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Instability property of homeomorphisms on surfaces

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Abstract. Let S be a compact surface without boundary, with Euler characteristic $\chi(S) > 0$, and let $K \subset S$ be a compact set such that each component of K is cellular. The aim of this paper is to prove that, if $f : S \setminus K \to S \setminus K$ is a homeomorphism which has a certain instability property in the proximity of K, then the set of periodic orbits of f is non-empty. This result give us some corollaries:

- there is no minimal homeomorphism $f : S^2 \setminus K \to S^2 \setminus K$ for K an arbitrary compact set;
- if $f : S^2 \to S^2$ is an area-preserving homeomorphism, then $K = \mathrm{cl}(\mathrm{Per}(f))$ is not an isolated invariant set.

1. *Introduction*

Given a homeomorphism $f : X \to X$, we say that f is *minimal* if every full orbit of f is dense. The question about the existence of minimal homeomorphisms of $\mathbb{R}^m$ with a single point omitted was a problem included by Ulam in [**13**]. There are several partial answers (see [**13**]) but the problem remains open. By the Brouwer translation theorem it follows that there are no minimal homeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$. For the multipunctured plane Handel (see [**9**]) proved the non-existence of minimal homeomorphisms of $\mathbb{R}^2 \setminus K$, where K is a finite set with at least two points and more recently Le Calvez and Yoccoz (see [**10**]) have completely solved this problem by using fixed-point index theory. The proof is obtained by computing the fixed-point index of any iteration of an orientation preserving homeomorphism at a fixed point which is an isolated invariant set (see [**16**] for an extension to the orientation reversing case of this computation in an elementary way and see [**11**] for a generalization to the computation of the Lefschetz index of an isolating set which is a union of discs of $\mathbb{R}^2$). Later Franks, in [**7**], gave a short alternative proof using Conley index techniques.

Following these steps, we prove the non-existence of minimal homeomorphisms of $S^2 \setminus K$ for an arbitrary compact set K.

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Let S be a compact connected surface without boundary and let $K \subset S$ be a compact set such that each component K_c of K is cellular, that is, there exists a sequence $\{B_i\}$ of 2-cells in S such that $B_{i+1} \subset \text{int}(B_i)$ and $\bigcap_{i=1}^{\infty} B_i = K_c$. This concept of cellularity on surfaces is equivalent to the term *trivial shape* in the sense of Borsuk [**2**]. It is not difficult to prove that for every neighborhood U of K we can obtain a finite union of disjoint discs $D = \bigcup_{i=1}^{m} D_i$ in U such that $K \subset \text{int}(D)$. Let us consider a homeomorphism $f : S \setminus K \to S \setminus K$ with S a compact connected surface with Euler characteristic > 0 (a sphere or a projective plane). The main goal of this paper is to prove that if there exists a finite family of disjoint discs D, with $K \subset \text{int}(D)$, such that there is no full orbit $\gamma \subset D \setminus K$ and for each subfamily of D, f and f^{-1} are not stable, then the set of periodic points of f, $\text{Per}(f)$, is non-empty.

A corollary of this result is that if $S = S^2$ or $S = P^2(\mathbb{R})$, there is no minimal homeomorphism of $S \setminus K$ for K a compact set with cellular components. In the case of $S = S^2$, K is an arbitrary compact set.

Another corollary is that, if $f : S^2 \to S^2$ is an area-preserving homeomorphism and $K \neq S^2$ is a compact and invariant set with $\text{Per}(f) \subset K$, then K is not isolated with respect to f and for every neighborhood U of K there exists a component of $\bigcap_{n\in\mathbb{Z}} f^{-n}(\text{cl}(U))$ which meets both K and $\partial(U)$.

The paper is organized as follows. Section 2 contains the preliminary definitions and results, the statement of the Main Theorem and the proof of the above corollaries. In §3 we prove the Main Theorem.

2. *Preliminary results and the Main Theorem*

Let X be a topological space. By a *semidynamical system* we mean a continuous map $f : X \to X$. A function $\sigma : \mathbb{Z} \to X$ is said to be a *solution to f through x* in $N \subset X$ if $f(\sigma(i)) = \sigma(i+1)$ for all $i \in \mathbb{Z}$, $\sigma(0) = x$ and $\sigma(i) \in N$ for all $i \in \mathbb{Z}$. The *invariant part* of N, $\text{Inv}(N, f)$, is defined as the set of all $x \in N$ that admit a solution to f through x in N, i.e. the set of all $x \in N$ such that there is a full orbit γ with $x \in \gamma \subset N$.

