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THE LARGE-TIME DEVELOPMENT OF THE SOLUTION TO AN INITIAL-VALUE PROBLEM FOR THE KORTEWEG–DE VRIES EQUATION. II. INITIAL DATA HAS A DISCONTINUOUS COMPRESSIVE STEP

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Abstract. In this paper, we consider an initial-value problem for the Korteweg–de Vries equation. The normalized Korteweg–de Vries equation considered is given by

$$u_\tau + uu_x + u_{xxx} = 0, \quad -\infty < x < \infty,\ \tau > 0,$$

where x and τ represent dimensionless distance and time, respectively. In particular, we consider the case when the initial data has a discontinuous compressive step, where $u(x,0) = u_0 > 0$ for $x \leqslant 0$ and $u(x,0) = 0$ for $x > 0$. The method of matched asymptotic coordinate expansions is used to obtain the detailed large-τ asymptotic structure of the solution to this problem, which exhibits the formation of a dispersive shock wave.

§1. *Introduction.* In this paper we consider the following initial-value problem for the normalized Korteweg–de Vries equation:

$$u_\tau + uu_x + u_{xxx} = 0, \quad -\infty < x < \infty, \tau > 0, \tag{1.1}$$

$$u(x,0) = \begin{cases} 0, & x > 0, \\ u_0, & x \leqslant 0, \end{cases} \tag{1.2}$$

$$u(x,\tau) \to \begin{cases} 0, & x \to \infty, \\ u_0, & x \to -\infty, \end{cases} \quad \tau \geqslant 0, \tag{1.3}$$

where $u_0 > 0$ and the initial distribution (1.2) is a discontinuous compressive step. In what follows we label initial-value problem (1.1)–(1.3) as **IVP**. Our objective is to obtain a uniform asymptotic approximation to the solution to **IVP** as $\tau \to \infty$, uniform in $-\infty < y < \infty$, where

$$y = \frac{x}{\tau} \tag{1.4}$$

is the natural spatial variable as $\tau \to \infty$.

We note that the Korteweg–de Vries equation is a canonical equation combining both convective nonlinearity and dispersion, and as such arises in the modelling of many physical phenomenon, including shallow-water gravity waves, ion-acoustic waves and waves in the atmosphere and ocean. Clearly, the literature relating to the Korteweg–de Vries equation is vast and we make no

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attempt here to summarize it, rather we make reference only to the most salient literature relevant to the present paper.

In the case of initial data which has compact support, in the form of a rectangular well, the corresponding form of **IVP** is exactly solvable via inverse scattering theory (see, for example, [12, Ch. 17]), and the detailed asymptotic structures as $\tau \to \infty$ may be obtained directly from this exact solution. Inverse scattering theory has also been used to obtain the large-time solution of the initial-value problem for the Korteweg–de Vries equation when the initial data is of the form of a smoothed step (see, for example, [6, 7]). In addition to this, a number of exact and approximate solutions to the Korteweg–de Vries equation have been obtained (for example, via [12, Ch. 16]) which represent fundamental coherent structures associated with the Korteweg–de Vries equation. These include a solution representing a train of interacting solitary waves, and, more pertinent to the present paper, an approximate solution representing what has been termed a dispersive shock wave. The dispersive shock wave solution is an approximate solution which has been obtained via Whitham's quasi-classical method (see, for example, [3, 4, 12]) and represents an asymptotic solution to the Korteweg–de Vries equation as $\tau \to \infty$, in the spatial domain $-u_0 + O(1) \leqslant y \leqslant \frac{2}{3}u_0 - O(1)$ as $\tau \to \infty$, and is in general given by $u \sim u_s(y, \tau)$ as $\tau \to \infty$, where

$$u_s(y,\tau) = 2u_0\, dn^2\left[\sqrt{\frac{u_0}{6}}\tau\left(y - \frac{(1+m(y)^2)u_0}{3}\right), m(y)^2\right] - u_0(1-m(y)^2), \tag{1.5}$$

with $-u_0 \leqslant y \leqslant \frac{2}{3}u_0$, and the function $m(y)$ is determined implicity by

$$\frac{(1+m(y)^2)u_0}{3} - \frac{2u_0 m(y)^2(1-m(y)^2)\,\mathrm{K}(m(y)^2)}{3[\mathrm{E}(m(y)^2) - (1-m(y)^2)\,\mathrm{K}(m(y)^2)]} = y, \tag{1.6}$$

for $-u_0 \leqslant y \leqslant \frac{2}{3}u_0$. Here $\mathrm{K}(s)$ and $\mathrm{E}(s)$ for $s \in [0, 1]$ are the standard complete elliptic integrals. We note that (1.6) describes $m(y)$ as a monotone increasing function of $-u_0 \leqslant y \leqslant 2u_0/3$, with

$$m(y) \to \begin{cases} 1 & \text{as } y \to \dfrac{2u_0}{3}, \\ 0 & \text{as } y \to -u_0. \end{cases} \tag{1.7}$$

An examination of (1.5) reveals that, with $\tau \gg 1$ and $-u_0 \leqslant y \leqslant 2u_0/3$, $u_s(y, \tau)$ has a rapidly oscillating term (in the spatial variable $\tilde{y} = y - \frac{1}{3}(1 + m(y)^2)u_0$) which is modulated by $m(y)$, together with a monotone decreasing background term $u_0(1 - m(y)^2)$. In particular, as $y \to (2u_0/3)^-$,

$$u_s(y,\tau) \sim 8u_0 \exp\left(\sqrt{\frac{2u_0}{3}}\left(y - \frac{2u_0}{3}\right)\tau\right), \tag{1.8}$$

whilst, as $y \to (-u_0)^+$,

$$u_s(y,\tau) \sim u_0 - \lambda(u_0)\,(y + u_0)\,\cos\left(\sqrt{\frac{2u_0}{3}}\left(y - \frac{u_0}{3}\right)\tau\right), \tag{1.9}$$

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where $\lambda(u_0)$ is fixed via (1.6). We observe that in the rapidly oscillating part of $u_s(y,\tau)$, the local wave number (in y) is $k(y)\tau$, where

$$k(y) = \sqrt{\frac{u_0}{6}}\,\frac{\pi}{2\,K(m(y)^2)}, \qquad y \in \left[-u_0, \frac{2u_0}{3}\right],$$

with

$$k(y) \to 0 \quad \text{as } y \to \left(\frac{2u_0}{3}\right)^-,$$

and

$$k(y) \to \sqrt{\frac{2u_0}{3}} \quad \text{as } y \to (-u_0)^+.$$

We observe that the form (1.8), as $y \to (2u_0/3)^-$, represents the tail of the leading soliton in the dispersive shock, whilst the form (1.9), as $y \to (-u_0)^+$, represents the oscillatory, harmonic tail of the dispersive shock, about the mean value u_0. In [**3**] an efficient numerical method (pseudospectral) was developed for solving nonlinear dispersive wave evolution problems, and the Korteweg–de Vries equation was considered via **IVP**; the numerical solutions presented indicate that the large-τ structure of $u(y,\tau)$ exhibits the formation of the dispersive shock wave structure discussed above for $-u_0 + o(1) < y < 2u_0/3 - o(1)$ as $\tau \to \infty$, with an exponential decay region ahead when $y > 2u_0/3 + o(1)$ as $\tau \to \infty$, and a radiation region to the rear when $y < -u_0 - o(1)$ as $\tau \to \infty$. The analytical structure in the radiation and decay regions has not been accessible up to the present.

It is the purpose of this paper to obtain the detailed asymptotic structure of the initial-value problem **IVP** as $\tau \to \infty$, uniformly for $-\infty < y < \infty$. We use the methodology developed by Needham and Leach in the context of reaction–diffusion theory, and elucidated in [**11**]. This technique uses matched asymptotic coordinate expansions to transfer information from the initial data (1.2) as $\tau \to 0$, via asymptotic structures when $\tau = O(1)$ as $|x| \to \infty$, into the asymptotic structure as $\tau \to \infty$. The initial-value problem for the Korteweg–de Vries equation when the initial data has a discontinuous expansive step has recently been considered via this approach in [**10**]. Further, this approach has been used to obtain the large-time solution to initial-boundary-value problems for the Korteweg–de Vries equation on the positive [**8**] and negative [**9**] quarter-planes.

Since submitting this paper the authors have become aware of the recent work of Egorova *et al* [**2**] and Ablowitz and Baldwin [**1**] who have developed complementary results to those presented here for similar but distinct problems. In [**2**] the method of steepest descent is used to develop the large-time solution of the Korteweg–de Vries equation with continuous, monotonic, "step-like" initial data. The large-time asymptotic structure of the solution to the initial-value problem in this case differs from that presented here for discontinuous step initial data, in that a sequence of distinct solitons develop ahead of the dispersive shock wave. The exact number in this soliton sequence depends on the detailed form of the initial data. In [**1**] the authors consider monotonic initial data that consists of a finite number of well-separated discontinuous steps. In

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contrast to [2], the large-time solution of the initial-value problem develops a single dispersive shock wave. This single dispersive shock wave develops from the interactions of the dispersive shock waves that are initially formed from each sub-step at intermediate times.

§2. *Asymptotic solution to* ***IVP*** *as* $\tau \to \infty$. In this section we develop the asymptotic structure of **IVP** as $\tau \to \infty$ with particular attention to solitary and cnoidal wave formation, radiation and decay. We must first begin by examining the asymptotic structure of the solution to **IVP** as $\tau \to 0$, which will ultimately transmit information to the structure when $\tau \to \infty$.

