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On the weak Harder–Narasimhan stratification on B_{dR}^{+}-affine Grassmannian

Miaofen Chen and Jilong Tong

In memory of Professor Linsheng Yin

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Abstract

We consider the Harder–Narasimhan formalism on the category of normed isocrystals and show that the Harder–Narasimhan filtration is compatible with tensor products which generalizes a result of Cornut. As an application of this result, we are able to define a (weak) Harder–Narasimhan stratification on the B^+_{dR}-affine Grassmannian for arbitrary (G, b, μ). When μ is minuscule, it corresponds to the Harder–Narasimhan stratification on the flag varieties defined by Dat, Orlik and Rapoport. Moreover, when b is basic, it has been studied by Nguyen and Viehmann, and Shen. We study the basic geometric properties of the Harder–Narasimhan stratification, such as non-emptiness, dimension and its relation with other stratifications.

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1. Introduction

Let F be a p-adic local field, with $\breve{F}$ the p-adic completion of the maximal unramified extension. Consider the p-adic flag variety $\mathcal{F}(G,\mu)$ attached to the pair (G,μ), where G is a connected reductive group over F and μ a geometric minuscule cocharacter of G. Let E be the reflex field of the conjugacy class $\{\mu\}$ of μ. Throughout this paper, we view the flag variety $\mathcal{F}(G,\mu)$ as an adic space over $\breve{E}$, the p-adic completion of the maximal unramified extension of E. Fix $b\in G(\breve{F})$ which satisfies the Kottwitz condition $[b]\in B(G,\mu)$ (cf. § 2.1.2). There are two open subspaces, both called *p-adic period domains*, inside $\mathcal{F}(G,\mu)$,

$$\mathcal{F}(G,\mu,b)^{a}\subset\mathcal{F}(G,\mu,b)^{wa}\subset\mathcal{F}(G,\mu),$$

where $\mathcal{F}(G,\mu,b)^{wa}$ is the *weakly admissible locus* and $\mathcal{F}(G,\mu,b)^{a}$ is the *admissible locus*. The weakly admissible locus $\mathcal{F}(G,\mu,b)^{wa}$ is of algebraic nature, and was constructed by Rapoport and Zink [RZ96] as the complement of a profinite union of explicit closed subspaces of $\mathcal{F}(G,\mu)$. Rapoport and Zink also conjectured the existence of the admissible locus $\mathcal{F}(G,\mu,b)^{a}$ which is of geometric nature, with the property that there exists a p-adic local system with additional structures over this space which interpolates the crystalline representations attached to all classical points of $\mathcal{F}(G,\mu)^{a}$. Since then, the admissible locus was first constructed only for several PEL type triples (G,μ,b) by Hartl [Har13, Har08] and by Faltings [Fal10]. Thanks to the recent revolutionary progress in p-adic Hodge theory, we are now able to define the admissible locus $\mathcal{F}(G,\mu,b)^{a}$ for an arbitrary triple (G,μ,b) based on the work of Fargues and Fontaine [FF18], Fargues [Far20], Kedlaya and Liu [KL15] and Scholze and Weinstein [SW20]. The weakly admissible locus $\mathcal{F}(G,\mu,b)^{wa}$ can be considered as an algebraic approximation of the admissible locus $\mathcal{F}(G,\mu,b)^{a}$ in the sense that they have the same classical points due to a fundamental result in p-adic Hodge theory by Colmez and Fontaine [CF00]. The study of the weakly admissible locus could reflect some information about the admissible locus whose geometry is in general still very mysterious (e.g. [CFS21]).

