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On Toda's fibrations

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In 1956 Toda[5] introduced two fibrations localized at $p > 2$:

$$S^{2n-1} \longrightarrow \Omega S^{2n}_{(p-1)} \xrightarrow{H'} \Omega S^{2np-1},$$

$$\Omega^2 S^{2np+1} \xrightarrow{\delta} S^{2n}_{(p-1)} \longrightarrow \Omega S^{2n+1} \xrightarrow{H} \Omega S^{2np+1},$$

where $S^{2n}_{(p-1)}$ is a subcomplex of the James construction S^{2n}_{∞}. The construction of H' was somewhat difficult and was discussed by Moore (see [3]). Moore's definition however is not natural (in the sense of Theorem 1 (*b*) below). It is our purpose to give another definition, closer to Toda's original definition, which is natural, is an H map* and behaves well with respect to the Dyer–Lashof map $\lambda: B\Sigma_p \to Q(S^0)$. We use this to settle an unanswered question in [2].

THEOREM 1. *Localize at a fixed prime $p > 2$. Then there is a map $H': \Omega S^{2n}_{(p-1)} \to \Omega S^{2np-1}$ such that*

(*a*) *the fibre of H' is homotopy equivalent to S^{2n-1}.*

(*b*) *For each map $f: S^{2n-1} \to S^{2m-1}$, there is a homotopy commutative diagram:*

$$\begin{array}{ccc} \Omega S^{2n}_{(p-1)} & \xrightarrow{H'} & \Omega S^{2np-1} \\ \downarrow{\scriptstyle \Omega f_{(p-1)}} & & \downarrow{\scriptstyle \Omega(S^{p-1} f^p)} \\ \Omega S^{2m}_{(p-1)} & \xrightarrow{H'} & \Omega S^{2mp-1} \end{array}$$

(*c*) *H' is an H map.*

(*d*) *Let B be the localization of $B\Sigma_p$. If $q = 2(p-1)$, $\lambda(B^{nq-1}) \subset \Omega^{2n} S^{2n}_{(p-1)}$ (see* [2], *Proposition* 11), *and we have a homotopy commutative diagram:*†

$$\begin{array}{ccc} \Omega S^{2n}_{(p-1)} & \xrightarrow{H'} & \Omega S^{2np-1} \\ \uparrow{\scriptstyle \Omega\lambda} & \nearrow{\scriptstyle \Omega j} & \\ \Omega S^{2n} B^{nq-1} & & \end{array}$$

where j is the projection onto the top cell.

* We wish to thank Fred Cohen for suggesting that this might be true.

† Here and throughout we use the same symbol for a map and for any of its adjoints or restrictions, provided no ambiguity can arise.

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(*e*) *The composite*

$$\Omega S^{2np-1} \xrightarrow{\iota} \Omega^3 S^{2np+1} \xrightarrow{\Omega\delta} \Omega S^{2n}_{(p-1)} \xrightarrow{H'} \Omega S^{2np-1}$$

is the loops on the degree p map.

We remark here that by [2], Proposition 11,

$$\begin{array}{ccc} \Omega S^{2n}_{(p-1)} & \xrightarrow{H'} & \Omega S^{2np-1} \\ \uparrow{\scriptstyle \lambda} & & \uparrow{\scriptstyle \iota} \\ S^{2n-1}B^{nq-1} & \xrightarrow{j} & S^{2np-2} \end{array}$$

homotopy commutes for anybody's choice of H'. This is not however as strong as (*d*).

As an application of 1(*d*), let us recall the elements $X_{r,s}$ constructed in [2]. $X_{r,s} \in \pi_{nq-2+2r+1}(S^{2r+1})$ where $n = r+s$ and $s > \nu_p(n)$. We showed there that $H(X_{r,s}) = 0$ iff $rp-2 \geqslant \nu_p(s)$ and $H'(X_{r,s}) = \bar{\alpha}_s$ when $rp-4 \geqslant \nu_p(s)$. We failed to calculate $H'(X_{r,s})$ when $rp-\nu_p(s) = 2$ or 3.

