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Pointwise-recurrent dendrite maps

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Abstract. Let D be a dendrite and $f : D \to D$ a continuous map. Denote by $E(D)$ and $B(D)$ the sets of endpoints and branch points of D, respectively. We show that if $E(D)$ is countable, then f is pointwise-recurrent if and only if f is a pointwise-periodic homeomorphism; also, if $B(D)$ is discrete, then f is pointwise-recurrent if and only if every point in $D\backslash E(D)$ is periodic.

1. *Introduction*

Recurrence and periodicity play an important role in the study of dynamical systems. It is interesting to study maps $f : X \to X$ from a topological space X to itself that are pointwise-periodic (i.e. such that all points in X are periodic) or pointwise-recurrent (i.e. all points in X are recurrent). Montgomery [**18**] showed that if X is a connected topological manifold, a pointwise-periodic homeomorphism $f : X \to X$ must be periodic. Weaver [**24**] showed that if X is a continuum embedded in an orientable 2-manifold, then an orientation-preserving C^1-homeomorphism $f : X \to X$ has this property. Gottschalk [**13**] proved that if X is a continuum, then a relatively recurrent homeomorphism $f : X \to X$ (i.e. a homeomorphism such that the closure of the recurrent point set is dense in X) has every recurrent cut point being periodic. In [**23**], Oversteegen and Tymchatyn showed that recurrent homeomorphisms of the plane are periodic. Kolev and Pérouème [**14**] proved that recurrent homeomorphisms of a compact surface with negative Euler characteristic are still periodic. Recently, Mai [**16**] showed that a graph map $f : G \to G$ is pointwise-recurrent if and only if one of the following statements holds:

(1) G is a circle, and f is a homeomorphism that is topologically conjugate to an irrational rotation;
(2) f is a periodic homeomorphism.

We mention here that Mai's result can also be obtained using the 'spectral decomposition' theorem on graphs due to Blokh (see [**6–9**] for more details).

In this paper we will study pointwise-recurrent dendrite maps, whose dynamical behavior is both important and interesting in the study of dynamical systems and continuum theory. Recent interest in the dynamics on dendrites has been motivated by

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the fact that dendrites often appear as Julia sets in complex dynamics (see [**4**]). In [**1, 3, 11, 17, 21, 22**] several results concerning dendrites were obtained. In [**22**], we proved that every relatively recurrent monotone dendrite map has all of its cut points being periodic. Before stating our main results, we recall some basic properties of dendrites and dendrite maps.

A continuum is a compact connected metric space. A topological space is arcwise connected if any two of its points can be joined by an arc. We use the terminology of Nadler [**20**]. An arc is any space homeomorphic to the compact interval $[0, 1]$. By a *dendrite* D we mean a locally connected continuum which contains no homeomorphic copy of a circle. Every sub-continuum of a dendrite is a dendrite [**20**, Theorem 10.10], and every connected subset of D is arcwise connected [**20**, Proposition 10.9]. In addition, any two distinct points x and y of a dendrite D can be joined by a unique arc with endpoints x and y; denote this arc by $[x, y]$, and let $[x, y) = [x, y]\backslash\{y\}$, $(x, y] = [x, y]\backslash\{x\}$ and $(x, y) = [x, y]\backslash\{x, y\}$. A point $x \in D$ is called an *endpoint* if $D\backslash\{x\}$ is connected. It is called a *branch point* if $D\backslash\{x\}$ has more than two connected components. Denote by $E(D)$ and $B(D)$ the sets of endpoints and branch points of D, respectively. A point $x \in D\backslash E(D)$ is called a *cut point*. The set of cut points of D is dense in D. A *tree* is a dendrite with a finite set of endpoints.

Let $\mathbb{Z}_+$ and $\mathbb{N}$ be the sets of non-negative integers and positive integers, respectively. Let X be a compact metric space with metric d, and let $f : X \longrightarrow X$ be a continuous map. Denote by f^n the nth iterate of f; that is, f^0 is the identity map, and $f^n = f \circ f^{n-1}$ for $n \geq 1$. For any $x \in X$, the subset $O_f(x) = \{f^n(x) : n \in \mathbb{Z}_+\}$ is called the *f-orbit of x*. A point $x \in X$ is said to be *periodic* of prime period $n \in \mathbb{N}$ if $f^n(x) = x$ and $f^i(x) \neq x$ for $1 \leq i \leq n-1$. A subset A of X is said to be *f-invariant* if $f(A) \subset A$. It is called *a minimal set of f* if it is non-empty, closed, f-invariant and minimal (in the sense of inclusion) with respect to these properties. For a subset A of X, denote by $\overline{A}$ the closure of A and by $\mathrm{diam}(A)$ the diameter of A. We define the *ω-limit set* of a point $x \in X$ to be the set