The admissible locus $\mathcal{F}(G,\mu,b)^{a}$ could be also understood as the image of an étale morphism of rigid analytic spaces over $\breve{E}$, i.e., the *p-adic period mapping* [RZ96, SW20]

$$\pi:(\mathcal{M}_{G,\mu,b,K})_{K\subset G(\mathbb{Q}_p)}\longrightarrow\mathcal{F}(G,\mu),$$

where $(\mathcal{M}_{G,\mu,b,K})_{K\subseteq G(\mathbb{Q}_p)}$ is the local Shimura variety attached to the triple (G,μ,b). For a general (not necessarily minuscule) geometric cocharacter μ, the p-adic period mapping above could be upgraded to an étale morphism of diamonds over $\breve{E}$ [SW20]:

$$\pi^{\diamond}:(\mathrm{Sht}_{G,\mu,b,K})_{K\subset G(\mathbb{Q}_p)}\longrightarrow\mathrm{Gr}_{G,\leq\mu},$$

where $(\mathrm{Sht}_{G,\mu,b,K})_{K\subset G(\mathbb{Q}_p)}$ is the moduli space of local Shtukas attached to (G,μ,b) and

$$\mathrm{Gr}_{G,\leq\mu}=\bigcup_{\lambda\leq\mu}\mathrm{Gr}_{G,\lambda}$$

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is the B_{dR}^{+}-affine Grassmannian attached to (G,μ) (cf. § 2.2). There exists a natural Bialynicki–Birula map

$$\mathrm{BB}:\mathrm{Gr}_{G,\mu}\longrightarrow\mathcal{F}(G,\mu)^{\diamond},$$

where $\mathcal{F}(G,\mu)^{\diamond}$ denotes the diamond associated with $\mathcal{F}(G,\mu)$. When μ is minuscule, the Bialynicki–Birula map is an isomorphism [CS17] and the upgraded p-adic period mapping $\pi^{\diamond}$ is the morphism of diamonds associated with the usual p-adic period mapping π above. For non-minuscule μ, the Bialynicki–Birula map is no longer an isomorphism. Moreover, when μ is non-minuscule, the points in the B_{dR}^{+}-affine Grassmannian $\mathrm{Gr}_{G,\mu}$ but not the points in $\mathcal{F}(G,\mu)$ are naturally related to the modifications of G-bundles on the Fargues–Fontaine curve (cf. [SW20]). This suggests that, when we work with general μ, the flag variety should be replaced by the B_{dR}^{+}-affine Grassmannian $\mathrm{Gr}_{G,\leq\mu}$ (or $\mathrm{Gr}_{G,\mu}$). Inside the flag varieties, for $[b]\in B(G,\mu)$, we still have subspaces

$$\mathrm{Gr}_{G,\leq\mu,b}^{a}\subset\mathrm{Gr}_{G,\leq\mu,b}^{wa}\subset\mathrm{Gr}_{G,\leq\mu}.$$

Here the admissible locus $\mathrm{Gr}_{G,\leq\mu,b}^{a}:=\mathrm{Im}(\pi^{\diamond})$ could be also defined directly using modification of G-bundles on the Fargues–Fontaine curve for arbitrary triple (G,μ,b) with the Kottwitz condition. On the other hand, the weakly admissible locus $\mathrm{Gr}_{G,\leq\mu,b}^{wa}$ is only defined for the triple (G,μ,b) with the condition that μ is minuscule [DOR10, CFS21], $b=1$ [Vie24, She23] or $G=\mathrm{GL}_n$ [NV23]. When μ is minuscule, the admissible locus $\mathrm{Gr}_{G,\leq\mu,b}^{a}=\mathrm{Gr}_{G,\mu,b}^{a}$ (respectively the weakly admissible locus $\mathrm{Gr}_{G,\leq\mu,b}^{wa}=\mathrm{Gr}_{G,\mu,b}^{wa}$) is the diamond associated to $\mathcal{F}(G,\mu,b)^{a}$ (respectively $\mathcal{F}(G,\mu,b)^{wa}$) through the identification $\mathrm{BB}:\mathrm{Gr}_{G,\mu}\xrightarrow{\sim}\mathcal{F}(G,\mu)^{\diamond}$.