COROLLARY 2. *If* $rp-\nu_p(s) = 3$, $H'(X_{r,s}) = \bar{\alpha}_s$.

In the case $rp-\nu_p(s) = 2$, $\bar{\alpha}_s$ is not defined on S^{2rp-1} but we conjectured then and still do that $H'(X_{r,s}) = \alpha_s(\nu_p(s))$.

1. The construction of H' is based on two general phenomena. First, given a fibration $F \to E \xrightarrow{\pi} B$ and a map $f: E \to X$ such that $f|_F \sim *$, we may construct a 'colifting' $l: \Omega B \to \Omega X$ so that the diagram

(1)
$$\begin{array}{ccc} \Omega B & \xrightarrow{l} & \Omega X \\ {\scriptstyle \Omega\pi}\uparrow & \nearrow{\scriptstyle \Omega f} & \\ \Omega E & & \end{array}$$

commutes up to homotopy. This is easily seen since $\Omega B \simeq \Omega(E; F, *)$.

The second construction involves $(X, A)_\infty$, the combinatorial model for the fibre of the pinch map $\mu: X \cup CA \to SA$ [1]. We need three facts about this construction:

(i) There is a Hopf invariant $H_2: (X, A)_\infty \to (X \wedge A)_\infty$ which is natural for maps of pairs and agrees with the James–Hopf invariant H_2 when $X = A$ (see [1], theorem 7·2). In particular there is a commutative diagram

(2)
$$\begin{array}{ccc} A_\infty & \xrightarrow{i_1} & (X, A)_\infty \\ \downarrow{\scriptstyle H_2} & & \downarrow{\scriptstyle H_2} \\ (A \wedge A)_\infty & \xrightarrow{i_2} & (X \wedge A)_\infty \end{array}$$

where i_1 and i_2 are the induced inclusion maps.

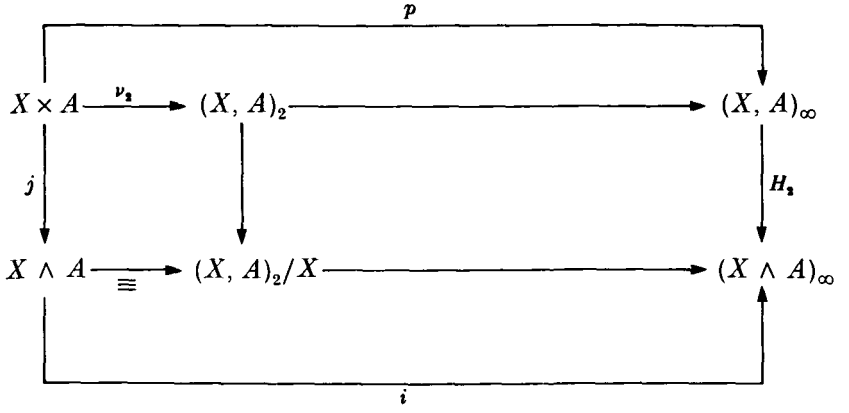
(ii) The map $i_1: A_\infty \to (X, A)_\infty$ fits into a fibre sequence (see [1], lemma 4·1):

$$\ldots \longrightarrow A_\infty \xrightarrow{i_1} (X, A)_\infty \xrightarrow{\rho_\infty} X \cup CA \xrightarrow{\mu} SA.$$

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(iii) There is a map $X \times A \xrightarrow{p} (X, A)_\infty$ so that the diagram

(3)



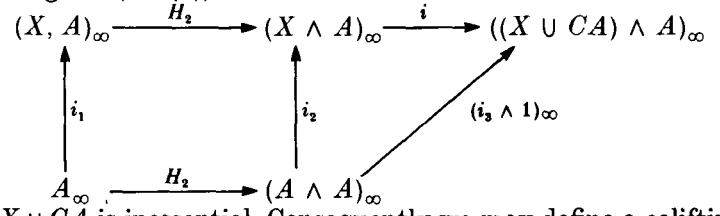
commutes, where j is the obvious quotient map and i is the inclusion (see [1], theorem 7·2).