$$\omega_f(x) = \left\{y \in X : \exists n_i \in \mathbb{N},\, n_i \to \infty,\, \lim_{i\to\infty} d(f^{n_i}(x),\, y) = 0\right\}$$
$$= \bigcap_{n\in\mathbb{N}} \overline{\{f^k(x) : k \geq n\}}.$$

The set $\omega_f(x)$ is a non-empty, closed and strongly invariant set, i.e. $f(\omega_f(x)) = \omega_f(x)$. A point $x \in X$ is said to be:

- *recurrent* for f if $x \in \omega_f(x)$;
- *almost periodic* if for any neighborhood U of x there exists $N \in \mathbb{N}$ such that $\{f^{n+i}(x) : i = 0, 1, \ldots, N\} \cap U \neq \emptyset$ for all $n \in \mathbb{N}$;
- *regularly recurrent* if for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x, f^{kN}(x)) < \varepsilon$ for all $k \in \mathbb{N}$.

It is easy to see that if x is regularly recurrent, then it is almost periodic. Note also that a recurrent point x is almost periodic if and only if $\omega_f(x)$ is minimal (see [**10**, Ch. V, Proposition 5]).

Let $\mathrm{Fix}(f)$, $P(f)$, $AP(f)$ and $R(f)$ denote the set of fixed points, periodic points, almost periodic points and recurrent points, respectively. Then we have the inclusion relations $\mathrm{Fix}(f) \subset P(f) \subset AP(f) \subset R(f)$. We say that f is:

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- *pointwise-periodic* if $P(f) = X$;
- *pointwise-recurrent* if $R(f) = X$;
- *relatively recurrent* if $\overline{R(f)} = X$.

A continuous map from a dendrite into itself is called a *dendrite map*.

Our main results are the following.

THEOREM 1.1. *Let $f: D \to D$ be a dendrite map. If $B(D)$ is discrete, then f is pointwise-recurrent if and only if every cut point is periodic.*

Remark 1.2. For a dendrite map $f: D \to D$, one can show easily that if all cut points of D are periodic, then f is a homeomorphism.

By [**2**, Corollary 3.6], for any dendrite D we have that $B(D)$ is discrete whenever $E(D)$ is closed. Therefore, a consequence of Theorem 1.1 is the following.

COROLLARY 1.3. *Let $f: D \to D$ be a dendrite map. If $E(D)$ is closed, then f is pointwise-recurrent if and only if every cut point is periodic.*

Remark 1.4. If $E(D)$ is closed and $f: D \to D$ is a pointwise-recurrent dendrite map, then an endpoint of D may not, in general, be periodic; for an example, see the one given by Efremova and Makhrova in [**11**] on the Gehman dendrite.

One can also prove the following corollary.

COROLLARY 1.5. *If $B(D)$ is discrete and $f: D \to D$ is a pointwise-recurrent dendrite map, then every endpoint of D is regularly recurrent.*

THEOREM 1.6. *Let $f: D \to D$ be a dendrite map. If $E(D)$ is countable, then f is pointwise-recurrent if and only if f is a pointwise-periodic homeomorphism.*

COROLLARY 1.7. [**16**] *Let T be a tree and $f: T \to T$ a continuous map. If f is pointwise-recurrent, then f is periodic.*

Recall that a map f is *periodic* if $f^n = \mathrm{id}_T$ for some $n \in \mathbb{N}$.

2. *Preliminaries*

LEMMA 2.1. [**17**, Lemma 2.3] *Let $(C_i)_{i\in\mathbb{N}}$ be a sequence of connected subsets of a dendrite (D, d). If $C_i \cap C_j = \emptyset$ for all $i \neq j$, then*

$$\lim_{n\to+\infty} \operatorname{diam}(C_n) = 0.$$

LEMMA 2.2. *Let D be a dendrite, and let $(p_n)_{n\in\mathbb{N}}$ be a sequence of D such that $p_{n+1} \in (p_n, p_{n+2})$ for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty} p_n = p_\infty$. Let U_n be the connected component of $D\backslash\{p_n, p_\infty\}$ that contains the open arc (p_n, p_∞). Then $\lim_{n\to+\infty} \operatorname{diam}(U_n) = 0$*

Here, the points p_n are located on the arc $[p_1, p_\infty]$ in a monotone way.