2.1. *Asymptotic solution to* ***IVP*** *as* $\tau \to 0$. Consideration of the initial data (1.2) indicates that the structure of the asymptotic solution to **IVP** as $\tau \to 0$ has three asymptotic regions in x, labelled as follows:

$$\left.\begin{array}{lll} \text{region } \mathbf{I}: & x = o(1), & u(x,\tau) = O(1) \\ \text{region } \mathbf{II}: & x = O(1)\ (>0), & u(x,\tau) = o(1) \\ \text{region } \mathbf{III}: & x = O(1)\ (<0), & u(x,\tau) = u_0 - o(1) \end{array}\right\} \quad \text{as } \tau \to 0. \tag{2.1}$$

We first consider region **I**, in which $x = o(1)$ and $u(x,\tau) = O(1)$ as $\tau \to 0$. To examine region **I**, we introduce the scaled coordinate $\eta = x\tau^{-\alpha} = O(1)$ as $\tau \to 0$, where $\alpha > 0$, and look for an expansion of the form

$$u(\eta,\tau) = \bar{u}(\eta) + o(1), \tag{2.2}$$

as $\tau \to 0$, with $\eta = O(1)$. On substitution of expansion (2.2) into equation (1.1) (when written in terms of η and τ) we find that the most structured leading order balance requires $\alpha = \frac{1}{3}$. At leading order we then obtain the equation

$$\bar{u}_{\eta\eta\eta} - \frac{\eta}{3}\bar{u}_\eta = 0, \quad -\infty < \eta < \infty. \tag{2.3}$$

Equation (2.3) is to be solved subject to matching with regions **II** (as $\eta \to \infty$) and **III** (as $\eta \to -\infty$) and initial condition (1.2), that is,

$$\bar{u}(\eta) = o(1) \quad \text{as } \eta \to \infty, \tag{2.4}$$

$$\bar{u}(\eta) = u_0 - o(1) \quad \text{as } \eta \to -\infty. \tag{2.5}$$

On making the substitution

$$\bar{u}_\eta = V(\xi), \qquad \eta = 3^{1/3}\xi, \tag{2.6}$$

equation (2.3) becomes Airy's equation

$$V_{\xi\xi} - \xi V = 0, \quad -\infty < \xi < \infty. \tag{2.7}$$

The solution to (2.7) is readily obtained as

$$V(\xi) = C_0\mathrm{Ai}(\xi) + D_0\mathrm{Bi}(\xi), \quad -\infty < \xi < \infty, \tag{2.8}$$

where Ai[.] and Bi[.] are the standard Airy functions and C_0, D_0 are constants to be determined on matching with regions **II** and **III**. Consideration of (2.8) and (2.4) as $\xi \to \infty$ (as we move into region **II**) indicates that we must

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set $D_0 = 0$ since $\mathrm{Bi}(\xi)$ grows exponentially as $\xi \to \infty$, with $\mathrm{Bi}(\xi) = O(\mathrm{e}^{2/3\xi^{3/2}})$ as $\xi \to \infty$. On setting $D_0 = 0$, equation (2.8) becomes

$$V(\xi) = C_0 \mathrm{Ai}(\xi), \quad -\infty < \xi < \infty, \tag{2.9}$$

Hence, via (2.6) and (2.9), we have that

$$\bar{u}(\eta) = \left(A_0 + C_0 3^{1/3} \int_0^{\eta 3^{-1/3}} \mathrm{Ai}(s)\, ds \right), \quad -\infty < \eta < \infty, \tag{2.10}$$

where A_0 is a constant of integration. Matching to leading order with region **II** (as $\eta \to \infty$) and with region **III** (as $\eta \to -\infty$) then requires, via (2.4) and (2.5), that

$$C_0 = -u_0 3^{-1/3} \quad \text{and} \quad A_0 = \tfrac{1}{3} u_0.$$

Therefore, in region **I** we have

$$u(\eta, \tau) = u_0 \left(\frac{1}{3} - \int_0^{\eta 3^{-1/3}} \mathrm{Ai}(s)\, ds \right) + o(1), \tag{2.11}$$

as $\tau \to 0$, with $\eta = O(1)$. We observe that

$$u(\eta, \tau) \sim \frac{u_0 3^{1/4}}{2\sqrt{\pi}} \eta^{-3/4} \exp\left(-\frac{2}{3\sqrt{3}} \eta^{3/2} \right) \quad \text{as } \eta \to \infty, \tag{2.12a}$$

$$u(\eta, \tau) \sim u_0 - \frac{u_0 3^{1/4}}{\sqrt{\pi}} (-\eta)^{-3/4} \cos\left(\frac{2}{3\sqrt{3}} (-\eta)^{3/2} + \frac{\pi}{4} \right) \quad \text{as } \eta \to -\infty. \tag{2.12b}$$

As $\eta \to \infty$ we move into region **II**, where $x = O(1)$ (> 0) as $\tau \to 0$. The form of expansion (2.11) for $\eta \gg 1$, given by (2.12*a*), suggests that in region **II** we expand as

$$u(x, \tau) = \mathrm{e}^{-\phi(x,\tau)} \quad \text{as } \tau \to 0, \tag{2.13}$$

with

$$\phi(x, \tau) = \phi_0(x) \frac{1}{\tau^{1/2}} + \phi_1(x) \ln \tau + \phi_2(x) + o(1), \tag{2.14}$$

where $x = O(1)$ (> 0) as $\tau \to 0$, and $\phi_0(x) > 0$. On substituting (2.13) and (2.14) into equation (1.1) and solving at each order in turn, we find (after matching with (2.11) as $x \to 0^+$) that

$$u(x, \tau) = \exp\left(-\frac{2x^{3/2}}{3\sqrt{3}} \tau^{-1/2} + \frac{1}{4} \ln \tau - \frac{3}{4} \ln x + \ln\left(\frac{u_0 3^{1/4}}{2\sqrt{\pi}} \right) + o(1) \right), \tag{2.15}$$

as $\tau \to 0$, with $x = O(1)$ (> 0).

Finally, we consider region **III**, where $x = O(1)$ (< 0) as $\tau \to 0$. The form of expansion (2.11) for $-\eta \gg 1$, given by (2.12*b*), suggests that in region **III** we expand as

$$u(x, \tau) = u_0 - (\mathrm{e}^{\psi^+(x,\tau)} + \mathrm{e}^{\psi^-(x,\tau)}) \quad \text{as } \tau \to 0, \tag{2.16}$$

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with

$$\psi^{\pm}(x,\tau) = \pm \mathrm{i}\psi_0(x)\tau^{-1/2} + \psi_1 \ln\tau \pm \mathrm{i}\psi_2(x) + \psi_3(x) \pm \mathrm{i}\psi_4(x)\tau^{1/2} + o(\tau^{1/2}), \tag{2.17}$$

with $x = O(1)$ (< 0) as $\tau \to 0$, and $\psi_0(x) > 0$. On substituting (2.16) and (2.17) into equation (1.1) and solving at each order in turn, we find (after matching with (2.11) as $x \to 0^-$) that

$$\begin{aligned} u(x,\tau) = u_0 - \exp\Bigg(& \mathrm{i}\frac{2}{3\sqrt{3}}(-x)^{3/2}\tau^{-1/2} + \frac{1}{4}\ln\tau \\ & + \left[\mathrm{i}\frac{\pi}{4} - \frac{3}{4}\ln(-x) + \ln\left(\frac{u_0 3^{1/4}}{2\sqrt{\pi}}\right)\right] \\ & + \tau^{1/2}\left[\mathrm{i}\frac{u_0}{\sqrt{3}}(-x)^{1/2}\right] + o(\tau^{1/2})\Bigg) \\ & - \exp\Bigg(-\mathrm{i}\frac{2}{3\sqrt{3}}(-x)^{3/2}\tau^{-1/2} + \frac{1}{4}\ln\tau \\ & + \left[-\mathrm{i}\frac{\pi}{4} - \frac{3}{4}\ln(-x) + \ln\left(\frac{u_0 3^{1/4}}{2\sqrt{\pi}}\right)\right] \\ & + \tau^{1/2}\left[-\mathrm{i}\frac{u_0}{\sqrt{3}}(-x)^{1/2}\right] + o(\tau^{1/2})\Bigg) \end{aligned} \tag{2.18}$$

as $\tau \to 0$, with $x = O(1)$ (< 0). We finally observe that expansion (2.15) remains uniform for $x \gg 1$ as $\tau \to 0$, and expansion (2.18) remains uniform for $(-x) \gg 1$ as $\tau \to 0$, so that the asymptotic structure as $\tau \to 0$ is now complete, with the expansions in regions **I**, **II** and **III** providing a uniform approximation to the solution of **IVP** as $\tau \to 0$.