To better understand the admissible locus (respectively weakly admissible locus) on the B_{dR}^{+}-affine Grassmannian $\mathrm{Gr}_{G,\mu}$, it is useful to have a natural stratification on $\mathrm{Gr}_{G,\mu}$ having the admissible locus (respectively the weakly admissible locus) as its unique open stratum. The stratification on $\mathrm{Gr}_{G,\mu}$ with the required property for the admissible locus is called the *Newton stratification*. Like the construction of the admissible locus, it is defined by the isomorphism classes of modifications of G-bundles on the Fargues–Fontaine curve for any triple (G,μ,b). Basic properties of the Newton stratification are studied by many people, such as Rapoport [Rap18], Caraiani and Scholze [CS17], Chen, Fargues and Shen [CFS21], Hansen [Han] and Viehmann [Vie24].

On the weakly admissible side, the corresponding stratification on the B_{dR}^{+}-affine Grassmannian $\mathrm{Gr}_{G,\mu}$ is called the *weak Harder–Narasimhan stratification*. Like the situation of the weakly admissible locus, in the literature, it is not yet defined for all the triples (G,μ,b). When μ is minuscule, this is done by Dat, Orlik and Rapoport [DOR10]. Indeed, they construct and study the Harder–Narasimhan stratification on the flag variety $\mathcal{F}(G,\mu)$ for arbitrary (G,μ,b). When μ is minuscule, recall that the Bialynicki–Birula map BB is an isomorphism, so their construction gives the desired weak Harder–Narasimhan stratification on $\mathrm{Gr}_{G,\mu}$. When b is basic while μ is general, the corresponding weak Harder–Narasimhan stratification on $\mathrm{Gr}_{G,\mu}$ is constructed and studied by Nguyen and Viehmann [NV23] and Shen [She23] based on a crucial result of Cornut and Peche Irissarry [CPI19] about the compatibility of certain notion of semistability with tensor products that we will explain later. Moreover, Nguyen and Viehmann also completely settles the case when $G=\mathrm{GL}_n$.

The reason we name it weak Harder–Narasimhan stratification here and also in the title is twofold. One is that it generalizes the weak admissibility condition, and the other is that historically the Newton stratification is also called Harder–Narasimhan stratification (e.g. [CFS21, 5.3]). In the following, for simplicity, we will omit the word 'weak' and call it simply the Harder–Narasimhan stratification if there is no confusion.

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The purpose of this paper is to propose a construction of a Harder–Narasimhan stratification on $\mathrm{Gr}_{G,\mu}$ for arbitrary (G,μ,b) that coincides with the previous constructions. A key ingredient is to generalize the result of Cornut and Peche Irissarry in a more general setting. To explain the main idea of our construction, let us first recall how to define the Harder–Narasimhan stratification on the flag variety $\mathcal{F}(G,\mu)$. We know that the weakly admissible locus $\mathcal{F}(G,\mu,b)^{wa}$ parametrizes weakly admissible filtered isocrystals with G-structures, with the isocrystal determined by b and the filtration of type μ. Let C be a field extension of $\breve{F}$. Consider the category

$$\mathbf{FilIsoc}^{C}_{\breve{F}|F}$$

of filtered isocrystals over $C|\breve{F}$ whose objects consist of an isocrystal D over $\breve{F}|F$ together with a separated exhaustive decreasing filtration by C-subspaces of $D\otimes_{\breve{F}} C$. We can define a Harder–Narasimhan formalism on this category, so that the semi-stable objects of degree 0 are precisely the weakly admissible filtered isocrystals introduced by Fontaine. By classifying the filtered isocrystals according to the type of their Harder–Narasimhan filtration, we obtain the Harder–Narasimhan stratification on $\mathcal{F}(\mathrm{GL}_n,\mu)$ for any triple of the form (GL_n,μ,b). Moreover, Faltings [Fal94] and Totaro [Tot94] show that the Harder–Narasimhan filtration is compatible with tensor products. Based on this result, using Tannakian formalism, Dat, Orlik and Rapoport are able to define the Harder–Narasimhan stratification on the flag variety $\mathcal{F}(G,\mu)$ for arbitrary G.