Proof. It is easy to see that $U_{n+1} \subset U_n$ for all $n \in \mathbb{N}$. Suppose that Lemma 2.2 is not true; then there is $\delta > 0$ such that for all $n \in \mathbb{N}$, $\operatorname{diam}(U_n) > \delta$. We will construct an infinite sequence $(I_{n_i})_{i\in\mathbb{N}}$ of pairwise disjoint arcs such that $\operatorname{diam}(I_{n_i}) > \delta/2$ for all $i \in \mathbb{N}$, contradicting Lemma 2.1. Fix $n \in \mathbb{N}$ and let $a_n, b_n \in U_n$ be such that $d(a_n, b_n) > \delta$. Then we have either $d(p_n, a_n) > \delta/2$ or $d(p_n, b_n) > \delta/2$. So let $I_n = [p_n, a_n]$

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if $d(p_n, a_n) > \delta/2$ and $I_n = [p_n, b_n]$ otherwise. Choose an integer $m > n$ such that $I_n \cap [p_m, p_\infty] = \emptyset$; then $I_n \cap \overline{U}_m = \emptyset$. In the same way as for I_n, we construct an arc $I_m \subset \overline{U}_m$. Then, by induction, we construct an infinite sequence of arcs $(I_{n_i})_{i\in\mathbb{N}}$ with $\operatorname{diam}(I_i) > \delta/2$ such that for all $i \in \mathbb{N}$, $I_{n_i} \subset \overline{U}_{n_i}$ and $I_{n_i} \cap \overline{U}_{n_{i+1}} = \emptyset$. This implies that the arcs (I_{n_i}), for $i \in \mathbb{N}$, are pairwise disjoint. □

THEOREM 2.3. [**17**, Theorem 2.13] *Let $f : D \to D$ be a dendrite map, $[x, y]$ an arc in D, and U the connected component of $D\backslash\{x, y\}$ containing the open arc (x, y). If there exist $k, m \in \mathbb{N}$ such that $\{f^k(x), f^m(y)\} \subset U$, then $U \cap P(f) \neq \emptyset$.*

We say that a dendrite map $f : D \to D$ is *monotone* if the preimage of any point under f is connected. Notice that if f is monotone, then so is f^n for any $n \in \mathbb{N}$.

THEOREM 2.4. [**22**, Theorem 1.6] *Let $f : D \to D$ be a dendrite map. If f is monotone, then the following statements are equivalent:*

(i) *f is pointwise-recurrent;*

(ii) *every cut point is a periodic point.*

LEMMA 2.5. *Let $f : D \to D$ be a dendrite map. Then the following properties are equivalent:*

(i) *any arc $I \subset f(D)$ contains a point with unique preimage under f;*

(ii) *the map f is monotone.*

Proof. (i) $\Longrightarrow$ (ii). If f is not monotone, then there exists $z \in D$ such that $f^{-1}(z)$ is not connected. So one can find $a, b \in D$ with $a \neq b$ and $w_{-1} \in (a, b)$ such that $f(a) = f(b) = z$ and $w := f(w_{-1}) \neq z$. By continuity of f, we have $[z, w] \subset f([a, w_{-1}]) \cap f([b, w_{-1}]) \subset f(D)$. Hence each point in (z, w) has at least two preimages under f.

(ii) $\Longrightarrow$ (i). Let $I \subset f(D)$ be an arc. Then $(f^{-1}(\{x\}))_{x\in I}$ is a family of uncountably many pairwise-disjoint connected non-empty subsets of D. Suppose that for every $x \in D$, $f^{-1}(\{x\})$ is not reduced to a point; then there is an arc $I_x \subset f^{-1}(\{x\})$ containing no endpoints. By [**20**, Corollary 10.28], D can be written as $D = \bigcup_{n\in\mathbb{N}} A_n \cup E(D)$ where $(A_n)_{n\in\mathbb{N}}$ is a family of arcs with pairwise-disjoint interiors. Hence, for each arc I_x there is $n(x) \in \mathbb{N}$ such that $I_x \cap A_{n(x)}$ is an arc. Thus, necessarily, there is an arc A_{n_0} that contains uncountably many pairwise-disjoint arcs of $(I_x)_{x\in I}$, which is a contradiction. □