Now consider the composition

$$(X, A)_\infty \xrightarrow{H_2} (X \wedge A)_\infty \xrightarrow{i} ((X \cup CA) \wedge A)_\infty,$$

where i is the obvious inclusion. This map is inessential when restricted to A_∞, by the commutative diagram (see (2)):

(2·5)

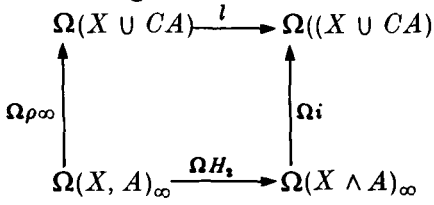


since $i_3 : A \to X \cup CA$ is inessential. Consequently we may define a colifting

$$l : \Omega(X \cup CA) \to \Omega((X \cup CA) \wedge A)_\infty,$$

and we have a commutative diagram

(4)

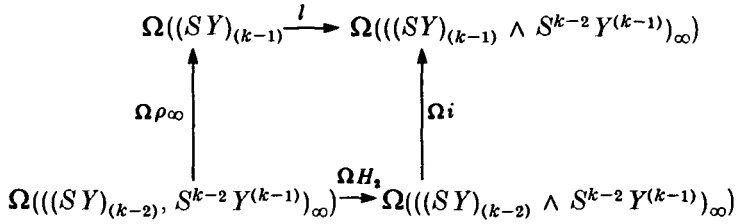


Let us now consider the James construction $Z_\infty \simeq \Omega SZ$ in case $Z = SY$. In this case we have

$$(SY)_{(k-1)} = (SY)_{(k-2)} \cup_{\gamma_k} C(S^{k-2}Y^{(k-1)}),$$

where $\gamma_k : S^{k-2}Y^{(k-1)} \to (SY)_{(k-2)}$ is natural in Y. Let us apply diagram (4) with $X = (SY)_{(k-2)}$ and $A = S^{k-2}Y^{(k-1)}$ to get the diagram:

(4·5)



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Now suppose that $k > 2$, so that $S^{k-2}Y^{(k-1)}$ is a suspension. Then we may define an evaluation map

$$e\colon (SY)_{(k-1)} \wedge S^{k-2}Y^{(k-1)} \to (\Omega S^2 Y) \wedge S^{k-2}Y^{(k-1)} \to SY \wedge S^{k-2}Y^{(k-1)} = S^{k-1}Y^{(k)}$$

Now define

$$\bar{H} = \Omega(e_\infty) \cdot l\colon \Omega((SY)_{(k-1)}) \to \Omega(S^{k-1}Y^{(k)})_\infty \simeq \Omega^2(SY)^{(k)}.$$

$\bar{H}$ is natural in the sense that for all $f\colon Y \to Y'$ we have a commutative diagram:

(5)
$$\begin{array}{ccc} \Omega((SY)_{(k-1)}) & \xrightarrow{\bar{H}} & \Omega^2(SY)^{(k)}. \\ \downarrow{\scriptstyle \Omega(Sf)_{k-1}} & & \downarrow{\scriptstyle \Omega^2(Sf)^{(k)}} \\ \Omega((SY')_{(k-1)}) & \xrightarrow{\bar{H}} & \Omega^2(SY')^{(k)} \end{array}$$

We are interested in the case $Y = S^{2n-1}$ and need to combine $\bar{H}$ with a suitably natural retraction.

LEMMA 3. *Localize away from 2. Then one may choose retractions $\phi_n\colon \Omega S^{2n} \to S^{2n-1}$ which are natural in the sense that, if $f\colon S^{2n-1} \to S^{2m-1}$, the diagram*

$$\begin{array}{ccc} \Omega S^{2n} & \xrightarrow{\phi_n} & S^{2n-1} \\ \downarrow{\scriptstyle \Omega Sf} & & \downarrow{\scriptstyle f} \\ \Omega S^{2m} & \xrightarrow{\phi_m} & S^{2m-1} \end{array}$$

homotopy commutes.