2.2. *Asymptotic solution to **IVP** as* $|x| \to \infty$. We next investigate the asymptotic structure of the solution to **IVP** as $|x| \to \infty$ with $\tau = O(1)$. We first investigate the structure of the solution to **IVP** as $x \to \infty$ with $\tau = O(1)$. The form of expansion (2.15) of region **II** for $x \gg 1$ as $\tau \to 0$ suggests that in this region, which we label as region **IV**, we expand as

$$u(x,\tau) = \mathrm{e}^{-\theta(x,\tau)} \quad \text{as } x \to \infty, \tag{2.19}$$

with

$$\theta(x,\tau) = \theta_0(\tau)x^{3/2} + \theta_1(\tau)\ln x + \theta_2(\tau) + o(1), \tag{2.20}$$

where $\tau = O(1)$ as $x \to \infty$. On substituting (2.19) and (2.20) into equation (1.1) and solving at each order in turn, we find (after matching with (2.15) as $\tau \to 0^+$) that

$$u(x,\tau) = \exp\left(-\frac{2}{3\sqrt{3}}\tau^{-1/2}x^{3/2} - \frac{3}{4}\ln x + \left(\frac{1}{4}\ln\tau + \left[\frac{u_0 3^{1/4}}{2\sqrt{\pi}}\right]\right) + o(1)\right) \tag{2.21}$$

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as $x \to \infty$, with $\tau = O(1)$. We observe that expansion (2.21) remains uniform for $\tau \gg 1$ provided that $x \gg \tau$, but becomes non-uniform when $x = O(\tau)$ as $\tau \to \infty$. Finally, we investigate the structure of the solution to **IVP** as $x \to -\infty$ with $\tau = O(1)$, which we label as region **V**. The details in this case follow, after minor modification, those given above in region **IV**, with

$$u(x,\tau) = u_0 - \exp\left(\mathrm{i}\frac{2}{3\sqrt{3}}\tau^{-1/2}(-x)^{3/2} + \mathrm{i}\frac{u_0}{\sqrt{3}}\tau^{1/2}(-x)^{1/2} - \frac{3}{4}\ln(-x) + \left[\mathrm{i}\frac{\pi}{4} + \frac{1}{4}\ln\tau + \ln\left(\frac{u_0 3^{1/4}}{2\sqrt{\pi}}\right)\right] + o(1)\right) - \exp\left(-\mathrm{i}\frac{2}{3\sqrt{3}}\tau^{-1/2}(-x)^{3/2} - \mathrm{i}\frac{u_0}{\sqrt{3}}\tau^{1/2}(-x)^{1/2} - \frac{3}{4}\ln(-x) + \left[-\mathrm{i}\frac{\pi}{4} + \frac{1}{4}\ln\tau + \ln\left(\frac{u_0 3^{1/4}}{2\sqrt{\pi}}\right)\right] + o(1)\right) \tag{2.22}$$

as $x \to -\infty$, with $\tau = O(1)$. We observe that expansion (2.22) remains uniform for $\tau \gg 1$ provided that $(-x) \gg \tau$, but becomes non-uniform when $(-x) = O(\tau)$ as $\tau \to \infty$.

2.3. *Asymptotic solution to **IVP** as* $\tau \to \infty$. As $\tau \to \infty$, the asymptotic expansions (2.21) and (2.22) of regions **IV** ($x \to \infty, \tau = O(1)$) and **V** ($x \to -\infty, \tau = O(1)$) respectively continue to remain uniform provided $|x| \gg \tau$. However, as already noted, a non-uniformity develops in expansions (2.22) and (2.21) when $|x| = O(\tau)$.

We begin by considering the asymptotic structure as $\tau \to \infty$ for $x > 0$. To proceed we introduce a new region, region **VI**. To examine region **VI** we introduce the scaled coordinate $y = (x/\tau)\ (> 0)$, where $y = O(1)$ as $\tau \to \infty$, and look for an expansion of the form (as suggested by (2.21))

$$u(y,\tau) = \mathrm{e}^{-\tau\sigma(y,\tau)} \quad \text{as } \tau \to \infty, \tag{2.23}$$

with

$$\sigma(y,\tau) = \sigma_0(y) + \sigma_1(y)\frac{\ln\tau}{\tau} + \sigma_2(y)\frac{1}{\tau} + o\left(\frac{1}{\tau}\right), \tag{2.24}$$

as $\tau \to \infty$, with $y = O(1)$. It is instructive to consider first the leading order problem in region **VI**. On substituting (2.23) and (2.24) into equation (1.1) (when written in terms of y and τ) we obtain the leading order problem as

$$(\sigma_0')^3 - y\sigma_0' + \sigma_0 = 0, \quad y > 0, \tag{2.25}$$

$$\sigma_0(y) > 0, \quad y > 0, \tag{2.26}$$

$$\sigma_0(y) \sim \frac{2}{3\sqrt{3}}y^{3/2} \quad \text{as } y \to \infty. \tag{2.27}$$

The final condition, (2.27), arises from matching expansion (2.23) ($y \gg 1$) with expansion (2.21) ($x = O(\tau)$). Equation (2.25) has a one-parameter family of linear solutions

$$\sigma_0(y) = \alpha(y - \alpha^2), \quad -\infty < y < \infty, \tag{2.28}$$

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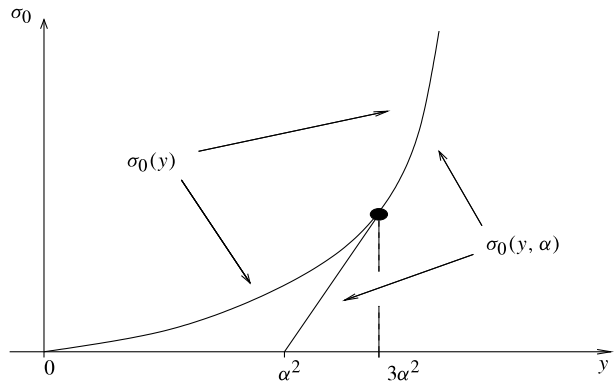


Figure 1: A sketch illustrating the envelope touching solution $\sigma_0(y,\alpha)$ (given by (2.31)) and the full envelope solution $\sigma_0(y)$ (given by (2.30)).

for any $\alpha \in \mathbb{R}$, together with the associated pair of envelope solutions,

$$\sigma_0(y) = \pm\frac{2}{3\sqrt{3}}y^{3/2}, \quad y > 0. \tag{2.29}$$

Combinations of (2.28) and (2.29) which remain continuous and differentiable also provide solutions to (2.25) (envelope touching solutions).

The problem (2.25)–(2.27) thus has the envelope solution given by

$$\sigma_0(y) = \frac{2}{3\sqrt{3}}y^{3/2}, \quad y > 0, \tag{2.30}$$

together with the family of envelope touching solutions given by

$$\sigma_0(y,\alpha) = \begin{cases} \dfrac{2}{3\sqrt{3}}y^{3/2}, & y \geqslant 3\alpha^2, \\ \alpha(y-\alpha^2), & \alpha^2 \leqslant y < 3\alpha^2, \end{cases} \tag{2.31}$$

for each $\alpha > 0$. Each of the above solutions is illustrated in Figure 1.

Since the solution to (2.25)–(2.27) approaches zero as

$$y \to y_c^+(\alpha) = \begin{cases} 0, & \alpha = 0, \\ \alpha^2, & \alpha > 0, \end{cases} \tag{2.32}$$

the expansion (2.23) with (2.24) in region **VI** becomes non-uniform as $y \to y_c^+(\alpha)$, and in particular when

$$y = y_c(\alpha) + O(\delta(\tau,\alpha)) \tag{2.33}$$

with

$$\delta(\tau,\alpha) = \begin{cases} \tau^{-1}, & \alpha > 0, \\ \tau^{-2/3}, & \alpha = 0. \end{cases} \tag{2.34}$$

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We note that, when $\alpha > 0$, although $\sigma_0(y, \alpha)$ and $\sigma_0'(y, \alpha)$ are continuous over the range of definition $y \geqslant y_c(\alpha)$, the second derivative $\sigma_0''(y, \alpha)$ is discontinuous at the point where the linear solution meets the envelope solution, that is, at $y = 3\alpha^2$. This indicates that a thin transition region exists in the neighbourhood of the point of contact of the linear solution and the envelope solution in which third-order derivatives are retained at leading order to smooth out these discontinuities in curvature.

There are now two distinct cases to consider: when $\alpha = 0$ and $\sigma_0(y)$ is given by (2.30); and when $\alpha > 0$ and $\sigma_0(y, \alpha)$ is given by (2.31). We consider each case in turn.