LEMMA 2.6. *Let X be a compact metric space and $f : X \to X$ a pointwise-recurrent continuous map. Then every periodic point of f has a unique preimage under f^n for all $n \in \mathbb{N}$.*

Proof. Suppose that for some periodic point p of f, $f^{-1}(\{p\})$ contains more than one point. Then there is a $q \notin O_f(p)$ such that $f(q) = p$, and so $\omega_f(q) = \omega_f(p) = O_f(p)$. Hence q is not recurrent, which is a contradiction. As a result, any periodic point of f has a unique preimage under f which is a periodic point of f. So we deduce by induction that any periodic point of f has a unique preimage under f^n for all $n \in \mathbb{N}$. □

LEMMA 2.7. *Let $f : D \to D$ be a pointwise-recurrent dendrite map, and let $(p_n)_{n\in\mathbb{N}} \subset D$ be a sequence of periodic points of f such that $p_{n+1} \in (p_n, p_{n+2})$ for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty} p_n = p_\infty$. Then p_∞ is a regularly recurrent point.*

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Proof. For $n \in \mathbb{N}$, let U_n be defined as in Lemma 2.2, let $V_n := U_n \cup \{p_n, p_\infty\}$, and denote by N_n the period of the point p_n. It is easy to see that if there is a sub-sequence of $(p_n)_{n\in\mathbb{N}}$ with bounded periods, then by the continuity of f (and hence the continuity of its iterations) p_∞ is periodic; in particular, it is a regularly recurrent point. Otherwise, we have to prove Lemma 2.7 in the case where p_∞ is not periodic. We can assume that

$$(p_n, p_\infty] \cap \mathrm{Fix}(f^{N_n}) = \emptyset. \tag{2.1}$$

We will prove that for all $n \in \mathbb{N}$, the orbit of the point p_∞ under the map $f^{N_{n+1}}$ is contained in the set V_n. Indeed, if not, then for some $n \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that $f^{kN_{n+1}}(p_\infty) \notin V_n$; so we have the following two possibilities, and will show that each of them leads to a contradiction:

$$p_\infty \in (p_{n+1}, f^{kN_{n+1}}(p_\infty)) \quad \text{or} \quad p_n \in (p_{n+1}, f^{kN_{n+1}}(p_\infty)).$$

Let $m := kN_{n+1}$. Suppose that $p_\infty \in (p_{n+1}, f^m(p_\infty))$. As $p_{n+1} \in \mathrm{Fix}(f^m)$, by the continuity of f^m we have

$$f^m([p_{n+1}, p_\infty]) \supset [p_{n+1}, f^m(p_\infty)] \ni p_\infty.$$

Hence there is a point $p_{\infty,-1} \in (p_{n+1}, p_\infty)$ such that $f^m(p_{\infty,-1}) = p_\infty$. Similarly, there is a point $p_{\infty,-2} \in (p_{n+1}, p_{\infty,-1})$ such that $f^m(p_{\infty,-2}) = p_{\infty,-1}$. Thus, by induction, we construct a sequence $(p_{\infty,-k})_{k\in\mathbb{N}}$ in (p_{n+1}, p_∞) such that for all $k \in \mathbb{N}$, $p_{\infty,-(k+1)} \in [p_{n+1}, p_{\infty,-k}]$ and

$$f^{km}(p_{\infty,-k}) = p_\infty. \tag{2.2}$$

As $p_{\infty,-1} \neq p_\infty$, there exists $p_s \in (p_{\infty,-1}, p_\infty)$ for some $s \in \mathbb{N}$. Let $r \in \mathbb{N}$ be such that $f^r(p_{n+1}) = p_{n+1}$ and $f^r(p_s) = p_s$. Hence $f^{rm}(p_{n+1}) = p_{n+1}$ and $f^{rm}(p_s) = p_s$. By (2.2), $f^{rm}(p_{\infty,-r}) = p_\infty$, and by the continuity of f^{rm} we have $f^{rm}([p_{n+1}, p_{\infty,-r}]) \supset [p_{n+1}, p_\infty] \ni p_s$; hence p_s has a preimage q under the map f^{rm} in the arc $[p_{n+1}, p_{\infty,-r}]$. As $p_s \notin [p_{n+1}, p_{\infty,-r}]$, $q \neq p_s$, and this contradicts Lemma 2.6 since p_s is a fixed point of f^{rm}.