Given $A \subset B \subset N$, $\text{cl}(A)$, $\text{cl}_B(A)$, $\text{int}(A)$, $\text{int}_B(A)$, $\partial(A)$ and $\partial_B(A)$ will denote the closure of A, the closure of A in B, the interior of A, the interior of A in B, the boundary of A and the boundary of A in B respectively.

A compact set $K \subset X$ is *invariant* if $f(K) = K$. An invariant compact set K is *isolated with respect to f* if there exists a compact neighborhood N of K such that $\text{Inv}(N, f) = K$. The neighborhood N is called an *isolating neighborhood of K*. A compact set N is an *isolating block* if $f(N) \cap N \cap f^{-1}(N) \subset \text{int}(N)$.

The *exit set* of N is defined as

$$N^- = \{x \in N : f(x) \notin \text{int}(N)\}.$$

A *decomposition* G of a topological space X is a partition of X. By identifying each element of G to a point we obtain the *decomposition space* X/G. The topology on X/G is the quotient space topology induced by the *decomposition map* $\pi : X \to X/G$ which sends each $x \in X$ to the element of G containing x.

A decomposition G of a topological space X is *upper semicontinuous* (usc) if each $g \in G$ is a compact set of X and for each $g \in G$ and each open subset U of X containing g, there exists an open neighborhood V of X containing g such that every

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$g' \in G$ intersecting V is contained in U. The last condition is equivalent to requiring that the map $\pi : X \to X/G$ be closed.

From now onwards a surface S will be compact, connected and without boundary. A *cell-like decomposition* G of a surface S is an upper semicontinuous decomposition such that each $g \in G$ is cellular in S. In the rest of the paper we suppose that $K \subset S$ is a compact set and each component of K is cellular.

LEMMA 1. *Let $K \subset S$ be a compact set such that each component of K is cellular. Let G be the decomposition of S given by the partition consisting of the components of K and the degenerate elements of $S \setminus K$ (if $g \in G$ contains more than one point, g is a component of K). Then G is a cell-like decomposition of S.*

Proof. We want to prove that G is usc but this follows in an easy way by using the fact that the set of continua is closed for the Hausdorff topology. □

Remark 1. In the setting of the above lemma, since G is a cell-like decomposition of S, the works of Moore [**14**], Roberts-Steenrod [**15**] and Youngs [**18**] permit us to say that the decomposition map $\pi : S \to S/G$ is approximable by homeomorphisms (see also [**3, 4**]). Then, the decomposition space S/G is homeomorphic to S.

We say that a topological space X is *zero-dimensional* if for each $x \in X$ and each neighborhood V of x there exists an open neighborhood $U \subset V$ of x such that $\partial(U) = \emptyset$. If X is a regular space, we can replace the condition $\partial(U) = \emptyset$ by U open and closed (see [**6**]).

LEMMA 2. *In the setting of the above lemma, the compact set $\pi(K) \subset S/G$ is zero-dimensional and $S/G \setminus \pi(K)$ is dense in S/G.*

Proof. If K_i is a connected component of the compact set K, then K_i is a quasicomponent of K and $K_i = \bigcap_{j \in J} U_j$, with U_j open and closed in K for every $j \in J$. Given an open neighborhood V in K of K_i, there exists $U = \bigcap_{j=1}^{n} U_j$, open and closed in K, with $K_i \subset U \subset V$.

We have $q_i = \pi(K_i) \in \pi(U) \subset \pi(V) \subset \pi(K)$, with $\pi(U)$ open and closed in $\pi(K)$ which give us the zero-dimensionality of $\pi(K)$. Since $\pi(K)$ is zero-dimensional, then $S/G \setminus \pi(K)$ is dense in S/G. □

PROPOSITION 1. *Let $K \subset S$ be a compact set such that each component of K is cellular. Given a neighborhood U of K in S, there exists a finite family of disjoint discs $D = \bigcup_{i=1}^{m} D_i$ in U such that $K \subset \operatorname{int}(D)$.*

Proof. Let N be a compact neighborhood of K in U and let $\varphi : \operatorname{cl}(U) \to \mathbb{R}^+$ be a smooth function which vanishes exactly on N. Given a small enough regular value $\delta > 0$, the set $M = \varphi^{-1}([0, \delta]) \subset U$ is a compact surface with boundary. Then M is a finite family of 2-cells with strips (for a classification of compact connected surfaces with boundary see [**12**]). Let M_1 be a connected component of M. Since $\pi(K) \subset S/G$ is zero-dimensional, it is easy to cut M_1 to obtain a finite family of disjoint discs $D_{1,1}, \ldots, D_{1,r}$ with $K \cap M_1 \subset \bigcup_{i=1}^{r} D_{1,i} \subset M_1$. The family of discs D is obtained in this way. □

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Definition 1. Given a family of disjoint discs $D' \subset S$, we say that $f : S \setminus K \to S \setminus K$ is *stable* in D' if for all $x \in D' \setminus K$ there exists $n_x \in \mathbb{N}$ and a neighborhood V_x of x in D' such that $f^n(x') \in \operatorname{int}(D')$ for all $n \geq n_x$ and all $x' \in V_x$.