Case $\alpha = 0$. In this case we have

$$\sigma_0(y) = \frac{2}{3\sqrt{3}} y^{3/2}, \quad y > 0. \tag{2.35}$$

Continuing expansion (2.23), (2.24) in region **VI** gives

$$\sigma_1(y) = \tfrac{1}{2}, \quad y > 0, \tag{2.36}$$

and

$$\sigma_2(y) = -M(y), \quad y > 0, \tag{2.37}$$

with $M : (0, \infty) \to \mathbb{R}$ undetermined, but having

$$M(y) \sim \frac{3}{4} \ln y - \ln\left(\frac{u_0 3^{1/4}}{2\sqrt{\pi}}\right) \quad \text{as } y \to \infty. \tag{2.38}$$

Thus we have in region **VI**,

$$u(y, \tau) = \exp\left(-\frac{2}{3\sqrt{3}} y^{3/2}\tau - \frac{1}{2} \ln \tau - M(y) + o(1)\right) \tag{2.39}$$

as $\tau \to \infty$ with $y = O(1)(\in (0, \infty))$. We must now introduce a localized region **O** in which, following (2.33) and (2.34), $y = O(\tau^{-2/3})$ as $\tau \to \infty$. Thus we write

$$y = \frac{\xi}{\tau^{2/3}} \tag{2.40}$$

in region **O**, with $\xi = O(1)$ as $\tau \to \infty$. It follows from (2.40) and expansion (2.39) in region **VI**, that we should expand as

$$u(\xi, \tau) = \phi(\tau) G(\xi) + o(\phi(\tau)) \quad \text{as } \tau \to \infty \tag{2.41}$$

with $\xi = O(1)$, whilst the gauge function

$$\phi(\tau) = o(1) \quad \text{as } \tau \to \infty \tag{2.42}$$

is to be determined. On rewriting equation (1.1) in terms of ξ and substituting from (2.41), we obtain

$$\phi'(\tau) G - \frac{1}{3} \frac{\phi(\tau)}{\tau} \xi G_\xi + \frac{\phi^2(\tau)}{\tau^{1/3}} G G_\xi + \frac{\phi(\tau)}{\tau} G_{\xi\xi\xi} = 0, \tag{2.43}$$

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with $\xi = O(1)$ as $\tau \to \infty$. A non-trivial balance requires

$$\frac{\phi^2(\tau)}{\tau^{1/3}} \sim \frac{\phi(\tau)}{\tau} \quad \text{as } \tau \to \infty, \tag{2.44}$$

and so, without loss of generality, we take

$$\phi(\tau) = \tau^{-2/3}. \tag{2.45}$$

We observe that all terms in (2.43) are retained at leading order as $\tau \to \infty$ and (2.43) becomes

$$G_{\xi\xi\xi} + GG_\xi - \frac{\xi}{3}G_\xi - \frac{2}{3}G = 0, \quad -\infty < \xi < \infty. \tag{2.46}$$

We remark that equation (2.46) admits the solution $G(\xi) = \xi$. Now, matching expansion (2.39) ($y \to 0^+$) with expansion (2.41) ($\xi \to \infty$) requires first that

$$M(y) \sim -\tfrac{1}{4}\ln y - \ln D \quad \text{as } y \to 0^+, \tag{2.47}$$

with $D > 0$ as yet undetermined, after which we require that

$$G(\xi) \sim D\xi^{1/4}\exp\left(-\frac{2}{3\sqrt{3}}\xi^{3/2}\right) \quad \text{as } \xi \to \infty. \tag{2.48}$$

Finally, for u to remain bounded in τ as $\tau \to \infty$ when $(-\xi) \gg 1$, we require, via (2.41) and (2.45), that

$$\xi^{-1}G(\xi) \text{ is bounded} \quad \text{as } \xi \to -\infty. \tag{2.49}$$

The leading order problem is now complete and is given by (2.46), (2.48) and (2.49). The boundary-value problem (2.46)–(2.49) is both nonlinear and non-autonomous. However, apart from $D > 0$, it is parameter-free. A numerical study of initial-value problem (2.46) and (2.48) using a shooting method reveals that there exits a value $D = D^* > 0$ such that boundary condition (2.49) is satisfied for each $D \in (0, D^*]$, whilst for each $D \in (D^*, \infty)$ the solution blows up at finite-ξ, say $\xi = \xi_0(D)$, with

$$G(\xi) \sim -k(\xi - \xi_0(D))^{-2} \quad \text{as } \xi \to \xi_0(D)^+, \tag{2.50}$$

and $k > 0$ being a constant. Hence, boundary-value problem (2.46), (2.48) and (2.49) has a unique solution for each $D \in (0, D^*]$ but no solution for $D \in (D^*, \infty)$. In particular, we have:

(i) $D = D^*(> 0)$: A unique solution exists, say $G = G^*(\xi)$, $-\infty < \xi < \infty$, with

$$G^*(\xi) = \xi + \delta(-\xi)^{-1/2} + O((-\xi)^{-1/2}\mathrm{e}^{-(\frac{2}{3})^{3/2}(-\xi)^{3/2}}) \tag{2.51}$$

as $\xi \to -\infty$, with $\delta = 1.22\ldots$.

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(ii) $0 < D < D^*$: A unique solution exists for each $D \in (0, D^*)$, say $G = G^o(\xi)$, $-\infty < \xi < \infty$. Moreover, $G^o(\xi)$ is oscillatory in $\xi < 0$ with

$$G^o(\xi) \sim A_1(-\xi)^{1/4}\cos\left(\frac{2}{3\sqrt{3}}(-\xi)^{3/2} + A_2\right) \tag{2.52}$$

as $\xi \to -\infty$, where A_1 (> 0) and A_2 are constants fixed in terms of D.

Further, on making the substitution

$$G(\xi) = 2\,3^{1/3}\,S(\eta), \qquad \xi = 3^{1/3}\,\eta, \tag{2.53}$$

equation (2.46) becomes

$$S_{\eta\eta\eta} + 6SS_\eta - \eta S_\eta - 2S = 0, \quad -\infty < \eta < \infty, \tag{2.54}$$

after which, on writing

$$S(\eta) = W_\eta(\eta) - W^2(\eta), \tag{2.55}$$

equation (2.54) becomes the second Painlevé equation (P_{II})

$$W_{\eta\eta} = \eta W + 2W^3 + \varphi, \tag{2.56}$$

where φ is a constant of integration. In terms of W, the boundary conditions associated with equation (2.56) become, via (2.48), (2.49), (2.53) and (2.55),

$$W(\eta) \sim -\frac{D}{2.3^{1/4}}\eta^{-1/4}\exp\left(-\frac{2}{3}\eta^{2/3}\right) \quad \text{as } \eta \to \infty, \tag{2.57}$$

$$(-\eta)^{1/2}W(\eta) \text{ is bounded} \quad \text{as } \eta \to -\infty. \tag{2.58}$$

It follows from (2.57) and (2.58) that we must take $\varphi = 0$ in equation (2.56). Moreover, it has been established in [**5**] that equation (2.56) (with $\varphi = 0$) has a solution $W = W^*(\eta)$, $-\infty < \eta < \infty$, for which

$$W^*(\eta) \sim -\frac{1}{2\sqrt{\pi}}\eta^{-1/4}\exp\left(-\frac{2}{3}\eta^{3/2}\right) \quad \text{as } \eta \to \infty, \tag{2.59a}$$

$$W^*(\eta) \sim -\left(-\frac{\eta}{2}\right)^{1/2} \quad \text{as } \eta \to -\infty. \tag{2.59b}$$

A comparison with (2.57) and (2.59*a*) above then establishes that D^* and δ (in (2.51)) are given by

$$D^* = \frac{3^{1/4}}{\sqrt{\pi}}, \quad \delta = \sqrt{\frac{3}{2}}.$$

A graph of $G(\xi)$ against ξ illustrating the solutions to initial-value problem (2.46) and (2.48) for representative values of the parameter D in the ranges $D^* < D < \infty$, $0 < D < D^*$ and $D = D^*$ is shown in Figure 2. Thus we have two distinct cases to consider, namely when $D = D^*$ or when $D \in (0, D^*)$.

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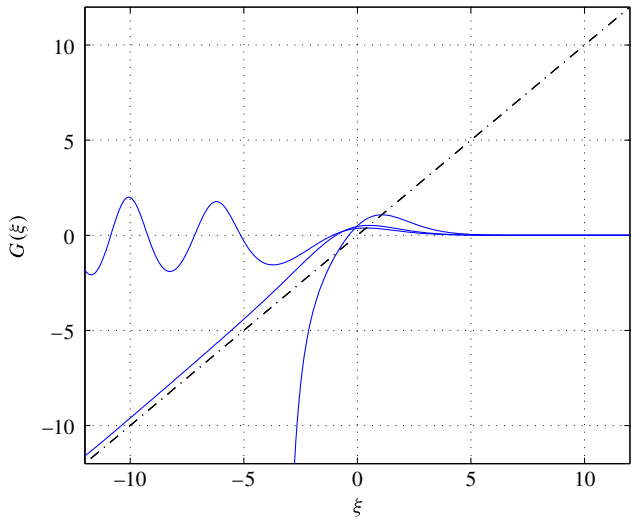


Figure 2: A graph of $G(\xi)$ against ξ illustrating solutions to (2.46) for representative values of D. We recall that the solution is oscillatory in $\xi < 0$ for $D \in (0, 3^{1/4}/\sqrt{\pi})$, while the solution blows up at finite ξ when $D \in (3^{1/4}/\sqrt{\pi}, \infty)$. We note that the dashed line represents the solution $G(\xi) = \xi$.

As $\xi \to -\infty$ we move out of region **O** into $y < 0$ and we have via (2.41), (2.51) and (2.52) that

$$u(y,\tau) \sim \begin{cases} y + o(1), & D = D^*, \\ \dfrac{A_1(-y)^{1/4}}{\tau^{1/2}} \cos\left(\dfrac{2}{3\sqrt{3}}\,\tau\,(-y)^{3/2} + A_2\right) + o(\tau^{-1/2}), & 0 < D < D^*, \end{cases} \tag{2.60}$$

as $\tau \to \infty$ with $(-y) \ll 1$. However, as will be seen later, consideration of the structure of $u(y,\tau)$ in $y < 0$ as $\tau \to \infty$ gives

$$u(y,\tau) = u_0 + o(1) \tag{2.61}$$

as $\tau \to \infty$ with $(-y) \ll 1$. It is clear that asymptotic matching between (2.60) and (2.61) cannot be achieved for either $D = D^*$ or $D \in (0, D^*)$. Hence we must rule out the case $\alpha = 0$. It remains to consider the case $\alpha > 0$.