Now consider the second case, $p_n \in (p_{n+1}, f^m(p_\infty))$. By the continuity of f^m, $f^m([p_{n+1}, p_\infty]) \supset [p_{n+1}, f^m(p_\infty)] \ni p_n$. So p_n has a preimage q under the map f^m in the arc $[p_{n+1}, p_\infty]$. By Lemma 2.6, q is a periodic point that belongs to the orbit of p_n, hence q has the same period as p_n. Thus, $q \in (p_n, p_\infty] \cap \mathrm{Fix}(f^{N_n})$, which contradicts (2.1).

It follows that for any $n \in \mathbb{N}$, the orbit of the point p_∞ under the map $f^{N_{n+1}}$ is included in the set V_n, and as $\mathrm{diam}(V_n) = \mathrm{diam}(U_n)$, by Lemma 2.2 we have that $\lim_{n\to+\infty} \mathrm{diam}(V_n) = 0$. This implies that p_∞ is a regularly recurrent point. □

3. *Proof of Theorem 1.1 and Corollary 1.5*

Proof of Theorem 1.1. If every cut point is periodic, then one can easily show that f is a homeomorphism. Hence, by Theorem 2.4, f is pointwise-recurrent.

If f is pointwise-recurrent, then f is surjective. We will use Lemma 2.5. Let $I \subset D$ be an arc. Since $B(D)$ is discrete, there exists an open arc $J \subset I$ containing no branch points; hence J is an open subset in D. So let $x \in J$. If x is periodic, then by Lemma 2.6 we know

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that x has a unique preimage under f. If x is not periodic, then since f is pointwise-recurrent, one can find $n, m \in \mathbb{N}$ such that $x, f^m(x) \in J$ and $f^{n+m}(x) \in (x, f^m(x))$. Thus, $(x, f^m(x))$ is the connected component of $D\backslash\{x, f^m(x)\}$ that contains the open arc $(x, f^m(x))$. By Theorem 2.3, $(x, f^m(x)) \subset I$ contains a periodic point p. By Lemma 2.6, p has a unique preimage under f. We conclude by Lemma 2.5 that f is monotone. Therefore, by Theorem 2.4, every cut point is periodic. The proof is complete. □

Proof of Corollary 1.5. Let $e \in E(D)$. Take any cut point $p \in D$. By Theorem 1.1, $[p, e) \subset P(f)$. Hence, by Lemma 2.7, e is regularly recurrent. □

4. *Proof of Theorem 1.6 and Corollary 1.7*

We need the following result.

THEOREM 4.1. [**17**] *Let $f : D \to D$ be a dendrite map. If $E(D)$ is countable, then $\overline{R(f)} = \overline{P(f)}$.*

LEMMA 4.2. *Let $f : D \to D$ be a pointwise-recurrent dendrite map. If $E(D)$ is countable, then $\omega_f(x) \cap P(f) \neq \emptyset$ for all $x \in D$.*

Proof. If $\omega_f(x) \subset E(D)$, then $\omega_f(x)$ is compact countable. Thus $\omega_f(x)$ is finite, because otherwise $\omega_f(x)$ would be perfect and hence uncountable, which is a contradiction. Therefore, $\omega_f(x)$ is a periodic orbit and so x is a periodic point (see [**10**, Ch. IV, Lemma 4]). One can thus assume that $\omega_f(x)\backslash E(D) \neq \emptyset$ and take $y \in \omega_f(x)\backslash E(D)$. Since $R(f) = D$, it follows from Theorem 4.1 that $\overline{P(f)} = D$, so one can find $p, q \in P(f)$ such that $y \in [p, q]$. Let $N \in \mathbb{N}$ be such that $f^N(p) = p$ and $f^N(q) = q$. We already have $y \in [p, f^N(y)] \cup [q, f^N(y)]$. One can suppose that $y \in [p, f^N(y)]$, the same proof being valid with p replaced by q. In this case, we have $y \in f^N([p, y])$, so there is $y_{-1} \in [p, y]$ such that $f^N(y_{-1}) = y$. Again, $y_{-1} \in [p, y] = [p, f^N(y_{-1})] \subset f^N([p, y_{-1}])$, so there is $y_{-2} \in [p, y_{-1}]$ such that $f^N(y_{-2}) = y_{-1}$. Thus, we construct by induction a sequence $(y_{-n})_{n\in\mathbb{N}} \subset [p, y]$ such that