From now on, let C be a complete algebraically closed non-archimedean field over $\breve{F}$. Let B_{dR}^{+} and B_{dR} be Fontaine's de Rham period rings corresponding to C. Recall that, for an arbitrary cocharacter μ (possibly non-minuscule), we replace the flag variety $\mathcal{F}(G,\mu)$ (which parametrizes filtrations of type μ) by the B_{dR}^{+}-affine Grassmannian $\mathrm{Gr}_{G,\mu}$ (which parametrizes lattices having relative position μ with the standard lattice). Accordingly, we replace the category $\mathbf{FilIsoc}^{C}_{\breve{F}|F}$ by the category

$$\mathbf{BunIsoc}^{B_{\mathrm{dR}}}_{\breve{F}|F},$$

whose objects consist of isocrystals D over $\breve{F}|F$ equipped with a B_{dR}^{+}-lattice inside $D\otimes_{\breve{F}} B_{\mathrm{dR}}$. There exists a natural Harder–Narasimhan formalism on this category, which leads to the construction of the Harder–Narasimhan stratification on $\mathrm{Gr}_{G,\mu}$ for $G=\mathrm{GL}_n$ by classifying the objects in $\mathbf{BunIso}^{B_{\mathrm{dR}}}_{\breve{F}|F}$ according to the type of their Harder–Narasimhan filtration. In order to work with arbitrary G, we then need a result parallel to the result of Faltings [Fal94] and Totaro [Tot94] mentioned above. More precisely, let

$$\mathcal{F}_{\mathrm{HN}}:\mathbf{BunIsoc}^{B_{\mathrm{dR}}}_{\breve{F}|F}\longrightarrow \mathbf{F}(\mathbf{Isoc}_{\breve{F}|F})$$

be the functor sending any object to its Harder–Narasimhan filtration, where $\mathbf{F}(\mathbf{Isoc}_{\breve{F}|F})$ denotes the category of $\mathbb{R}$-descending filtrations of isocrystals over $\breve{F}|F$ by sub-isocrystals.

Theorem (Theorem 3.23). *The functor $\mathcal{F}_{\mathrm{HN}}:\mathbf{BunIsoc}^{B_{\mathrm{dR}}}_{\breve{F}|F}\to\mathbf{F}(\mathbf{Isoc}_{\breve{F}|F})$ is compatible with tensor products.*

When we restrict the functor $\mathcal{F}_{\mathrm{HN}}$ to the subcategory consisting of objects whose underlying isocrystals are isoclinic of slope 0, this result is proved by Cornut and Peche Irissarry ([CPI19], compare also [Cor18]). To show the desired compatibility with tensor products without any restriction on the slopes, we follow the general approach developed by Cornut in [Cor18]. Indeed, generalizing the notion of normed vector spaces considered by Cornut in *loc. cit.*, we introduce

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the notion of normed isocrystals, so that the isocrystals with lattices are precisely those normed isocrystals equipped with the gauge norm induced from a lattice (cf. § 3.4). We show that one can also develop a Harder–Narasimhan formalism for normed isocrystals, and the resulting Harder–Narasimhan filtration is compatible with tensor products (cf. Theorem 3.10).