Proof. There is a homotopy equivalence

$$e_n\colon S^{2n-1} \times \Omega S^{4n-1} \to \Omega S^{2n}$$

(localized away from 2), given by adding the inclusion $S^{2n-1} \to \Omega S^{2n}$ and $\beta_n = \Omega[\iota, \iota]\colon \Omega S^{4n-1} \to \Omega S^{2n}$ using the H space structure of ΩS^{2n}. Now each of these maps is natural, i.e. the diagrams

$$\begin{array}{ccccc} \Omega S^{2n} \times \Omega S^{2n} & \to & \Omega S^{2n} & \Omega S^{4n-1} \xrightarrow{\beta_n} & \Omega S^{2n} \\ \downarrow{\scriptstyle \Omega Sf \times \Omega Sf} & & \downarrow{\scriptstyle \Omega Sf} & \downarrow{\scriptstyle \Omega Sf \wedge f} & \downarrow{\scriptstyle \Omega Sf} \\ \Omega S^{2m} \times \Omega S^{2m} & \to & \Omega S^{2m} & \Omega S^{4m-1} \xrightarrow{\beta_m} & \Omega S^{2m} \end{array} \qquad \begin{array}{ccc} S^{2n-1} & \to & \Omega S^{2n} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \Omega Sf} \\ S^{2m-1} & \to & \Omega S^{2m} \end{array}$$

each commute. Thus e_n is natural in the same sense. Since e_n is an equivalence, its inverse is natural. Define ϕ_n to be the first component of $(e_n)^{-1}$.

Let us introduce here some notation that will be used throughout. We write $F = (S^{2n}_{(k-2)}, S^{2n(k-1)-1})_\infty$. With this notation, the fibre sequence (ii) becomes

$$\ldots \to \Omega S^{2n(k-1)} \to F_{k-1} \to S^{2n}_{(k-1)} \to S^{2n(k-1)}.$$

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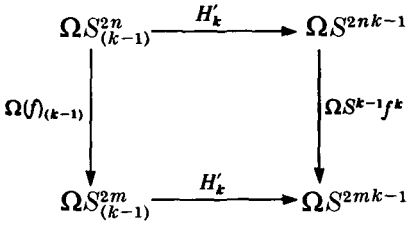
In summary, we have:

PROPOSITION 4. *Localize away from* 2. *Then there is a Toda–Hopf invariant*:

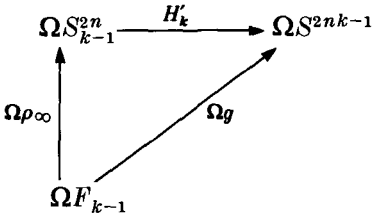
$$H'_k: \Omega S^{2n-1}_{(k-1)} \to \Omega S^{2nk-1}$$

*satisfying**:

(*a*) (*Naturality*) *If* $f: S^{2n-1} \to S^{2m-1}$, *we have a homotopy commutative diagram*:



(*b*) *There is a map* $g: F_{k-1} \to S^{2nk-1}$ *such that the diagram*:



homotopy commutes.

Proof. Define $H'_k = \Omega\phi_{nk} \cdot \bar{H}$ with $Y = S^{2n-1}$.

(*a*) follows from (5) and Lemma 3. For (*b*), take g to be the composite

$$F \xrightarrow{H_s} (S^{2n}_{(k-2)} \wedge S^{2n(k-1)-1})_\infty \xrightarrow{e_\infty} (S^{2nk-1})_\infty \xrightarrow{\phi_{nk}} S^{2nk-1},$$

and apply (4·5) and the definition of H'_k.

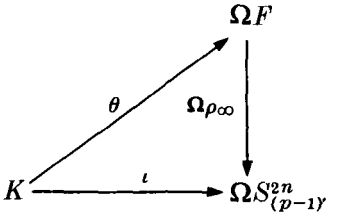
2. In this section all spaces will be localized at a fixed prime $p > 2$. All groups will be implicitly considered as modules over the p-local integers, and all spheres and cells will be so localized. We will write F for F_{p-1} and H' for H'_p.