Definition 2. Let $f : S \setminus K \to S \setminus K$ be a homeomorphism and let us suppose that there exists a finite family of disjoint discs $D = \bigcup_{i=1}^{m} D_i$, with $K \subset \operatorname{int}(D)$, such that there is no full orbit γ in $D \setminus K$ and for every subfamily

$$D' = \bigcup_{j \in J \subset \{1,\ldots,m\}} D_j$$

f and f^{-1} are not stable in D'. Then we say that f is *unstable in* K.

The next theorem is the main result of this work and will be proved in the next section.

MAIN THEOREM. *Let S be a surface with Euler characteristic $\chi(S) > 0$ and let K be a compact subset of S such that each component of K is cellular. If $f : S \setminus K \to S \setminus K$ is an unstable homeomorphism in K, then* $\operatorname{Per}(f) \neq \emptyset$.

COROLLARY 1. *There is no minimal homeomorphism $f : S \setminus K \to S \setminus K$ for S a sphere or a projective plane and K a compact set with cellular components.*

The proof is obvious because if there exists such a minimal homeomorphism f, then f is unstable in K and $\operatorname{Per}(f) = \emptyset$ which is a contradiction to the Main Theorem.

COROLLARY 2. *Let K be a compact subset of S^2, $K \neq S^2$. There is no minimal homeomorphism $f : S^2 \setminus K \to S^2 \setminus K$.*

Proof. If there exists a minimal homeomorphism $f : S^2 \setminus K \to S^2 \setminus K$, then $S^2 \setminus K$ has finitely many connected components, $S^2 \setminus K = \bigcup_{i=1}^{n} A_i$. In fact, given a connected component A_1 of $S^2 \setminus K$, since f is minimal, there exist $n \in \mathbb{N}$ and connected components $\{A_2, \ldots, A_n\}$ of $S^2 \setminus K$ such that f acts as a permutation in the following way:

$$f(A_i) = A_{i+1} \quad \text{for } i = 1, \ldots, n-1 \quad \text{and} \quad f(A_n) = A_1.$$

Moreover, the only connected components of $S^2 \setminus K$ are $\{A_1, \ldots, A_n\}$. Let us observe that the map $g = f^n|_{A_1} : A_1 \to A_1$ is minimal. The set $S^2 \setminus A_1 = K'$ is a compact set which does not separate S^2, with $K \subset K'$, and each component of K' is cellular. The map $g : S^2 \setminus K' \to S^2 \setminus K'$ is minimal and we obtain a contradiction to the Main Theorem. □

Another consequence of the Main Theorem is the following result, which give us some information about the properties of the set $K = \operatorname{cl}(\operatorname{Per}(f))$ with f an area-preserving homeomorphism of S^2.

PROPOSITION 2. *Let $f : S^2 \to S^2$ be an area-preserving homeomorphism of the sphere and let $K \neq S^2$ be a compact subset of S^2, with $f(K) = K$, and* $\operatorname{Per}(f) \subset K$. *Then K is not an isolated set with respect to f and for any neighborhood U of K there exists a connected component of* $\operatorname{Inv}(\operatorname{cl}(U), f)$ *which meet both K and $\partial(U)$.*

Proof. Let us prove that K is not an isolated set with respect to f. Given a connected component A_1 of $S^2 \setminus K$, since f is area-preserving, there exist $r \in \mathbb{N}$ and connected

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components $\{A_2, \dots, A_r\}$ of $S^2 \setminus K$ such that f acts as a permutation in the following way:

$$f(A_i) = A_{i+1} \quad \text{for } i = 1, \dots, r-1 \quad \text{and} \quad f(A_r) = A_1.$$

Let g be the area-preserving homeomorphism

$$g = f^r|_{A_1} : A_1 \to A_1.$$

Since $\mathrm{Per}(g) = \emptyset$ we have, by the Main Theorem, that g is not unstable in the compact set with cellular components $K' = S^2 \setminus A_1$.