Based on this result, we are able to define the Harder–Narasimhan stratification on $\mathrm{Gr}_{G,\mu}$ for arbitrary triple (G,μ,b). Then we study the basic properties of the Harder–Narasimhan stratification and its relations to other stratifications. We show that the Harder–Narasimhan stratification, the Newton stratification on the B^+_{dR}-affine Grassmannian and the Harder–Narasimhan stratification on the flag variety (via the Bialynicki–Birula map) coincide on classical points (Theorem 5.10) which generalizes the result of Viehmann [Vie24] for $b=1$. We give a combinatorial criterion for the non-emptiness of any Harder–Narasimhan stratum (Theorem 6.2) which is compatible with the result of Nguyen and Viehmann [NV23] for $b=1$ and is parallel to the result of Orlik [Orl06] for the Harder–Narasimhan stratum on flag varieties for $G=\mathrm{GL}_n$. When μ is minuscule, we also give a dimension formula for a Harder–Narasimhan stratum (Theorem 6.9) which generalizes the result of Fargues [Far19] for $G=\mathrm{GL}_n$ and $b=1$.

We briefly describe the structure of this article. In § 1, we review the preliminaries such as G-bundles on the Fargues–Fontaine curve and the B^+_{dR}-affine Grassmannian. In § 2, we show the fact that the Harder–Narasimhan filtration is compatible with tensor products in the category of normed isocrystals, which generalizes a result of Cornut for normed vector spaces. As an application, in § 3, we define the Harder–Narasimhan stratification on the B^+_{dR}-affine Grassmannian for arbitrary triple (G,μ,b). In § 4, we compare the Harder–Narasimhan stratification with the Newton stratification and the Harder–Narasimhan stratification on the flag variety. In § 5, we discuss some basic properties of a Harder–Narasimhan stratum such as non-emptiness and dimension formula.

2. Preliminaries on G-bundles on the Fargues–Fontaine curve

We denote by **Perf** the category of perfectoid spaces over $\mathbb{F}_q$ which is the residue field of the p-adic local field F. Let G be a connected reductive group over F. Unless otherwise specified, we fix two algebraic subgroups $A\subset T\subset G$ of G defined over F, where A is a maximal split torus and T is a maximal torus containing A in G. Let $W=W_G=(N_G(T)/T)(\bar{F})$ be the absolute Weyl group of T in G and let $w_0\in W$ be the longest element. In the case that G is quasi-split, we also fix a Borel subgroup $B\subset G$ containing the maximal F-torus $T\subset G$.

2.1 G-bundles on the Fargues–Fontaine curve

2.1.1 *Fargues–Fontaine curve.* We briefly recall the construction of the (relative) Fargues–Fontaine curve (cf. [Far25, § 1] or [FS21, § II.1]). Let π_F be a uniformizer of F and let $\mathbb{F}_q$ be the residue field of F. For $S=\mathrm{Spa}(R,R^+)\in\mathbf{Perf}$ a perfectoid affinoid space over $\mathbb{F}_q$, fix a pseudo-uniformizer $\omega\in R^+\subset R$. Let

$$\mathcal{Y}_S:=\mathrm{Spa}(\mathbb{A},\mathbb{A})\backslash V(\pi_F[\omega]),$$

where

$$\mathbb{A}=W_{\mathcal{O}_F}(R^+)=\left\{\sum_{n\geq 0}[x_n]\pi_F^n \mid x_n\in R^+\right\}$$

is the ramified Witt ring with coefficients in R^+ equipped with the $(\pi_F,[\omega])$-adic topology. It is known that $\mathcal{Y}_S$ is an analytic adic space over $\mathcal{O}_F$ [FS21, Proposition II.1.1]. The qth Frobenius

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automorphism on R^+ induces an automorphism ϕ_S on $\mathcal{Y}_S$ over $\mathcal{O}_F$, and this action is free and totally discontinuous [FS21, Proposition II.1.16]. The *relative adic Fargues–Fontaine curve* $\mathcal{X}_S$ is by definition the following quotient

$$\mathcal{X}_S := \mathcal{Y}_S/\phi_S^{\mathbb{Z}}.$$

We also have the *relative schematic Fargues–Fontaine curve* X_S defined as

$$X_S := \mathrm{Proj}\left(\bigoplus_{d=0}^{\infty} H^0(\mathcal{Y}_S, \mathcal{O}_{\mathcal{Y}_S})^{\phi_S=\pi_F^d}\right).$$

When $S = \mathrm{Spa}(K, K^+)$ with K a perfectoid field, X_S (sometimes also denoted by X or X_K since it does not depend on the choice of K^+) is a noetherian scheme of dimension 1.