$$y_{-(k+1)} \in (y_{-k}, y_{-(k+2)}), \quad f^N(y_{-(k+1)}) = y_{-k} \quad \text{for every } k \in \mathbb{N}$$

and $\lim_{n\to+\infty} y_{-n} = y_\infty \in [p, y]$. Therefore one has $f^N(y_\infty) = y_\infty$ (by the continuity of f^N), and so $y_\infty \in P(f)$. Since $R(f) = D$ and, for any $k \in \mathbb{N}$, $f^{kN}(y_{-k}) = y$, it follows that $y_{-k} \in \omega_f(y_{-k}) = \omega_f(y)$. Therefore $y_\infty \in \omega_f(y)$, and as $\omega_f(y) \subset \omega_f(x)$, we conclude that $y_\infty \in \omega_f(x) \cap P(f)$. □

LEMMA 4.3. *Let $f : D \to D$ be a pointwise-recurrent dendrite map and let p_∞ be as in Lemma 2.7. If $E(D)$ is countable, then p_∞ is a periodic point.*

Proof. By Lemma 2.7, p_∞ is regularly recurrent, and hence $\omega_f(p_\infty)$ is a minimal set. It follows, by Lemma 4.2, that $\omega_f(p_\infty)$ is a periodic orbit, and so p_∞ is a periodic point since $p_\infty \in \omega_f(p_\infty)$. □

LEMMA 4.4. *Let D be a dendrite with a countable set of endpoints. Then every subdendrite of D has a countable set of endpoints.*

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Proof. Let Y be a sub-dendrite of D. By [**12**, p. 157] (see also [**20**]), Y is a monotone retraction of D by $r : D \to Y$. Then we have $r(E(D)) = E(Y)$. Indeed, $r(E(D)) \subset E(Y)$ follows from the fact that $r(x)$ lies in any arc joining x to any point of Y. Let $a \in E(Y)$ and let Z be the connected component of $D \setminus Y$ such that $\overline{Z}$ contains a. Then $r(\overline{Z}) = \{a\}$. As $\overline{Z} \cap E(D) \neq \emptyset$, there is $b \in \overline{Z} \cap E(D)$ with $r(b) = a$. It follows that $\mathrm{Card}(E(Y)) \leq \mathrm{Card}(E(D))$ and hence $E(Y)$ is countable. □

Proof of Theorem 1.6. The 'if' part of the Theorem is clear. Let us prove the 'only if' part. By Theorem 4.1, one has $\overline{P(f)} = D$.

Claim 1. If $x \in D$ is not periodic, then for any $z \in D$ with $z \neq x$, (x, z) contains a non-periodic branch point.

Take $z \in D$ with $z \neq x$; let us show that $[x, z]$ contains an infinite sequence of branch points that converges to x. Indeed, if not, then there is $y \in (x, z]$ such that $(x, y) \cap B(D) \neq \emptyset$. So (x, y) is an open subset of D and hence, as $\overline{P(f)} = D$, one can find a sequence $(p_n)_{n\in\mathbb{N}}$ of periodic points in (x, y) such that $p_{n+1} \in (p_n, p_{n+2})$ for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty} p_n = x$. By Lemma 4.3, x is periodic, which is a contradiction. Now suppose that every branch point in (x, z) is periodic; then, again by Lemma 4.3, x is periodic. Therefore, necessarily one branch point in (x, z) must be non-periodic.

Claim 2. If there exists $x \in D$ which is not periodic, then D contains a sub-dendrite with uncountable set of endpoints.