On the other hand, given a finite union of disjoint discs $D = \bigcup_{i=1}^{m} D_i$, with $K' \subset \mathrm{int}(D)$, and given a subfamily of D

$$D' = \bigcup_{j \in J \subset \{1,\dots,m\}} D_j,$$

if for all $x \in D' \setminus K'$ there exists $n_x \in \mathbb{N}$ and a neighborhood V_x of x in D' such that $g^n(V_x) \subset \mathrm{int}(D')$ for all $n \geq n_x$, then there exists $n_{D'} \in \mathbb{N}$ such that $g^n(D') \subset \mathrm{int}(D')$ for all $n \geq n_{D'}$, but this is not possible because f is area-preserving. Then g is not stable in D' for every subfamily D' of D. In the same way we prove that g^{-1} is not stable in D'. Since g is not unstable in K', there exists a full orbit γ for f^r in $D \setminus K'$ and this argument is valid for every neighborhood D of K'. Then we have that K' is not an isolated set with respect to f^r. Since $\partial(K') \subset K$, it follows that the set K is not isolated with respect to f.

Let us prove that for any neighborhood U of K there exists a connected component of $\mathrm{Inv}(\mathrm{cl}(U), f)$ which meets both K and $\partial(U)$. Let U be a neighborhood of K and let us suppose that there are no components of $K_{\mathrm{cl}(U)} = \mathrm{Inv}(\mathrm{cl}(U), f)$ which meet both K and $\partial(U)$. Given $x \in K$, let K_x be the connected component of $K_{\mathrm{cl}(U)}$ which contains x. Let us see that

$$K'' = \bigcup_{x \in K} K_x$$

is a compact set. Let $\{x_n\}$ be a sequence in K'' with $\{x_n\} \to x_0$ and $x_n \in K_n$ with K_n a connected component of K''. Let $\{y_n\}$ be a sequence with $y_n \in K_n \cap K$. Since K is a compact set, there exists a subsequence $\{y_{n_k}\} \to y_0 \in K$ and $\{x_{n_k}, y_{n_k}\} \subset K_{n_k}$. As the set of continua is closed for the Hausdorff topology we have that there exists a continuum $K_{x_0,y_0} \subset K_{\mathrm{cl}(U)}$ with $\{x_0, y_0\} \subset K_{x_0,y_0}$. Therefore $x_0 \in K_{x_0,y_0} \subset K''$.

Since K'' is a compact set, we can select a compact manifold $M \subset U$ such that $K'' \subset \mathrm{int}(M)$. Each connected component of K'' can be separated from the finite family of connected components of $\partial(M)$ by closed arcs in $M \setminus [\partial(M) \cup (K_{\mathrm{cl}(U)} \cap M)]$. This follows from the fact that, given two connected components A_1 and A_2 of a compact set A in $\mathbb{R}^2$, there exists a closed arc in $R^2 \setminus A$ separating A_1 and A_2. In our situation $A = \partial(M) \cup (K_{\mathrm{cl}(U)} \cap M)$, A_1 is a connected component of K'' and A_2 is a connected component of A which contains a connected component of $\partial(M)$. In this way we construct for each K_x of K'' a compact, connected manifold $M(K_x) \subset \mathrm{int}(M)$ such that $K_x \subset \mathrm{int}(M(K_x))$ and $\partial(M(K_x)) \cap K_{\mathrm{cl}(U)} = \emptyset$. Since K'' is a compact set, the cover $\{M(K_x)\}_{x \in K}$ has a finite subcover which permits us to obtain a compact set $M' \subset \mathrm{int}(M)$ with $K'' \subset \mathrm{int}(M')$ and such that $K_{\mathrm{cl}(U)} \cap \partial(M') = \emptyset$. Then $\mathrm{Inv}(M', f)$, which is an invariant and compact set containing $\mathrm{Per}(f)$, can be isolated and we obtain a contradiction. □

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PROPOSITION 3. *Let $f : S \setminus K \to S \setminus K$ be a homeomorphism with K a zero-dimensional compact subset of the surface S. Then there exists a homeomorphism $\overline{f} : S \to S$ with $\overline{f}|_{S\setminus K} = f$.*