Remark 2.1. Let $S \in \mathbf{Perf}$ be a perfectoid space over $\mathbb{F}_q$.

(i) By gluing, the local construction above defines the relative Fargues–Fontaine curves $\mathcal{X}_S$ [FS21, Proposition II.1.3].

(ii) Suppose moreover that S is affinoid. There is a morphism of locally ringed spaces:

$$(\mathcal{X}_S, \mathcal{O}_{\mathcal{X}_S}) \longrightarrow X_S.$$

The GAGA functor associated to the morphism of ringed spaces above gives an equivalence of categories [KL15]:

$$\mathrm{Bun}_{X_S} \longrightarrow \mathrm{Bun}_{(\mathcal{X}_S, \mathcal{O}_{\mathcal{X}_S})},$$

where Bun_{X_S} (respectively $\mathrm{Bun}_{(\mathcal{X}_S, \mathcal{O}_{\mathcal{X}_S})}$) denotes the category of vector bundles on X_S (respectively on the locally ringed space $(\mathcal{X}_S, \mathcal{O}_{\mathcal{X}_S})$). See [FS21, Proposition II.2.7] for more details.

Let $S = \mathrm{Spa}(R, R^+) \in \mathbf{Perf}/\mathrm{Spd}(F)$ be a perfectoid affinoid space lying over the diamond $\mathrm{Spd}(F)$, which corresponds to an untilt $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp +})$ over F of S. The untilt $S^\sharp$ gives rise to a natural closed immersion of adic spaces

$$S^\sharp \hookrightarrow \mathcal{Y}_S$$

that presents $S^\sharp$ as a closed Cartier divisor in $\mathcal{Y}_S$ [FS21, Proposition II.3.1], with image $V(\xi) \subset \mathcal{Y}_S$ for some element $\xi \in \mathbb{A}$. Furthermore, locally on S the closed Cartier divisor above can be also defined by an element in $H^0(\mathcal{X}_S, \mathcal{O}_{\mathcal{X}_S}(1)) = H^0(\mathcal{Y}_S, \mathcal{O}_{\mathcal{Y}_S})^{\phi_S=\pi_F}$ [FS21, Proposition II.2.3]. Consequently, it defines a Cartier divisor of the relative schematic Fargues–Fontaine curve

$$\mathrm{Spec}(R^\sharp) \hookrightarrow X_S.$$

The Fontaine's de Rham period ring $B_{\mathrm{dR}}^+(R^\sharp)$ is defined to be the completion of $\mathcal{O}_{X_S}$ along this Cartier divisor (cf. [Far25, 1.33]) which is also the ξ-adic completion of $\mathbb{A}[1/\pi_F]$. We write $B_{\mathrm{dR}}(R^\sharp) = B_{\mathrm{dR}}^+(R^\sharp)[1/\xi]$.

2.1.2 *The Kottwitz set.* Let $B(G)$ be the set of σ-conjugacy classes of $G(\breve{F})$, where σ is the Frobenius with respective to $\breve{F}|F$. For $b \in G(\breve{F})$, let $[b] \in B(G)$ be the σ-conjugacy class of b. The Kottwitz set is characterized by two invariants. One is called the Newton map (cf. [Kot85, § 4]). For any $b \in G(\breve{F})$, we can define a slope morphism

$$\nu_b : \mathbb{D}_{\breve{F}} \longrightarrow G_{\breve{F}},$$

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whose conjugacy class is defined over F, where $\mathbb{D}$ is the pro-torus with characters $X^*(\mathbb{D}) = \mathbb{Q}$. This defines the *Newton map*