Using the Serre spectral sequence, one easily sees that

$$H^*(\Omega S^{2n}_{(p-1)}) \simeq \Lambda(e_{2n-1}) \otimes DP(x_{2np-2}) \tag{6}$$

– the tensor product of an exterior and a divided power algebra. In particular, the $(2n(p+1)-4)$-skeleton of a minimal cellular approximation to $\Omega S^{2n}_{(p-1)}$ is $K = S^{2n-1} \cup e^{2np-2}$. Let $\iota: K \to \Omega S^{2n}_{(p-1)}$ be the inclusion.

PROPOSITION 5. *There is a lifting* $\theta: K \to \Omega F$ *in the diagram*:

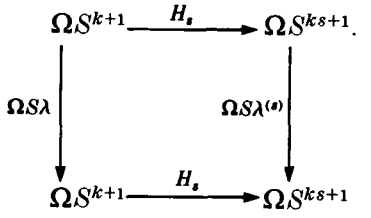


LEMMA 6. *Suppose* k *and* s *are even and* $H_s = \Omega S^{sk+1} \to \Omega S^{sk+1}$ *is the James–Hopf invariant. Then* $0 = \Omega H_s \in [\Omega^2 S^{k+1}, \Omega^2 S^{ks+1}]$.

* Other formulas seem likely, e.g. $H'_{kl} = H'_k \cdot \Omega H_l$, but we have no proof.

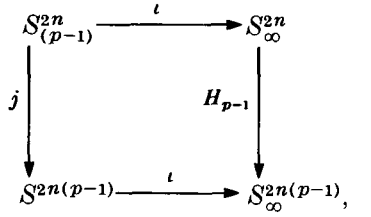
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Proof of Lemma 6. By naturality, we have for each map $\lambda: S^k \to S^k$ a commutative diagram

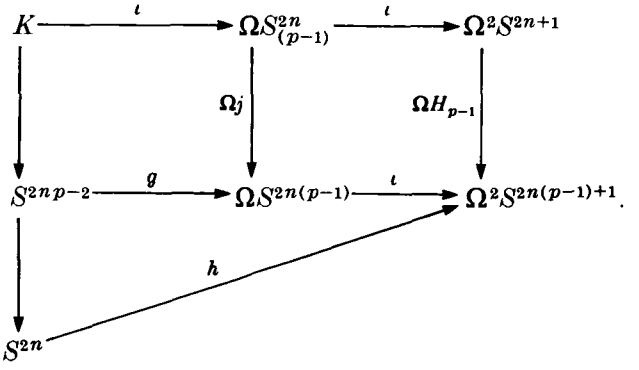


Suppose now that $\deg \lambda = d$. We claim that for all X $(\Omega S\lambda)_*: [X, \Omega S^{k+1}] \to [X, \Omega S^{k+1}]$ is multiplication by d. This homomorphism is the same as $(S\lambda)_*: [SX, S^{k+1}] \to [SX, S^{k+1}]$. For $u \in [SX, S^{k+1}]$, $S((S\lambda)_*(u)) = (S\lambda \cdot u) \wedge 1 = S(u) \cdot (d\iota \wedge 1) = S(u)(1 \wedge d\iota) = d(S(u))$. Since k is even $S: [SX, S^{k+1}] \to [S^2X, S^{k+2}]$ is 1–1 by Serre [4] or Lemma 3. Thus $(S\lambda)_*(u) = du$. Now ΩH_s is a homomorphism, so $(d^s - d) \cdot \Omega H_s = 0$. Choose λ with $d = -1$. Since s is even, $2 \cdot \Omega H_s = 0$, hence $\Omega H_s = 0$.

Proof of Proposition 5. Localize the following diagram involving the James–Hopf invariant H_{p-1}:



and insert the loops on this diagram into the following:



The left hand sequence is the cofibre sequence for K. g exists because $(\Omega j) \cdot \iota|_{S^{2n-1}} \sim *$. h exists because $\Omega H_{p-1} \sim *$. Since $\pi_{2n}(\Omega^2 S^{2n(p-1)+1}) = 0$, $h \simeq *$. Thus $g \in \pi_{2np-1}(S^{2n(p-1)})$ and $Sg \sim *$. However, $2np - 1 < 4n(p-1) - 1)$, so this group is stable and $g \sim *$. Thus $(\Omega j) \cdot \iota \sim *$ and the lifting θ exists.