Proof. Let us see that there exists a continuous map $\overline{f} : S \to S$ with $\overline{f}|_{S\setminus K} = f$. We only have to prove that for each $x \in K$ and $\epsilon > 0$ there exists a neighborhood U of x such that $\delta(f(U \setminus K)) < \epsilon$ with $\delta(f(U \setminus K)) = \sup\{d(x, y) : x, y \in f(U \setminus K)\}$ (see [**5**, Lemma 4.3.16]). Let $x \in K$ and let $B_n(x)$ be small enough 2-cells such that $B_{n+1}(x) \subset \text{int}(B_n(x))$ and $\bigcap_{n=1}^{\infty} B_n(x) = \{x\}$. Then $\text{cl}(f(B_n(x) \setminus K))$ are compact, connected sets with $\text{cl}(f(B_{n+1}(x) \setminus K)) \subset \text{cl}(f(B_n(x) \setminus K))$. The set $L = \bigcap_{n=1}^{\infty} \text{cl}(f(B_n(x) \setminus K))$ is a non-empty continuum. If $\delta(L) > 0$ then $L \not\subset K$ and there is $z_0 \in L \setminus K$ and a sequence $\{z_n\} \to z_0$ with $z_n \in f(B_n(x) \setminus K)$ for all n. Then, $\{f^{-1}(z_n)\} \to f^{-1}(z_0) \in S \setminus K$ and $\{f^{-1}(z_n)\} \to x \in K$, which is a contradiction. Consequently, we have that $\delta(L) = 0$ and the continuous map $\overline{f}$ is well defined.

Let us prove that $\overline{f}$ is a homeomorphism. The map $\overline{f}$ is unique and we can construct, in the same way, a unique continuous map $\overline{f^{-1}} : S \to S$ such that $\overline{f^{-1}}|_{S\setminus K} = f^{-1}$. Then $\overline{f^{-1}} \circ \overline{f} : S \to S$ is the identity map because it is the unique continuous extension of the identity map $\text{Id} : S \setminus K \to S \setminus K$. Therefore, we obtain that $\overline{f}$ is a homeomorphism. □

The last proposition can be proved by using Freudenthal compactifications [**1**]. It is not difficult to see that S is the Freudenthal compactification of $S \setminus K$ (see [**1**, Theorem 9.10]). By the theory of Freudenthal compactifications we have that the map $f : S \setminus K \to S \setminus K$ has a unique continuous extension $\overline{f} : S \to S$ (see [**1**, Proposition 9.11]).

If K is a compact subset of S such that each component of K is cellular and if $f : S \setminus K \to S \setminus K$ is an unstable homeomorphism in K then, by Remark 1 and Lemma 2, we have an induced unstable homeomorphism $f' : S \setminus K' \to S \setminus K'$, with K' a compact zero-dimensional set, which we can extend to a homeomorphism $g : S \to S$. From now onwards we will work with g.

Since $g|_{S\setminus K'} : S \setminus K' \to S \setminus K'$ is an unstable homeomorphism in K', we have that K' is an isolated invariant set for g with the finite family of disjoint discs D as isolating neighborhood ($\text{Inv}(D, g) = K' \subset \text{int}(D)$).

It is easy to obtain an open set $U \subset S$ such that $U \cup g(U)$ is contained in an open ball $V \subset S$ and $K' \subset \text{int}(D) \subset D \subset U$. In fact, let us consider the surface S as the topological space obtained from a polygonal disc by identification of the edges in an adequate way. We can suppose without loss of generality that $K' \subset V$ with V the interior of the polygonal disc. If the family of discs D is a small enough neighborhood of K', then we can select an open neighborhood U of D such that $U \cup g(U) \subset V$.

Let us consider the map $g|_U$ as a map locally defined in $\mathbb{R}^2$, $g|_U : U \subset \mathbb{R}^2 \to g(U)$. We can prove that there exists an isolating block $N \subset D$, with N a finite union of disjoint discs such that $K' \subset \text{int}(N)$ (see [**17**, Proposition 1]). We can select $N = \bigcup_{i=1}^{n} N_i$ with $K' \cap N_i \neq \emptyset$ for $i = 1, \ldots, n$.

For every subfamily of N,

$$N' = \bigcup_{j\in J\subset\{1,\ldots,n\}} N_j,$$

g and g^{-1} are not stable in N'.

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Definition 3. (See [**8**]) Let $L \subset N$ be a compact pair contained in U. The pair (N, L) is called a *filtration pair* for K' if N and L are each the closure of their interiors and:

(1) $\mathrm{cl}(N \setminus L)$ is an isolating neighborhood of K';

(2) L is a neighborhood of N^- in N;

(3) $g(L) \cap \mathrm{cl}(N \setminus L) = \emptyset$.

By a theorem of Franks and Richeson (see [**8**]) we have that there exist filtration pairs (N, L) for K' with L a manifold that admits a decomposition $L = L_1 \cup \cdots \cup L_m$ where every L_i is a disc with a finite amount of holes.

Let us consider a filtration pair (N, L) for K' as above such that L is a small enough neighborhood of N^- and has a minimum amount of holes, with $L_i \cap N^- \neq \emptyset$ for $i = 1, \ldots, m$.