$$\begin{aligned} \nu : B(G) &\longrightarrow \mathcal{N}(G), \\ [b] &\longmapsto [\nu_b], \end{aligned}$$

where

$$\mathcal{N}(G) := [\mathrm{Hom}(\mathbb{D}_{\bar{F}}, G_{\bar{F}})/G(\bar{F})\text{-conjugacy}]^{\Gamma}$$

with $\Gamma = \mathrm{Gal}(\bar{F}|F)$. An element $[b]$ in $B(G)$ is said to be *basic* if ν_b is central. When G is quasi-split, there is a natural identification

$$\mathcal{N}(G) \simeq X_*(A)^+_{\mathbb{Q}}.$$

The other invariant is the *Kottwitz map* (cf. [Kot97, 4.9, 7.5]):

$$\begin{aligned} \kappa = \kappa_G : B(G) &\longrightarrow \pi_1(G)_{\Gamma}, \\ [b] &\longmapsto \kappa([b]), \end{aligned}$$

where $\pi_1(G)$ is the algebraic fundamental group of G and $\pi_1(G)_{\Gamma}$ is its Galois coinvariants.

By [Kot97, 4.13], the induced map

$$(\nu, \kappa) : B(G) \longrightarrow \mathcal{N}(G) \times \pi_1(G)_{\Gamma} \tag{2.1}$$

is injective.

Letting $[b] \in B(G)$ and $\mu \in X_*(T)^+$, we define the *Kottwitz set* [Kot97]

$$B(G, \mu) := \{[b] \in B(G) \mid [\nu_b] \leq \mu^{\diamond}, \kappa_G(b) = \mu^{\sharp}\},$$

where $\mu^{\diamond} \in \mathcal{N}(G)$ is the Galois average of μ, $\mu^{\sharp} \in \pi_1(G)_{\Gamma}$ is the image of μ via the natural quotient map $X_*(T) \to \pi_1(G)_{\Gamma}$ and the order $\leq$ on $\mathcal{N}(G)$ is the usual *dominance order*: $\nu_1 \leq \nu_2$ if and only if ν_1 lies in the convex hull of the Weyl orbit of ν_2.

We also need the following *generalized Kottwitz set* (cf. [CFS21]). For $\epsilon \in \pi_1(G)_{\Gamma}$ and $\delta \in \mathcal{N}(G)$ we set

$$B(G, \epsilon, \delta) = \{[b] \in B(G) \mid \kappa_G(b) = \epsilon \text{ and } [\nu_b] \leq \delta\}.$$

We will need the following well-known fact, which can be deduced easily from a general result of Kottwitz [Kot97, 3.6].

Lemma 2.2. *Let P be a parabolic subgroup of G over F with M the Levi component. Then $B(M) \simeq B(P)$.*

2.1.3 *G-bundles.* Recall that a *G-bundle* on an F-scheme Z is a (right) G-torsor on Z locally trivial for the étale topology. It could also be viewed as an exact tensor functor

$$\mathrm{Rep}G \longrightarrow \mathrm{Bun}_Z,$$

where $\mathrm{Rep}G$ is the category of rational algebraic representations of G and Bun_Z denotes the category of vector bundles (of finite rank) over Z.

Let S be a perfectoid affinoid space over the residue field $\bar{\mathbb{F}}_q$ of $\breve{F}$. For each $b \in G(\breve{F})$, we can associate a G-bundle

$$\mathcal{E}_b$$

on the relative Fargues–Fontaine curve X_S as follows. Suppose $S = \mathrm{Spa}(R, R^+)$. In particular, $X_S = \mathrm{Proj}(\oplus_d B^{\phi_S = \pi_F^d})$, with $B = H^0(\mathcal{Y}_S, \mathcal{O}_{\mathcal{Y}_S})$. As an exact tensor functor the G-bundle $\mathcal{E}_b$ is given by