Proof of Theorem 1 (*a*). Apply the Serre cohomology spectral sequence to the fibration equivalent to H'. Using (6) it is clear that it suffices to prove that $(H')^*$ is an isomorphism from $H^*(\Omega S^{2np-1})$ onto $DP(x_{2np-2})$. This will be the case if it is true in dimension $2np-2$, since $(H')^*$ is a ring homomorphism. Thus it suffices to show that

$$(H')_*: H_{2np-2}(\Omega S^{2n}_{(p-1)}) \to H_{2np-2}(\Omega S^{2np-1})$$

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is onto (for both groups are free of rank 1). Since $(\Omega\phi)$ and $\Omega(e_\infty)$ are retractions, it suffices to show that

$$l_*: H_{2np-2}(\Omega S^{2n}_{(p-1)}) \to H_{2np-2}(\Omega(S^{2n}_{(p-1)} \wedge S^{2n(p-1)-1})_\infty)$$

is onto. Using (4·5), this reduces to showing that

$$(\Omega H_2)_*: H_{2np-2}(\Omega F) \to H_{2np-2}(\Omega(S^{2n}_{(p-2)} \wedge S^{2n(p-1)-1})_\infty)$$

is onto. To this end we consider the suspension diagram:

$$\begin{array}{ccc} H_{2np-2}(\Omega F) & \xrightarrow{(\Omega H_2)_*} & H_{2np-2}(\Omega((S^{2n}_{(p-2)} \wedge S^{2n(p-1)-1})_\infty)) \\ \downarrow \sigma & & \downarrow \sigma \\ H_{2np-1}(F) & \xrightarrow{(H_2)_*} & H_{2np-1}((S^{2n}_{(p-2)} \wedge S^{2n(p-1)-1})_\infty) \end{array}$$

The suspension on the right is an isomorphism by the Hurewicz theorem. To show that the same is true for the suspension on the left, we apply [5], theorem 5·1 to verify that

$$H^r(F, S^{2n}_{(p-2)}) = \begin{cases} Z_{(p)}, & r = 2np-1 \\ 0, & r < 2n(p+1)-1, r \neq 2np-1. \end{cases}$$

Thus we may choose a cellular model for F in which the $2n(p+1)-2$ skeleton is $S^{2n}_{(p-2)} \cup e^{2np-1}$. Our task is then reduced to showing that

$$\sigma: H_{2np-2}(\Omega(S^{2n}_{(p-2)} \cup e^{2np-1})) \to H_{2np-1}(S^{2n}_{(p-2)} \cup e^{2np-1})$$

is an isomorphism. This follows directly from the Serre spectral sequence for the loop space fibration.

We have now reduced our problem to that of showing that $(H_2)_*$ is onto in dimension $2np-1$. Using (3) we have:

$$\begin{array}{ccc} H_{2np-1}(F) & \xrightarrow{(H_2)_*} & H_{2np-1}((S^{2n}_{(p-2)} \wedge S^{2n(p-1)-1})_\infty) \\ \uparrow p_* & & \uparrow i_* \\ H_{2np-1}(S^{2n}_{(p-2)} \times S^{2n(p-1)-1}) & \xrightarrow{j_*} & H_{2np-1}(S^{2n}_{(p-2)} \wedge S^{2n(p-1)-1}). \end{array}$$

In this diagram, i_* and j_* are clearly isomorphisms; hence $(H_2)_*$ is onto and we are done.