Remark 2. We can select each component of L intersecting N^- because, if there is a non-empty union of components of L, L', such that $L' \cap N^- = \emptyset$, the couple $(N, L \setminus L')$ also satisfies (1), (2) and (3). The only non-obvious property is (3) and follows from the fact that, if L_1 is a connected component of $L \setminus L'$ and $g(L_1) \cap L_2 \neq \emptyset$ for some $L_2 \subset L$ then, since $g(L) \cap \mathrm{cl}(N \setminus L) = \emptyset$, there exists $z \in L_1$ with $g(z) \in L_2 \cap \partial(N)$. If $L_2 \cap N^- = \emptyset$, $g^2(z) \in \mathrm{int}(N)$ and we obtain a contradiction because N is an isolating block. Then $L_2 \subset L \setminus L'$ and the proof of (3) is finished.

The next two lemmas follow from [**17**, Propositions 2 and 3].

LEMMA 3. *The components of L are discs without holes.*

Proof. Let us suppose that there exists a component L_1 of L, $L_1 \subset N_1$, with N_1 a component of N, such that L_1 has a hole G_1. One knows that $g(\partial(G_1)) \subset \mathrm{int}(N)$ by (2) and $g(\partial(G_1)) \cap \mathrm{cl}(N \setminus L) = \emptyset$ by (3). So $g(\partial(G_1)) \subset \mathrm{int}(L_2)$ with L_2 a connected component of L. If $g(G_1) \subset L_2$ we can add the hole G_1 to L_1 to get a new $L' = L \cup G_1$. In this case $G_1 \cap K' = \emptyset$ and the new couple (N, L') satisfies (1), (2) and (3). But this is a contradiction of the fact that L has the minimum possible number of holes. We can always suppose that for every hole G_i there is a hole G_{i+1} such that $\mathrm{cl}(G_{i+1}) \subset g(G_i)$.

We say that G_i is *minimal* if it does not contain any other G_j. Since the number of holes is finite, for each G_i there is a minimal hole $G_j \subset G_i$. We can suppose without loss of generality that the hole G_1 of L_1 is minimal. Then there is a minimal hole G_2 with $\mathrm{cl}(G_2) \subset g(G_1)$. In this way we can select a family of minimal holes $\mathcal{G} = \{G_1, \ldots, G_r\}$ such that $\mathrm{cl}(G_{i+1}) \subset g(G_i)$ for $i = 1, \ldots, r-1$ and $\mathrm{cl}(G_1) \subset g(G_r)$.

We have that

$$\begin{aligned}\mathrm{cl}(G_1) &\subset g(G_r) \subset g(\mathrm{cl}(G_r)) \subset g(g(G_{r-1})) \\ &= g^2(G_{r-1}) \subset g^2(\mathrm{cl}(G_{r-1})) \subset \cdots \subset g^r(G_1).\end{aligned}$$

Since $g^{-r}(\mathrm{cl}(G_i)) \subset G_i$, then $K_i' = K' \cap G_i$ are non-empty cellular sets. We have that $\bigcup_{i=1}^r K_i' \subset K'$ is an isolated invariant and repeller set, but this is not possible because $g|_{S \setminus K'}$ is unstable in K'. □

LEMMA 4. *$L \cap N_i \neq \emptyset$ for every component N_i of N.*

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Proof. Let us suppose that there exists a component of N, N_1, such that $N_1 \cap L = \emptyset$. Let us see that there is a disc $N_2 \subset \text{int}(g^{-1}(N_1))$. In fact, if the set $g^{-1}(\partial(N_1))$ meets a disc N_2 at a point z, then $z \in N^- \subset L$ by (2), but this is impossible because $g(z) \in N \setminus L$, which contradicts (3). Therefore, $g^{-1}(\partial(N_1)) \cap N = \emptyset$. Since $N_i \cap K' \neq \emptyset$ for $i = 1, \ldots, n$, then $g^{-1}(N_1) \cap K' \neq \emptyset$ and there is a disc $N_2 \subset \text{int}(g^{-1}(N_1))$. The disc N_2 does not intersect L by (3) and $K'_2 = K' \cap N_2 \neq \emptyset$ is such that $g(K'_2) \subset K'_1$. By repeating this argument we obtain $N_1, \ldots, N_r$, with $g(N_{i+1}) \subset \text{int}(N_i)$ for $i = 1, \ldots, r-1$, $g(N_1) \subset \text{int}(N_r)$, and $N_i \cap L = \emptyset$ for $i = 1, \ldots, r$.