Case $\alpha > 0$. In this case we have, in (2.23) and (2.24),

$$\sigma_0(y,\alpha) = \begin{cases} \dfrac{2}{3\sqrt{3}} y^{3/2}, & y > 3\alpha^2, \\ \alpha(y - \alpha^2), & \alpha^2 < y < 3\alpha^2. \end{cases} \tag{2.62}$$

We recall that with (2.62) a non-uniformity occurs in expansion (2.23) (with (2.24)) when $y = \alpha^2 + O(\tau^{-1})$. We must introduce a further region,

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which we denote as region **S**. In this region we write

$$y = \frac{s(\tau) + z}{\tau} \tag{2.63}$$

with $z = O(1)$ as $\tau \to \infty$, and

$$s(\tau) = \alpha^2\tau + \hat{\phi}(\tau) + \hat{\phi}_0 + \hat{\psi}(\tau) + o(\hat{\psi}(\tau)) \tag{2.64}$$

as $\tau \to \infty$, with $\hat{\phi}(\tau)$ unbounded as $\tau \to \infty$, but $\tau^{-1}\hat{\phi}(\tau) = o(1)$ as $\tau \to \infty$, whilst $\hat{\phi}_0$ is a constant, and $\hat{\psi}(\tau) = o(1)$ as $\tau \to \infty$. In region **S** expansions (2.23) and (2.24) demonstrate that $u = O(1)$ as $\tau \to \infty$, and so we expand in the form

$$u(z, \tau) = U(z) + o(1), \tag{2.65}$$

as $\tau \to \infty$, with $z = O(1)$. On substituting into equation (1.1) (when written in terms of z and τ) we obtain the leading order problem in region **S** as

$$U_{zzz} + UU_z - \alpha^2 U_z = 0, \quad -\infty < z < \infty, \tag{2.66}$$

$$U(z) \to 0^+ \quad \text{as } z \to \infty, \tag{2.67}$$

$$U(z) \text{ bounded} \quad \text{as } z \to -\infty, \tag{2.68}$$

with condition (2.67) arising from matching expansion (2.65) (as $z \to \infty$), with expansion (2.23), (2.24) (as $y \to (\alpha^2)^+$), and it is important to note that the limit from above must be imposed. We can integrate (2.66) once, and use the condition as $z \to \infty$, to obtain the second-order ordinary differential equation

$$U_{zz} - \alpha^2 U + \tfrac{1}{2}U^2 = 0, \quad -\infty < z < \infty, \tag{2.69}$$

in place of (2.66). A sketch of the (U, U_z) phase plane of the second-order autonomous nonlinear ordinary differential equation (2.69) is shown in Figure 3. The only phase path which satisfies the boundary conditions (2.67) and (2.68) is the homoclinic orbit which lies in the half-plane $U \geqslant 0$. Hence, the required solution to boundary-value problem (2.69), (2.67) and (2.68) is readily obtained as

$$U(z) = 3\alpha^2 \operatorname{sech}^2\left(\frac{\alpha}{2}(z - z_0)\right), \quad -\infty < z < \infty, \tag{2.70}$$

where, without loss of generality (simply by relabelling $\hat{\phi}_0$ in (2.64)), we may take $z_0 = 0$. This represents a soliton (or solitary wave solution), corresponding to the homoclinic orbit in the phase plane. We observe that the soliton has wave speed α^2 and amplitude $3\alpha^2$. However, α remains undetermined, a point to which we will return later after we have completed the large-τ structure of the solution to **IVP** for $y \geqslant \alpha^2 + O(1)$. We further remark that expansion (2.65) (as $z \to \infty$) matches at leading order with expansion (2.23), (2.24) (as $y \to (\alpha^2)^+$), as required.

Now, as indicated earlier, region **VI** must be replaced by three regions: region **VI(a)** ($3\alpha^2 + O(1) \leqslant y < \infty$), region **TRA** (transition region, $y = 3\alpha^2 \pm o(1)$) and region **VI(b)** ($\alpha^2 + O(1) \leqslant y \leqslant 3\alpha^2 - O(1)$). We consider each of these

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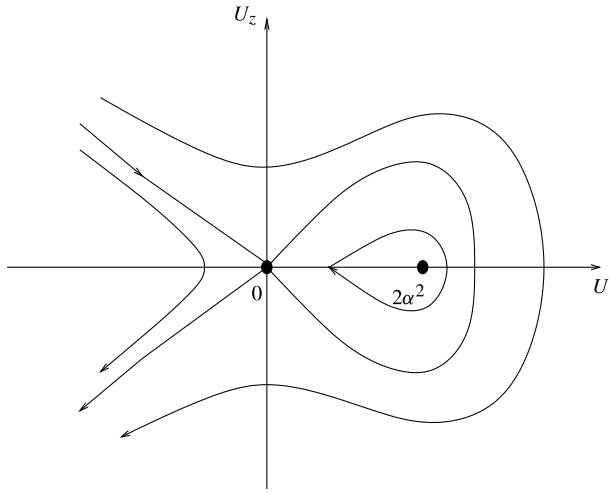


Figure 3: A sketch of the (U, U_z) phase plane for (2.69).

regions in turn. We begin with region **VI(a)**, where $(3\alpha^2 + O(1) \leqslant y < \infty)$. Substitution of (2.23), (2.24) into equation (1.1) (when written in terms of y and τ) gives, on solving at each order in turn and matching to expansion (2.21) $[x = O(\tau)]$ as $y \to \infty$, that

$$u(y, \tau) = \exp\left(-\frac{2}{3\sqrt{3}}\, y^{3/2}\, \tau - \frac{1}{2}\ln\tau - K(y) + o(1)\right) \tag{2.71}$$

as $\tau \to \infty$ with $3\alpha^2 + O(1) \leqslant y < \infty$, where

$$K(y) \sim \frac{3}{4}\ln y - \ln\left(\frac{u_0 3^{1/4}}{2\sqrt{\pi}}\right) \tag{2.72}$$

as $y \to \infty$. The function $K(y)$ remains undetermined. We next consider region **VI(b)**. The expansion in region **VI(b)** is given by

$$u(y, \tau) = \exp(-\alpha(y - \alpha^2)\tau - \beta_1 \ln\tau - \beta_1 \ln(3\alpha^2 - y) - \beta_2 + o(1)) \tag{2.73}$$

as $\tau \to \infty$, with $\alpha^2 + O(1) \leqslant y \leqslant 3\alpha^2 - O(1)$, where β_1 and β_2 are constants to be determined on matching with region **S** as $y \to (\alpha^2)^+$. Matching expansion (2.73) (as $y \to \alpha^2$) with expansion (2.65) (with (2.70)) (as $z \to \infty$) requires that

$$\beta_1 = 0 \quad \text{and} \quad \beta_2 = -\ln(12\alpha^2). \tag{2.74}$$

We now examine region **TRA**. An examination of expansion (2.71) (as $y \to (3\alpha^2)^+$) and expansion (2.73) (as $y \to (3\alpha^2)^-$) reveals that in this region $y = 3\alpha^2 + O(\tau^{-1/2})$ as $\tau \to \infty$. To examine region **TRA** we introduce the scaled coordinate

$$\eta = (y - 3\alpha^2)\tau^{1/2}, \tag{2.75}$$

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where $\eta = O(1)$ as $\tau \to \infty$, and expand as

$$u(\eta, \tau) = [F(\eta) + o(1)] \exp(-2\alpha^3\tau - \alpha\eta\tau^{1/2}), \tag{2.76}$$

as $\tau \to \infty$ with $F(\eta) > 0$ and $\eta = O(1)$. On substitution of expansion (2.76) into equation (1.1) (when written in terms of η and τ), we obtain at leading order

$$F_{\eta\eta} + \frac{\eta}{6\alpha} F_\eta = 0, \quad -\infty < \eta < \infty. \tag{2.77}$$

On writing

$$\eta = (3\alpha)^{1/2}\xi, \tag{2.78}$$

equation (2.77) becomes

$$F_{\xi\xi} + \frac{\xi}{2} F_\xi = 0, \quad -\infty < \xi < \infty. \tag{2.79}$$