Suppose that $x \in D$ is not periodic. By Claim 1, there is $a \in B(D)$ which is not periodic. Now, as a is a branch point, there exist two arcs I_0 and I_1, each with a as one endpoint, which together with $[x, a]$ form a family of three arcs having pairwise-disjoint interior. As a is not periodic, by Claim 1 there is $a_0 \in I_0$ (respectively, $a_1 \in I_1$), a non-periodic branch point distinct from a. Denote by J_0 the arc $[a, a_0]$ and by J_1 the arc $[a, a_1]$. At the second step, as a_0 and a_1 are branch points, there exist two arcs I_{00} and I_{01} (respectively, I_{10} and I_{11}), each with a (respectively, a_1) as one endpoint, which together with $[a, a_0]$ (respectively, $[a, a_1]$) form a family of three arcs having pairwise-disjoint interior. Similarly, by Claim 1 we find, for any $i, j \in \{0, 1\}$, a non-periodic branch point $a_{ij} \in I_{ij}$ distinct from a_i; write $J_{ij} := [a_i, a_{ij}]$. Thus, by induction, we construct for $n \in \mathbb{N}$ a sequence of arcs J_{α_n} and a sequence of non-periodic branch points a_{α_n}, where $\alpha_n \in \{0, 1\}^n$, which satisfy the following properties (see Figure 1):

(i) $J_{\alpha_n 0} = [a_{\alpha_n}, a_{\alpha_n 0}]$ and $J_{\alpha_n 1} = [a_{\alpha_n}, a_{\alpha_n 1}]$;

(ii) $J_{\alpha_n} \cap J_{\alpha_n 0} = J_{\alpha_n} \cap J_{\alpha_n 1} = J_{\alpha_n 0} \cap J_{\alpha_n 1} = \{a_{\alpha_n}\}$.

For any $n \in \mathbb{N}$, let $D_n = \bigcup_{\alpha_n\in\{0,1\}^n} J_{\alpha_n}$; then clearly D_n is a tree. Also, we have $D_n \subset D_{n+1}$ for all $n \in \mathbb{N}$. Hence $D_\infty = \overline{\bigcup_{n\in\mathbb{N}} D_n}$ is a sub-dendrite. Take $\beta \in \{0, 1\}^{\mathbb{N}}$. For each $n \in \mathbb{N}$, we have $[a, a_{\beta(n)}] \subset [a, a_{\beta(n+1)}]$ (where, for each $i \in \mathbb{N}$, we denote by $\beta(i)$ the first word of length i in the sequence β). Then $\overline{\bigcup_{n\in\mathbb{N}}[a, a_{\beta(n)}]}$ is an arc where the sequence $(a_{\beta(n)})_{n\in\mathbb{N}}$ is monotone in that arc; so it converges to a point, namely a_β, that belongs to the dendrite D_∞. The points a_β for $\beta \in \{0, 1\}^{\mathbb{N}}$ are the endpoints of the sub-dendrite D_∞. Hence, D_∞ has an uncountable set of endpoints.

We then deduce Theorem 1.6 from Claim 2 and Lemma 4.4. □

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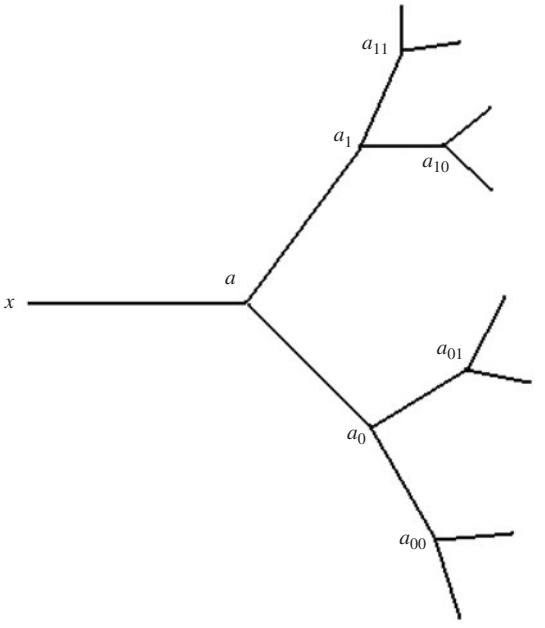


FIGURE 1. The tree D_2.

Proof of Corollary 1.7. By Theorem 1.6, f is a pointwise-periodic homeomorphism. As $E(T)$ is finite, there is $N \in \mathbb{N}$ such that every endpoint of T is fixed by f^N. Let $a, b \in T$ be two distinct endpoints of T. As f is a homeomorphism (and so is f^N), the arc $[a, b]$ is f^N-invariant; hence every point in $[a, b]$ is fixed by f^N. Since $T = \bigcup_{a,b \in E(T)} [a, b]$, every point in the tree T is fixed by f^N. So f is periodic.

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