The family $\{K'_i\}_{i=1}^r$ determines an attractor cycle of length r, i.e. there is $r \in \mathbb{N}$ such that

$$K'_1 = g(K'_2), \ldots, K'_{r-1} = g(K'_r), \quad K'_r = g(K'_1)$$

with $\bigcup_{i=1}^r K'_i \subset K'$ a compact isolated invariant and attractor set ($g^r(N_i) \subset \text{int}(N_i)$) and with K'_i non-empty cellular sets. But this is not possible because $g|_{S \setminus K'}$ is unstable in K'. □

Remark 3. By the last two lemmas, we can choose a filtration pair (N, L) with $L = L_1 \cup \cdots \cup L_m$ discs without holes, $L \cap N_i \neq \emptyset$ for every component N_i of N, and $L_j \cap N^- \neq \emptyset$ for every $j = 1, \ldots, m$. Each disc L_j intersects with $\partial(N)$ and we can suppose without loss of generality that $\partial_N(L_j)$ is a finite union of arcs for all j.

LEMMA 5. *Given a filtration pair (N, L) for K' under the conditions of the last remark, there exists a filtration pair (N', L') with N' a disc and L' with the same properties as L.*

Proof. Let us construct N'. The idea is to connect by arcs the components of $N = \bigcup_{i=1}^n N_i$ to obtain the disc N'. Let us observe that, by the last lemma, $N_i \not\subset g^{-1}(N)$ for $i = 1, \ldots, n$ and then $\partial(N_i) \setminus g^{-1}(N) \neq \emptyset$. Moreover, $\partial(N_i) \setminus g^{-1}(N) \subset L \cap N_i$. Let us consider an arc γ joining all the components of N and such that $\gamma \cap g^{-1}(N) = \emptyset$. It is easy to obtain a finite family of disjoint arcs $\Gamma = \bigcup_{i,j} \gamma_{i,j}$ such that $\Gamma \subset \gamma$ and:

(1) $\gamma_{i,j} \cap N = \{x_i, x_j\}$ with $x_i \in \partial(N_i) \cap L$, $x_j \in \partial(N_j) \cap L$;
(2) $\gamma_{i,j} \cap g^{-1}(N) = \emptyset$;
(3) $N \cup \Gamma$ is contractible.

Then, let us cover each $\gamma_{i,j}$ by a small enough disc $D_{i,j}$ and consider the disc $N' = \bigcup D_{i,j} \cup N$.

We define $L' = \bigcup D_{i,j} \cup L$. Let us prove that the pair (N', L') is the filtration pair we are looking for. Since $\text{cl}(N \setminus L) = \text{cl}(N' \setminus L')$, (1) is obvious and $g(\text{cl}(N' \setminus L')) = g(\text{cl}(N \setminus L)) \subset \text{int}(N) \subset \text{int}(N')$ give us (2). On the other hand $\bigcup D_{i,j} \cap g^{-1}(N) = \emptyset$ give us the proof of (3) because

$$\begin{aligned} g(L') \cap \text{cl}(N' \setminus L') &= g(L') \cap \text{cl}(N \setminus L) \\ &= \left(g\left(\bigcup D_{i,j}\right) \cap \text{cl}(N \setminus L)\right) \cup (g(L) \cap \text{cl}(N \setminus L)) = \emptyset. \end{aligned}$$ □

3. *Proof of the Main Theorem*

Let (N, L) be a filtration pair as in the last lemma, with N a disc and L a finite number of disjoint discs (N can also be selected as a finite union of disjoint discs). Let us consider the quotient space N/L obtained by identifying the set L with a point $\{*\}$ and the continuous

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map $G : N/L \to N/L$ defined as $G([x]) = [g(x)]$ if $g(x) \in N$ and $G([x]) = *$ if $g(x) \notin N$.

If L has l connected components, then N/L has the homotopy type of the pointed union of $l-1$ loops and we obtain the singular homology group with rational coefficients $H_1(N/L) \equiv \mathbb{Q} \oplus \cdots \oplus \mathbb{Q}$ with $l-1$ generators.

Let $G_*^k : H(N/L) \to H(N/L)$ be the induced homomorphisms with $k \in \mathbb{N}$. Then, the Lefschetz number of G_*^k is

$$\Lambda(G_*^k) = i_{N/L}(G^k, N/L) = i_{N/L}(G^k, *) + i_{N/L}(G^k, K') = 1 + i_{N/L}(G^k, K')$$

with $i_{N/L}(G^k, N/L)$, $i_{N/L}(G^k, *)$ and $i_{N/L}(G^k, K')$ the fixed-point indices of G^k in N/L, $\{*\}$ and K' respectively.

In the same way, the Lefschetz number of $g_*^k : H(S) \to H(S)$ is

$$\Lambda(g_*^k) = i_S(g^k, S) = i_{N/L}(G^k, K') + i_S(g^k, S \setminus U_k(K'))$$

with $U_k(K')$ a small enough open neighborhood of K' such that $\mathrm{Fix}(g^m) \cap \mathrm{cl}(U_k(K')) \subset K'$ for all $m \leq k$.