The general solution of (2.79) is readily obtained as

$$F(\xi) = A_0 \operatorname{erfc}\left(\frac{\xi}{2}\right) + B_0, \quad -\infty < \xi < \infty, \tag{2.80}$$

where erfc[.] is the complementary error function and A_0 and B_0 are constants to be determined via matching. In region **TRA** we have, via (2.76) and (2.78), that

$$u(\eta, \tau) = \left[A_0 \operatorname{erfc}\left(\frac{\eta}{2\sqrt{3\alpha}}\right) + B_0 + o(1)\right] \exp(-2\alpha^3\tau - \alpha\eta\tau^{1/2}) \tag{2.81}$$

as $\tau \to \infty$ with $-\infty < \eta < \infty$. Matching expansion (2.81) (as $\eta \to -\infty$) with expansion (2.73) (as $y \to (3\alpha^2)^-$) requires that

$$2A_0 + B_0 = 12\alpha^2. \tag{2.82}$$

Matching expansion (2.81) (as $\eta \to \infty$) with expansion (2.71) (as $y \to (3\alpha^2)^+$) requires that

$$K(y) \sim \ln(y - 3\alpha^2) - \ln\left(12\alpha^2\sqrt{\frac{3\alpha}{\pi}}\right) \quad \text{as } y \to (3\alpha^2)^+, \tag{2.83}$$

and

$$A_0 = 6\alpha^2, \qquad B_0 = 0. \tag{2.84}$$

As $\eta \to -\infty$ we move out of region **TRA** into region **VI(b)**, where $\alpha^2 + O(1) \leqslant y \leqslant 3\alpha^2 - O(1)$. The form of expansion (2.81) for $(-\eta) \gg 1$ indicates that we must develop the expansion in region **VI(b)** in the form

$$u(y, \tau) = 12\alpha^2 \exp(-\alpha[y - \alpha^2]\tau) + H(y)\tau^{-1/2} \exp\left(-\frac{2}{3\sqrt{3}} y^{3/2}\tau\right) \tag{2.85}$$

$$+ O\left(\tau^{-1/2} \exp\left(-\frac{2}{3\sqrt{3}} y^{3/2}\tau\right)\right), \tag{2.86}$$

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as $\tau \to \infty$ with $\alpha^2 + O(1) \leqslant y \leqslant 3\alpha^2 - O(1)$. On substitution of (2.85) into equation (1.1) (when written in terms of y and τ), we find $H(y)$ is indeterminate at this order. However, matching to region **TRA** (as $y \to (3\alpha^2)^-$) requires that

$$H(y) \sim \frac{12\alpha^2\sqrt{3\alpha}}{\sqrt{\pi}(y - 3\alpha^2)} \quad \text{as } y \to (3\alpha^2)^-. \tag{2.87}$$

This then completes the leading order structure of the solution of **IVP** as $\tau \to \infty$ for $\alpha^2 + O(1) \leqslant y < \infty$ as $\tau \to \infty$, via regions **VI(a)**, **TRA** and **VI(b)**, together with region **S** where $y = \alpha^2 \pm O(\tau^{-1})$ as $\tau \to \infty$. We recall that in region **S**, where $y = \alpha^2 + O(1/\tau)$ as $\tau \to \infty$, the solution $u(y, \tau)$ given by (2.65) and (2.70) represents, at leading order as $\tau \to \infty$, a solitory wave with speed α^2 and amplitude $3\alpha^2$. At this stage we depart from considering the asymptotic structure as $\tau \to \infty$ in $y > 0$ (to which we will return later), and turn our attention to the asymptotic structure as $\tau \to \infty$ when $y < 0$.

We introduce region **VII**, where $y = O(1)$ (< 0) as $\tau \to \infty$. It follows from the form of expansion (2.22) that we must expand in the form

$$u(y, \tau) = u_0 - (\mathrm{e}^{g^+(y,\tau)} + \mathrm{e}^{g^+(y,\tau)}) \tag{2.88}$$

as $\tau \to \infty$, with

$$g^{\pm}(y, \tau) = \pm \mathrm{i} g_0(y)\tau + g_1(y) \ln \tau \pm \mathrm{i} g_2(y) + g_3(y) + o(1) \tag{2.89}$$

as $\tau \to \infty$ with $y = O(1)$. On substitution from (2.88), (2.89) into equation (1.1) (when written in terms of y and τ), we obtain the leading order problem as

$$(g_0')^3 + (y - u_0)g_0' - g_0 = 0, \quad y < 0, \tag{2.90}$$

$$g_0(y) > 0, \quad y < 0, \tag{2.91}$$

$$g_0(y) = \frac{2}{3\sqrt{3}}(-y)^{3/2} + \frac{u_0}{\sqrt{3}}(-y)^{1/2} + o((-y)^{1/2}) \quad \text{as } y \to -\infty. \tag{2.92}$$

The final condition (2.92) arises from matching expansion (2.88) $((-y) \gg 1)$ with expansion (2.22) $((-x) = O(\tau))$. Equation (2.90) has a one-parameter family of linear solutions,

$$g_0(y) = A(y + A^2 - u_0), \quad y < 0, \tag{2.93}$$

for any $A \in \mathbb{R}$, together with the associated pair of envelope solutions

$$g_0(y) = \pm\frac{2}{3\sqrt{3}}(u_0 - y)^{3/2}, \quad y < 0. \tag{2.94}$$

Combinations of (2.93) and (2.94) which remain continuous and differentiable also provide solutions to (2.90) (envelope touching solutions). Applying condition (2.92) requires us to select the solution

$$g_0(y) = \frac{2}{3\sqrt{3}}(u_0 - y)^{3/2}, \quad y < 0, \tag{2.95}$$

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or

$$g_0(y, A) = \begin{cases} \dfrac{2}{3\sqrt{3}}(u_0 - y)^{3/2}, & y < u_0 - 3A^2, \\ A(y + A^2 - u_0), & u_0 - 3A^2 \leqslant y < 0, \end{cases} \tag{2.96}$$

for any $A < -\frac{1}{\sqrt{3}}u_0^{1/2}$. We observe at this point that if (2.96) is selected, then a weak non-uniformity will be present in expansion (2.88) and (2.89) due to the jump in curvature in $g_0(y, A)$ at $y = u_0 - 3A^2$, the envelope touching point. This will necessitate the inclusion of a transition region when $y = (u_0 - 3A^2) \pm o(1)$ as $\tau \to \infty$.

We must now connect the oscillatory form of expansion (2.88), (2.89) in region **VII** with the exponential tail leaving region **S** via expansion (2.65) and (2.70), when $y = \alpha^2 - O(1/\tau)$. The important step here is to recognize that this can be achieved by the dispersive shock wave structure given in (1.5). To consider this in detail we first return to region **S**, and write expansion (2.65), with (2.70), for $(-z) \gg 1$, which gives, when written in terms of y and τ,

$$u(y, \tau) \sim 12\alpha^2 \mathrm{e}^{\alpha(y-\alpha^2)\tau} \tag{2.97}$$

for $\tau^{-1} \ll (\alpha^2 - y) \ll 1$. Comparing (2.97) with (1.8), we see that asymptotic matching will be possible with (1.5) provided that we select

$$\alpha = \sqrt{\frac{2u_0}{3}}. \tag{2.98}$$

Thus, with α given by (2.98), we introduce region **DSW**, in which $-u_0 + O(1) \leqslant y \leqslant \frac{2}{3}u_0 - O(1)$ as $\tau \to \infty$, with expansion

$$u(y, \tau) = u_s(y, \tau) + o(1) \tag{2.99}$$

as $\tau \to \infty$, and $u_s(y, \tau)$ given by (1.5). Matching, at leading order, between expansion (2.99) (as $y \to \frac{2}{3}u_0^-$) in region **DSW** and expansion (2.65), (2.70) (as $z \to -\infty$) in region **S** has been achieved via the selection of α given in (2.98). The asymptotic structure as $\tau \to \infty$ can be completed, provided we can connect expansion (2.99) in region **DSW** with expansion (2.88), (2.89) in region **VII**. Now, for $0 < y + u_0 \ll 1$, the form of expansion (2.99) is, via (1.9),

$$u(y, \tau) \sim u_0 - (\mathrm{e}^{g^+(y,\tau)} + \mathrm{e}^{g^+(y,\tau)}) \tag{2.100}$$

with

$$g^{\pm}(y, \tau) \sim \pm \mathrm{i}\sqrt{\frac{2u_0}{3}}\left(y - \frac{u_0}{3}\right)\tau + \frac{1}{2}\ln(y + u_0) + \ln\left(\frac{1}{2}\lambda(u_0)\right). \tag{2.101}$$

The only possibility for connecting region **DSW** with region **VII** follows by comparing (2.100), (2.101) with (2.88), (2.89), (2.95), (2.96), which demands that in region **VII** we select $g_0(y, A)$ as given in (2.96), with the choice

$$A = -\sqrt{\frac{2u_0}{3}}, \tag{2.102}$$

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and introduce a transition region **C** when $y = -u_0 \pm o(1)$ as $\tau \to \infty$, to connect region **VII** with region **DSW**. Thus, region **VII** is now restricted to $-\infty < y \leqslant -u_0 - O(1)$, with expansion (2.88) and

$$g^{\pm}(y,\tau) \sim \pm \mathrm{i}\sqrt{\frac{2u_0}{3}}\left(y - \frac{u_0}{3}\right)\tau + O(\ln\tau) \tag{2.103}$$

as $\tau \to \infty$. We first develop the expansion in region **VII** to higher order. Substitution of (2.88), (2.89) into equation (1.1) (when written in terms of y and τ) gives, on solving at each order in turn and matching to expansion (2.22) $[(-x) = O(\tau)]$ as $y \to -\infty$, that in region **VII**,

$$\begin{aligned} u(y,\tau) = u_0 &- \exp\left(\mathrm{i}\frac{2}{3\sqrt{3}}(u_0 - y)^{3/2}\tau - \frac{1}{2}\ln\tau + [\mathrm{i}\chi(y) + \Theta(y)]\right) \\ &- \exp\left(-\mathrm{i}\frac{2}{3\sqrt{3}}(u_0 - y)^{3/2}\tau - \frac{1}{2}\ln\tau + [-\mathrm{i}\chi(y) + \Theta(y)]\right) \end{aligned} \tag{2.104}$$

as $\tau \to \infty$ with $-\infty < y < -u_0 - o(1)$. We note that the functions $\chi(y)$ and $\Theta(y)$ remain undetermined, but matching with the far field as $y \to -\infty$ requires that

$$\Theta(y) \sim -\frac{3}{4}\ln(-y) + \ln\left(\frac{u_0 3^{1/4}}{2\sqrt{\pi}}\right), \qquad \chi(y) \sim \frac{\pi}{4} \quad \text{as } y \to -\infty.$$