PROPOSITION 7. *There is a homotopy equivalence*

$$\Omega F \simeq \Omega S^{2n}_{(p-1)} \times \Omega^2 S^{2n(p-1)}.$$

Proof. Following Moore (see [3]), we construct from $\theta: K \to \Omega F$ and the H-space structure on ΩF, maps $\theta_m: K^m \to \Omega F$ (Cartesian product!). Define

$$\bar{\theta}: \bigvee_{m=1}^{\infty} K^m \to \Omega F$$

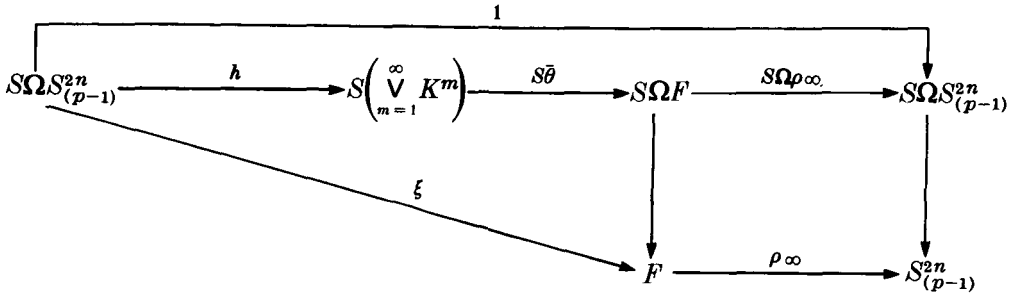
by $\bar{\theta}|_{K^m} = \theta_m$. Thus the composition $\Omega\rho_\infty \cdot \bar{\theta}: \bigvee_{m=1}^{\infty} K^m \to \Omega S^{2n}_{(p-1)}$ is onto in homology since $\iota: K \to \Omega S^{2n}_{(p-1)}$ is onto the indecomposables in homology. Consider now $S(\bigvee_{m=1}^{\infty} K^m)$.

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Since the suspension of a product breaks up as a wedge and $K \wedge K$ is a wedge of spheres (see [3]), we may choose a subcomplex $L \subset S(\bigvee_{m=1}^{\infty} K^m)$ such that

$$H_*(L) \to H_*(S\Omega S^{2n}_{(p-1)})$$

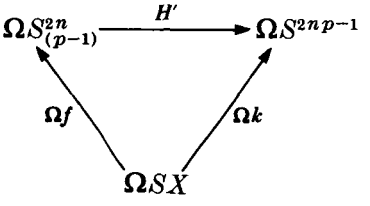
is an isomorphism. In other words, $S\Omega S^{2n}_{(p-1)}$ is a retract of $S(\bigvee_{m=1}^{\infty} K^m)$, and in particular there is a map $h: S\Omega S^{2n}_{(p-1)} \to S(\bigvee_{m=1}^{\infty} K^m)$ such that $S(\Omega\rho_\infty \cdot \bar{\theta}) \cdot h = 1$. From the diagram



we see that the adjoint of ξ is a right inverse to $\Omega\rho_\infty$. Thus the fibration splits as a product, and we are done.

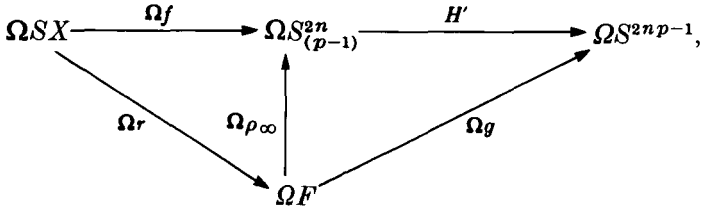
It is possible that a similar result holds for F_{p^k-1}. The analogue of Proposition 5 may be carried through where K is an appropriate skeleton.

COROLLARY 8. *Let X be any space and $f: X \to \Omega S^{2n}_{(p-1)}$ any map. Let $k = H'f: X \to \Omega S^{2np-1}$. Then the diagram*



homotopy commutes, where Ωf and Ωg are the loops on the adjoints of f and g.*

Proof. Combine the lifting ξ of Proposition 7 with the diagram of Proposition 4 (*b*):



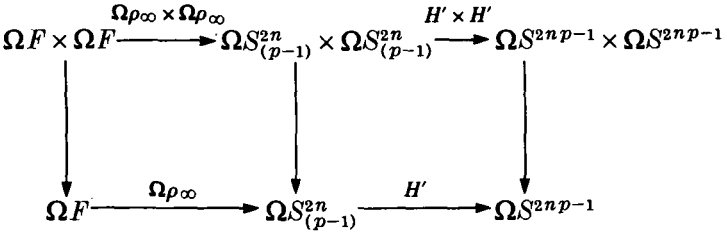
where $r = \xi \cdot f$. Thus the diagram of the corollary commutes for some map $k': SX \longrightarrow S^{2np-1}$. However k' is the adjoint to the composition $X \xrightarrow{\iota} \Omega SX \xrightarrow{\Omega k'} \Omega S^{2np-1}$, and since $H' \cdot \Omega f \cdot \iota = H' \cdot f = k$, we are done.

