} a_i$. Let $\mathbb{T} = \{0, 1, 2, \ldots, m\}$, $x(0) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$, $x(1) = \begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \end{pmatrix}$, $x(k) = \begin{pmatrix} \sum\limits_{j=1}^{k} x_{1j} \\ \vdots \\ \sum\limits_{j=1}^{k} x_{nj} \end{pmatrix}$,

$$u(k) = \begin{pmatrix} u_1(k) \\ \vdots \\ u_n(k) \end{pmatrix} = \begin{pmatrix} x_{1,k+1} \\ \vdots \\ x_{n,k+1} \end{pmatrix}, C_{k+1}(u(k)) = (1 \ \cdots \ 1) \begin{pmatrix} c_{1,k+1}\left(u_i(k)\right) \\ \vdots \\ c_{n,k+1}\left(u_n(k)\right) \end{pmatrix},$$

$$U_{ad} = \left\{ u : \mathbb{T} \longrightarrow R^n \;\middle|\; 0 \le u(k) \le \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} - \sum_{j=0}^{k-1} u(j), \quad (1 \ \cdots \ 1)\, u(k) \ge b_{k+1} \right\}.$$

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Then the mathematical programming problem (7.9) can be rewritten as the following optimal control problem

$$\min J(u) = \sum_{k=0}^{m-1} C_{k+1}\left(u(k)\right)$$

satisfying

$$\begin{cases} x(k+1) = x(k) + u(k), \quad k = 0, 1, 2, \ldots, n-1, \\ x(0) = 0, \\ u \in U_{ad}. \end{cases} \tag{7.10}$$

**Theorem 7.3.** *Suppose that $C_k : R^n \longrightarrow R$ is convex ($k = 1, 2, \ldots, m$). In order that the pair $(\bar{x}, \bar{u})$ is optimal, it is necessary that the following equality and inequality hold:*

$$\begin{cases} \bar{x}\left(k+1\right) = \bar{x}\left(k\right) + \bar{u}\left(k\right), \quad 0 \le k \le m-1, \\ \bar{x}(0) = 0; \end{cases} \tag{7.11}$$

$$\sum_{k=0}^{m-1} C'_{k+1}\left(\bar{u}(k)\right)\left[u\left(k\right) - \bar{u}\left(k\right)\right] \ge 0. \tag{7.12}$$

Specially, when $m = 1$, $n = 2$, $c_{11}(x_{11}) = 20x_{11}$, $c_{21}(x_{21}) = 25x_{21}$, $a_1 = 100$, $a_2 = 150$, $b_1 = 200$, the mathematical programming problem (7.9) is a linear programming problem and its solution is $(x_{11}, x_{21}) = (100, 100)$. On the other hand, we have $\mathbb{T} = \{0, 1\}$ in Theorem 6.3, control set

$$U_{ad} = \left\{ u : \mathbb{T} \longrightarrow R^2 \,|\, 0 \le u(0) = \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} \le \begin{pmatrix} 100 \\ 150 \end{pmatrix},\ u_1(0) + u_2(0) \ge 200 \right\}$$

is a nonempty closed convex set. The controlled dynamic equation

$$\begin{cases} x(k+1) = x(k) + u(k), \\ x(0) = 0, \\ u \in U_{ad} \end{cases} \tag{7.13}$$

with the cost functional

$$J(u) = (20 \quad 25)u(0).$$

Furthermore, $(\bar{x}, \bar{\bar{u}})$ is optimal if and only if the following equality and inequality hold:

$$\begin{cases} \bar{x}(k+1) = \bar{x}(k) + \bar{u}(k), \\ \bar{x}(0) = 0, \end{cases} \tag{7.14}$$

$$u(0) - \bar{u}(0) \ge 0. \tag{7.15}$$

Furthermore, we immediately obtain $\bar{u}(0) = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$, that is, $(x_{11}, x_{21}) = (100, 100)$.

**Example 7.4.** Consider variational problem

$$\min J(x) = \int_{[a,b)} g\left(x^{\sigma}(t)\right)\Delta t + \int_{[a,b)} h\left(x^{\Delta}(t)\right)\Delta t, \tag{7.16}$$

over all $x \in C^1_{rd}(\mathbb{T}, R)$ satisfying $x(a) = x_0$, where $g$, $h \in C^1(R, R)$.

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It is easy to rewrite the problems of the calculus of variations into optimal control problems. Consider

$$\begin{cases} x^{\Delta}(t) = u(t), \quad a \le t < b, \\ x(a) = x_0 \end{cases} \tag{7.17}$$

with the cost functional

$$J(u) = \int_{[a,b)} g\left(x^{\sigma}(t)\right) \Delta t + \int_{[a,b)} h\left(u(t)\right) \Delta t.$$

By Theorem 5.A and Remark 5.1, we have the following result.

**Theorem 7.4.** *In order that the pair $(\bar{x}, \bar{u})$ be optimal, it is necessary that there is $\varphi \in C_{rd}(\mathbb{T}, R)$ such that the following equations and inequality hold:*

$$\begin{cases} \bar{x}^{\Delta}(t) = \bar{u}(t), \quad a \le t < b, \\ \bar{x}(a) = x_0, \end{cases} \tag{7.18}$$

$$\begin{cases} \varphi^{\Delta}(t) = -g'\left(\bar{x}^{\sigma}(t)\right), \quad a \le t < b, \\ \varphi(b) = 0, \end{cases} \tag{7.19}$$

$$\int_{[a,b)} \left[\varphi(t) + h'(\bar{u}(t))\right] \left[u(t) - \bar{u}(t)\right] \Delta t \ge 0, \quad \forall u \in C_{rd}. \tag{7.20}$$

Define Hamiltonian function

$$H(t, x(t), u(t), \varphi(t)) = g\left(x^{\sigma}(t)\right) + h(u(t)) + \varphi(t)u(t).$$

Note that $U_{ad} = C_{rd}(\mathbb{T}, R)$, we have

$$\frac{\partial H}{\partial u}|_{u=\bar{u}(t)} = \varphi(t) + h'(\bar{u}(t)) = 0 \quad \forall t \in \mathbb{T}.$$

Hence,

$$h'(\bar{u}(t)) = -\int_{[t,b)} g'\left(\bar{x}^{\sigma}(\tau)\right) \Delta\tau. \tag{7.21}$$

This equation is the Euler-Lagrange equation (integral form).

Let $E : R \times R \longrightarrow R$ be the function with the values

$$E(r, u) = h(u) + h(r) - (u - r)h'(r).$$

Obviously, we have

$$E\left(\bar{x}^{\Delta}(t), u(t)\right) = H(t, x(t), u(t), \varphi(t)) - H(t, x(t), \bar{x}^{\Delta}(t), \varphi(t)).$$

By optimal condition, we obtain immediately the following inequality

$$E\left(\bar{x}^{\Delta}(t), u(t)\right) \ge 0, \quad \forall t \in \mathbb{T}, \tag{7.22}$$

that is, we obtain also the Weierstrass condition [9].

**Theorem 7.5** (see Thm. of 3.3 of [9])**.** *Assume that $g$, $h \in C^1(R, R)$. If $\bar{x}$ is optimal for (7.16), then for all $t \in \mathbb{T}$, we have*

$$h'(\bar{u}(t)) = -\int_{[t,b)} g'\left(\bar{x}^{\sigma}(\tau)\right) \Delta\tau, \tag{7.23}$$

$$E\left(\bar{x}^{\Delta}(t), u(t)\right) \ge 0. \tag{7.24}$$

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