Further, we note that in this region $u(y,\tau) = u_0 + O(\tau^{-1/2})$ as $\tau \to \infty$, with the oscillatory envelope being of $O(\tau^{-1/2})$ as $\tau \to \infty$. As $y \to -u_0^-$ $(y \to -u_0^+)$ we move into region **C** from region **VII** (**DSW**). An examination of expansions (2.104) and (2.99) (with (1.9)) reveals that in region **C** we require $y = -u_0 \pm O(\tau^{-1/2})$ as $\tau \to \infty$. To examine region **C** we introduce the scaled coordinate $\xi = (y + u_0)\tau^{1/2}$, with $\xi = O(1)$ as $\tau \to \infty$, and expand as

$$u(\xi,\tau) = u_0 - (\widehat{F}(\xi) + o(1))\tau^{-1/2}(\mathrm{e}^{\mathrm{i}G(\xi,\tau)} + \mathrm{e}^{-\mathrm{i}G(\xi,\tau)}) \tag{2.105}$$

with

$$G(\xi,\tau) = 2\widehat{A}^3\tau - \widehat{A}\xi\tau^{1/2} + \widehat{G}(\xi) + o(1) \tag{2.106}$$

as $\tau \to \infty$ with $\xi = O(1)$. Here

$$\widehat{A} = \sqrt{\frac{2u_0}{3}}. \tag{2.107}$$

On substituting from (2.105) and (2.106) into equation (1.1) (when written in terms of ξ and τ) we find that $\widehat{F}(\xi)$ and $\widehat{G}(\xi)$ are indeterminate at this order. However, the matching conditions between region **C** and region **DSW** require

$$\widehat{F}(\xi) \sim \frac{2}{3}\xi, \quad \widehat{G}(\xi) \sim \frac{2}{27\widehat{A}}\xi^2 \quad \text{as } \xi \to \infty,$$

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whilst the matching conditions between region **C** and region **VII** require

$$\widehat{F}(\xi) \to C, \quad \widehat{G}(\xi) \sim \frac{1}{12\widehat{A}}\xi^2 \quad \text{as } \xi \to -\infty,$$

where C is a positive constant, after which we require,

$$\theta(y) \to \ln C \quad \text{as } y \to u_0^-.$$

The leading order asymptotic structure as $\tau \to \infty$ is now complete. However, it is of interest to consider higher-order effects in regions **S** and **VI(b)**, before summarizing the full structure as $\tau \to \infty$. We recall that as $y \to (2u_0/3)^+$, we move out of region **VI(b)** into region **S**. We examine the form of expansion (2.85) in region **VI(b)**, when written in terms of z, as we move into region **S**. On substituting from (2.63) and (2.64) into (2.85), and expanding, with $z = O(1)$ as $\tau \to \infty$, we find that

$$\begin{aligned} u(z,\tau) \sim 8u_0 \exp\left\{-\sqrt{\frac{2u_0}{3}}\hat{\phi}(\tau) - \sqrt{\frac{2u_0}{3}}[z+\phi_0] - \sqrt{\frac{2u_0}{3}}\hat{\psi}(\tau)\right\} \\ + H\tau^{-1/2}\exp\left\{-\frac{2^{5/2}u_0^{3/2}}{3^3}\tau - \frac{\sqrt{2u_0}}{3}\hat{\phi}(\tau) - \frac{\sqrt{2u_0}}{3}[z+\phi_0]\right. \\ \left. - \frac{\sqrt{2u_0}}{3}\hat{\psi}(\tau) + \cdots\right\} + \cdots \end{aligned} \tag{2.108}$$

as $\tau \to \infty$ with $z \gg 1$. We now make the assumption (which we will verify as consistent) that

$$H(y) \sim H_c\left(y - \frac{2u_0}{3}\right)^{\gamma} \quad \text{as } y \to \left(\frac{2u_0}{3}\right)^+, \tag{2.109}$$

for some non-zero constants H_c and γ. With (2.109) we can rewrite (2.108) as

$$\begin{aligned} u(z,\tau) \sim (8u_0 - 8u_0\hat{\psi}(\tau) + \cdots)\exp\left\{-\sqrt{\frac{2u_0}{3}}\hat{\phi}(\tau) - \sqrt{\frac{2u_0}{3}}[z+\phi_0]\right\} \\ + [H_c\tau^{-1/2-\gamma}(\hat{\phi}(\tau) + [z+\phi_0])^{\gamma} + \cdots]\exp\left\{-\frac{2^{5/2}u_0^{3/2}}{3^3}\tau\right. \\ \left. - \frac{\sqrt{2u_0}}{3}\hat{\phi}(\tau) - \frac{\sqrt{2u_0}}{3}[z+\phi_0] + \cdots\right\} + \cdots \end{aligned} \tag{2.110}$$

as $\tau \to \infty$ with $z \gg 1$. Now matching expansion (2.65) (as $z \to \infty$) up to $O(1)$ with expansion (2.110) requires $\hat{\phi}(\tau) \equiv 0$ and $\phi_0 = 0$. On rewriting (2.110), we then have

$$\begin{aligned} u(z,\tau) \sim \left(8u_0\exp\left\{-\sqrt{\frac{2u_0}{3}}z\right\} + \cdots\right)(1 + O(\hat{\psi}(\tau))) \\ + \left(H_c\tau^{-1/2-\gamma}z^{\gamma}\exp\left\{-\frac{2^{5/2}u_0^{3/2}}{3^3}\tau - \frac{\sqrt{2u_0}}{3}z\right\} + \cdots\right) + \cdots \end{aligned} \tag{2.111}$$

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as $\tau \to \infty$ with $z \gg 1$. We conclude from (2.111) that in region **S** the correction term in expansion (2.65) must be such that

$$u(z, \tau) = U(z) + O(\hat{\psi}(\tau)) \tag{2.112}$$

as $\tau \to \infty$. On substitution of (2.112) into equation (1.1) (when written in terms of z and τ) we require, to obtain a non-trivial balance at $O(\hat{\psi}'(\tau))$, that $\hat{\psi}'(\tau) = O(\hat{\psi}(\tau))$ as $\tau \to \infty$ and we conclude that $\hat{\psi}(\tau)$ must be exponentially small in τ, as $\tau \to \infty$, and hence we set

$$\hat{\psi}(\tau) = v_1 \tau^{\bar{\gamma}} \mathrm{e}^{-\lambda\tau}[1 + o(1)] \tag{2.113}$$

as $\tau \to \infty$, where v_1, $\bar{\gamma}$ and $\lambda(> 0)$ are non-zero constants to be determined. We now continue the expansion in region **S** as

$$u(z, \tau) = U(z) + U_c(z)\hat{\psi}'(\tau) + o(\hat{\psi}'(\tau)) \tag{2.114}$$

as $\tau \to \infty$ with $z = O(1)$. On solving at $O(\hat{\psi}'(\tau))$ for $U_c(z)$, we find that

$$U_c(z) \sim E_1 \mathrm{e}^{(2\sqrt{2u_0}/3)z} + [E_2 + E_3 z]\mathrm{e}^{-(\sqrt{2u_0}/3)z} \tag{2.115}$$

as $z \to \infty$, where E_1, E_2 and E_3 are constants to be determined. Matching expansion (2.114) (with (2.115)) (as $z \to \infty$) with expansion (2.111), up to exponentially small terms of $O(\tau^{\bar{\gamma}}\mathrm{e}^{-\lambda\tau})$ as $\tau \to \infty$, requires that

$$\lambda = \frac{2^{5/2}u_0^{3/2}}{3^3}, \quad \gamma = 1, \quad \bar{\gamma} = -\frac{3}{2}, \quad E_1 = E_2 = 0, \quad E_3 = -\frac{H_c}{v_1}\frac{3^3}{2^{5/2}u_0^{3/2}},$$

after which matching is complete. Note that the non-zero constants H_c and v_1 remain undetermined at this order in the asymptotic development. Finally, we have in region **S**,

$$u(z, \tau) = 2u_0 \operatorname{sech}^2\left(\sqrt{\frac{u_0}{6}}z\right) + O\left(\tau^{-3/2}\exp\left[-\frac{2^{5/2}u_0^{3/2}}{3^3}\tau\right]\right) \tag{2.116}$$

as $\tau \to \infty$ with $z = O(1)$, and

$$s(\tau) = \frac{2u_0}{3}\tau + v_1\tau^{-3/2}\exp\left[-\frac{2^{5/2}u_0^{3/2}}{3^3}\tau\right] + o\left(\tau^{-3/2}\exp\left(-\frac{2^{5/2}u_0^{1/2}}{3^3}\tau\right)\right) \tag{2.117}$$

as $\tau \to \infty$. We conclude from (2.116) and (2.117) that the leading soliton is contracted onto in form and speed exponentially in τ as $\tau \to \infty$.