Therefore,

$$\Lambda(G_*^k) = 1 + i_{N/L}(G^k, K') = 1 + \Lambda(g_*^k) - m_k$$

with $m_k = i_S(g^k, S \setminus U_k(K'))$. On the other hand,

$$\Lambda(G_*^k) = (-1)^0 + (-1)^1 \operatorname{tr}((G_*^k)_1)$$

for all $k \in \mathbb{N}$. We have that

$$\operatorname{tr}((G_*^k)_1) = -\Lambda(g_*^k) + m_k \quad \text{for all } k \in \mathbb{N}.$$

If $\{\lambda_1, \ldots, \lambda_{l-1}\}$ are the eigenvalues of the $l-1 \times l-1$ matrix generated by $(G_*)_1$ it is easy to see that $\operatorname{tr}((G_*^k)_1) = \lambda_1^k + \cdots + \lambda_{l-1}^k$ and then

$$\lambda_1^k + \cdots + \lambda_{l-1}^k = -\Lambda(g_*^k) + m_k \quad \text{for all } k \in \mathbb{N}.$$

Before continuing let us state the next lemma.

LEMMA 6. (See [7]) *Let M be an arbitrary $n \times n$ real matrix. Then, for infinitely many integers $k > 0$, $\operatorname{tr}(M^k) \geq 0$.*

The key point in the proof of the lemma is that there are infinitely many integers k such that λ^k is non-negative if λ is a real eigenvalue and λ^k has argument very close to zero if λ is not real.

Continuing with the proof of the Main Theorem, if $S = S^2$ and $f : S^2 \setminus K \to S^2 \setminus K$ is an orientation-preserving homeomorphism, unstable in K, we can work with the induced unstable homeomorphism $g|_{S^2 \setminus K'} : S^2 \setminus K' \to S^2 \setminus K'$, with K' a compact zero-dimensional subset of S^2. Since the singular homology groups of S^2 with rational coefficients are

$$H_0(S^2) = \mathbb{Q}, \quad H_1(S^2) = 0, \quad H_2(S^2) = \mathbb{Q},$$

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then $\Lambda(g_*^k) = (-1)^0 + (-1)^2 = 2$ for all $k \in \mathbb{N}$. If $(S^2 \setminus K') \cap \mathrm{Per}(g) = \emptyset$ we have $m_k = 0$ for all $k \in \mathbb{N}$ and

$$\mathrm{tr}((G_*^k)_1) = \lambda_1^k + \cdots + \lambda_{l-1}^k = -2 \quad \text{for all } k \in \mathbb{N},$$

but this is not possible by the last lemma. Then, $\mathrm{Per}(g) \cap (S^2 \setminus K') \neq \emptyset$. In the same way, if $f : S^2 \setminus K \to S^2 \setminus K$ is orientation-reversing, we only have to consider the map f^2 (which is orientation-preserving) and

$$\mathrm{tr}((G_*^{2k})_1) = \lambda_1^{2k} + \cdots + \lambda_{l-1}^{2k} = -2 \quad \text{for all } k \in \mathbb{N},$$

which is not possible.

In the case of $S = P^2(\mathbb{R})$, since the singular homology groups with rational coefficients are

$$H_0(P^2(\mathbb{R})) = \mathbb{Q}, \quad H_1(P^2(\mathbb{R})) = 0, \quad H_2(P^2(\mathbb{R})) = 0,$$

then $\Lambda(g_*^k) = (-1)^0 = 1$ for all $k \in \mathbb{N}$ and we obtain

$$\mathrm{tr}((G_*^k)_1) = \lambda_1^k + \cdots + \lambda_{l-1}^k = -1 \quad \text{for all } k \in \mathbb{N},$$

which gives us a contradiction to the above lemma. Then, $\mathrm{Per}(g) \cap (P^2(\mathbb{R}) \setminus K') \neq \emptyset$ and the Main Theorem is proved. □

Remark 4. Let S be a surface and let $f : S \setminus K \to S \setminus K$ be a homeomorphism under the conditions of the Main Theorem. Since $\mathrm{tr}((G_*^k)_1) = -\Lambda(g_*^k) + m_k$ for all $k \in \mathbb{N}$, if there exists $k \in \mathbb{N}$ with $\mathrm{tr}((G_*^k)_1) \neq -\Lambda(g_*^k)$, then $\mathrm{Per}(f) \neq \emptyset$ and f is not minimal on $S \setminus K$.

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