Proof of Theorem 1 (*b*). This is a special case of 4*a* with $k = p$.

* Thus H' acts as if it was a loop map.

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Proof of Theorem 1(*c*). Using Proposition 4(*b*) and 7, we have a diagram



in which the outside and left rectangles commute up to homotopy and the right rectangle is in question. Since $\Omega\rho_\infty \times \Omega\rho_\infty$ has a right inverse $(\xi \times \xi)$, we are done.

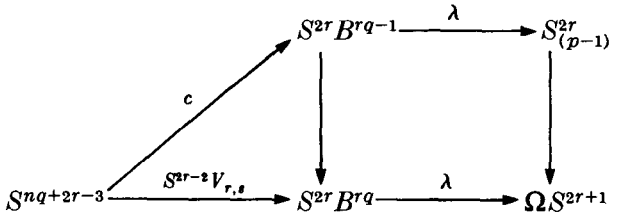
Proof of Theorem 1(*d*). Apply Corollary 8 with $X = S^{2n-1}B^{nq-1}$ and $f = \lambda$.

Proof of Theorem 1(*e*). Apply Corollary 8 with $X = S^{2np-2}$ and f the composite $S^{2np-2} \xrightarrow{\iota} \Omega^3 S^{2np+1} \xrightarrow{\Omega\delta} \Omega S^{2n}_{(p-1)}$.

Proof of Corollary 2. Referring to [2], let us suppose that $rp - \nu_p(s) = 3$. Then the composite

$$S^{nq+2r-3} \xrightarrow{S^{2n-3}V_{r,s}} S^{2r-1}B^{rq} \xrightarrow{j} S^{2rp-1}$$

is $jj_s = \alpha_{s-1}\alpha_1$ by [2], theorem 8(*b*), which is inessential by [2], theorem 12(*e*); consequently we map construct a coextension c in the diagram



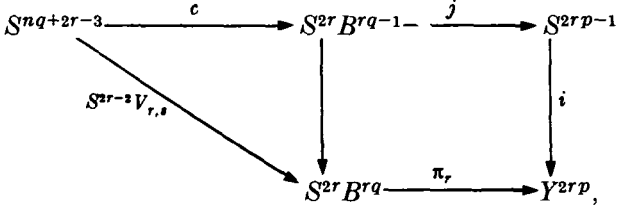
as in the proof of [2], theorem 12(*g*). The bottom composite is the adjoint of $X_{r,s}$ and $H'(X_{r,s})$ is thus the adjoint of the composite

$$S^{nq+2r-4} \xrightarrow{c} \Omega S^{2r}B^{rq-1} \xrightarrow{\Omega\lambda} \Omega S^{2r}_{(p-1)} \xrightarrow{H'} \Omega S^{2rp-1}.$$

Now apply Theorem 1(*d*). Since $H' \cdot \Omega\lambda = \Omega j$, $H'(X_{r,s})$ is homotopic to the composite

$$S^{nq+2r-3} \xrightarrow{c} S^{2r}B^{rq-1} \xrightarrow{j} S^{2rp-1}.$$

Now insert this into the diagram



where π_r is the suspension of the projection $B^{rq} \to B^{rq}/B^{(r-1)q} = Y^{rq}$. Since $\pi_r \cdot (S^{2r-2}V_{r,s}) = j_s$ by [2], theorem 8(*d*) and $\bar{\alpha}_s$ is defined solely by the criterion $i\bar{\alpha}_s = j_s$, we have $H'(X_{r,s}) = \bar{\alpha}_s$ as required.

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