§3. *Summary.* In this paper we have obtained, via the method of matched asymptotic coordinate expansions, the spatially uniform asymptotic structure of the solution to **IVP** as $\tau \to \infty$. The structure is composed of seven spatial

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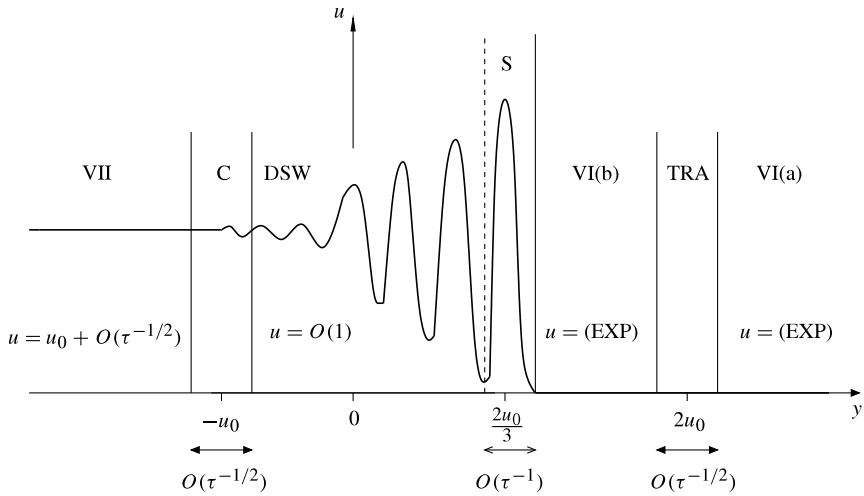


Figure 4: Schematic representation of the location and thickness of the asymptotic regions as $\tau \to \infty$, overlaid with the leading order solution to **IVP** as $\tau \to \infty$. The dispersive shock wave can clearly be seen in region **DSW** with the lead soliton being captured in region **S**. Further, we note that the solution in region **VII** is oscillatory. Here we recall that $y = x/\tau$ and note that (EXP) denotes terms exponentially small in τ as $\tau \to \infty$.

regions (sketched in Figure 4). The main regions are summarized as follows (in terms of $y = x/\tau$ as $\tau \to \infty$).

*Region **VII**:* $-\infty < y \leqslant -u_0 - O(1)$ *as* $\tau \to \infty$. This region accommodates radiation from the initial disturbance. Here

$$u(y,\tau) = u_0 - H(y)\tau^{-1/2}(e^{i\bar{\theta}(y,\tau)} + e^{-i\bar{\theta}(y,\tau)})$$

with

$$\bar{\theta}(y,\tau) = \frac{2}{3\sqrt{3}}(u_0 - y)^{3/2}\tau + \chi(y) + o(1)$$

as $\tau \to \infty$, and

$$\left.\begin{aligned} H(y) &\sim \frac{3^{1/4}u_0}{2\sqrt{\pi}}(-y)^{-3/4} \\ \chi(y) &\sim \frac{\pi}{4} \end{aligned}\right\} \quad \text{as } y \to -\infty,$$
$$H(y) \sim C \quad \text{as } y \to u_0^-.$$

Note that the amplitude of the radiation field is of $O(\tau^{-1/2})$ as $\tau \to \infty$, and decays like $(-y)^{-3/4}$ for $(-y) \gg 1$.

*Region **DSW**:* $-u_0 + O(1) \leqslant y \leqslant 2u_0/3 - O(1)$ *as* $\tau \to \infty$. This region encompasses the dispersive shock wave structure. Here

$$u(y,\tau) = u_s(y,\tau) + o(1)$$

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as $\tau \to \infty$, with $u_s(y,\tau)$ as given in (1.5), with $u_s(y,\tau) = O(1)$ as $\tau \to \infty$, and represents a train of modulated oscillations between the leading soliton and the radiation field.

Region ***S****:* $y = s(\tau)/\tau \pm O(\tau^{-1})$ *as* $\tau \to \infty$. This region captures the leading soliton structure. Here

$$s(\tau) = \frac{2u_0}{3}\tau + O\left(\tau^{-3/2}\exp\left(-\frac{2^{5/2}u_0^{3/2}}{3^3}\tau\right)\right)$$

as $\tau \to \infty$, with

$$u(z,\tau) = 2u_0\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{2u_0}{3}}z\right) + O\left(\tau^{-3/2}\exp\left(-\frac{2^{5/2}u_0^{3/2}}{3^3}\tau\right)\right)$$

as $\tau \to \infty$, and

$$z = \left(y - \frac{s(\tau)}{\tau}\right)\tau = O(1) \quad \text{as } \tau \to \infty.$$

Note that the leading soliton has amplitude $2u_0$ and propagation speed $2u_0/3$, and this quasi-steady structure is approached exponentially in τ as $\tau \to \infty$.

Region ***VI(b)****:* $2u_0/3 + O(1) \leqslant y \leqslant 2u_0 - O(1)$ *as* $\tau \to \infty$. In this region we have exponential decay ahead of the leading soliton. In particular,

$$u(y,\tau) = 8u_0\exp\left(-\sqrt{\frac{2u_0}{3}}\left(y - \frac{2u_0}{3}\right)\tau\right) + H(y)\tau^{-1/2}\exp\left(-\frac{2}{3\sqrt{3}}y^{3/2}\tau\right) + o\left(\tau^{-1/2}\exp\left(-\frac{2}{3\sqrt{3}}y^{3/2}\tau\right)\right)$$

as $\tau \to \infty$, with

$$H(y) \sim \begin{cases} \dfrac{8(6u_0^5)^{1/4}}{\sqrt{\pi}(y-2u_0)} & \text{as } y \to (2u_0)^-, \\ H_c\left(y - \dfrac{2u_0}{3}\right) & \text{as } y \to \left(\dfrac{2u_0}{3}\right)^+. \end{cases}$$

Region ***VI(a)****:* $2u_0 + O(1) \leqslant y < \infty$ *as* $\tau \to \infty$. In this region we have the far field exponential decay. In particular,

$$u(y,\tau) = \bar{K}(y)\tau^{-1/2}\exp\left(-\frac{2}{3\sqrt{3}}y^{3/2}\tau + o(1)\right)$$

as $\tau \to \infty$, with

$$\bar{K}(y) \sim \begin{cases} \dfrac{3^{1/4}u_0}{2\sqrt{\pi}\,y^{3/4}} & \text{as } y \to \infty, \\ \dfrac{8(6u_0^5)^{1/4}}{\sqrt{\pi}(y-2u_0)} & \text{as } y \to (2u_0)^+. \end{cases}$$

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The remaining regions are **C** and **TRA**. In region **C**, $y = -u_0 \pm O(\tau^{-1/2})$ as $\tau \to \infty$, with $u = u_0 + O(\tau^{-1/2})$ as $\tau \to \infty$, and this region provides a transition between the dispersive shock wave structure and the radiation field to its rear. In region **TRA**, $y = 2u_0 \pm O(\tau^{-1/2})$ as $\tau \to \infty$, with u being exponentially small in τ as $\tau \to \infty$. This region provides a transition between the envelope exponential decay structure as $\tau \to \infty$ in region **VI(a)** and the envelope touching exponential decay structure as $\tau \to \infty$ in region **VI(b)**. A schematic diagram of this asymptotic structure is given in Figure 4.

It is important to note that the approach we have adopted in this paper has enabled us to obtain the complete structure of the solution to **IVP** as $\tau \to \infty$, uniform for $-\infty < y < \infty$. In particular, it has enabled the dispersive shock wave structure (obtained previously as a generic coherent structure associated with the Korteweg–de Vries equation) to be rationally embedded into the large-τ solution to **IVP**.

Finally, it is worth noting that the present theory relates directly to the weakly dispersive Korteweg–de Vries equation

$$u_t + uu_{\bar{x}} + \epsilon u_{\bar{x}\bar{x}\bar{x}} = 0, \tag{3.1}$$

and the weakly nonlinear Korteweg–de Vries equation

$$u_t + \epsilon uu_{\bar{x}} + u_{\bar{x}\bar{x}\bar{x}} = 0. \tag{3.2}$$

In considering **IVP**, with either (3.1) (**IVP1**) or (3.2) (**IVP2**) replacing (1.1), we observe that the coordinate scalings

$$\tau = \epsilon^{-1/2} t, \qquad x = \epsilon^{-1/2} \bar{x},$$
$$\tau = \epsilon^{3/2} t, \qquad x = \epsilon^{1/2} \bar{x},$$

for (3.1) and (3.2) respectively, reduce (**IVP1**) and (**IVP2**) respectively to **IVP**. Thus, the asymptotic forms of the solution to (**IVP1**) and (**IVP2**) when $0 < \epsilon \ll 1$ can be obtained directly from the coordinate expansions obtained here for **IVP**. In particular, for (**IVP1**), we have obtained the asymptotic structure for $t \ll \epsilon^{1/2}$ and $t \gg \epsilon^{1/2}$ uniformly for $\bar{x} \in \mathbb{R}$, whilst for (**IVP2**) we have obtained the asymptotic structure for $t \ll \epsilon^{-3/2}$ and $t \gg \epsilon^{-3/2}$ uniformly for $\bar{x} \in \mathbb{R}$. For weak dispersion, the nonlinearity-dispersive interaction takes place when $t = O(\epsilon^{1/2})$. For weak nonlinearity, the nonlinearity-dispersive interaction takes place when $t = O(\epsilon^{-3/2})$.

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