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$$\mathcal{E}_b : \mathrm{Rep}G \longrightarrow \mathrm{Bun}_{X_S}, \quad (V,\rho) \mapsto \mathcal{E}_b(V,\rho),$$

with $\mathcal{E}_b(V,\rho)$ the vector bundle on X_S corresponding to the graded module

$$\bigoplus_{d\in\mathbb{Z}} (V \otimes_F B)^{\rho(b)\phi_S = \pi_F^d}$$

over the graded ring $\oplus_d B^{\phi_S=\pi_F^d}$. Here we use the same notation $\rho(b)$ to denote the B-linear automorphism on $V \otimes_F B = (V \otimes_F \breve{F}) \otimes_{\breve{F}} B$ induced by $\rho(b) \in \mathrm{GL}(V \otimes_F \breve{F})$. The isomorphism class of $\mathcal{E}_b$ only depends on the σ-conjugacy class of b.

If moreover $S = \mathrm{Spa}(C, C^+)$ with C an algebraically closed perfectoid field containing $\bar{\mathbb{F}}_q$ and we write $X = X_S$, this construction gives a bijection of pointed sets by a fundamental theorem of Fargues [Far20]:

$$\begin{aligned} B(G) &\xrightarrow{\sim} H^1_{\acute{e}t}(X, G), \\ [b] &\longmapsto \mathcal{E}_b. \end{aligned}$$

2.1.4 *Harder–Narasimhan reduction in the quasi-split case.* In this subsection, suppose G is quasi-split.

Recall the following definition of semi-stable G-bundles on X. See for example [CFS21, Definition 1.5].

Definition 2.3. A G-bundle $\mathcal{E}$ on the Fargues–Fontaine curve X is said to be semi-stable if, for any standard parabolic subgroup P of G, any reduction $\mathcal{E}_P$ of $\mathcal{E}$ to P and any $\chi \in X^*(P/Z_G)^+$, we have $\deg \chi_*(\mathcal{E}_P) \leq 0$.

Remark 2.4. Because any two Borel subgroups of G are $G(F)$-conjugate, the notion of semi-stability defined above does not depend on the choice of the Borel subgroup that we fixed before. Moreover, for a G-bundle $\mathcal{E} = \mathcal{E}_b$ on X with $[b] \in B(G)$, the following statements are equivalent (cf. [Far20, §§ 5.1 and 5.3] or [CFS21, § 1.3]):

(i) $\mathcal{E}$ is semi-stable;
(ii) the associated adjoint bundle

$$\mathcal{E} \times^{G,\mathrm{Ad}} \mathrm{Lie}(G)$$

of $\mathcal{E}$ on X is semi-stable (of degree 0);
(iii) $[b] \in B(G)$ is basic.

For any G-bundle $\mathcal{E}$ on X, by [CFS21, Theorem 1.7], there exist a unique standard parabolic subgroup P of G and a unique reduction $\mathcal{E}_P$ of $\mathcal{E}$ to P, such that the associated M-bundle $\mathcal{E}_P \times^P M$ is semi-stable and P is maximal among all the standard parabolic with the previous condition. The reduction $\mathcal{E}_P$ is called the *Harder–Narasimhan reduction* or *canonical reduction* of $\mathcal{E}$. This defines the Harder–Narasimhan polygon of $\mathcal{E}$:

$$\nu_{\mathcal{E}} \in X_*(A)^+_{\mathbb{Q}},$$

such that for any $\chi \in X^*(P)$, we have $\langle \chi, \nu_{\mathcal{E}} \rangle = \deg \chi_*(\mathcal{E}_P)$. By [Far20, Proposition 6.6], we have $\nu_{\mathcal{E}_b} = -w_0\nu_b$.

The Harder–Narasimhan reduction has the following characterization.

Proposition 2.5 [Sch15, Theorem 4.5.1]. *For $\mathcal{E}$ a G-bundle on X equipped with a reduction $\mathcal{E}_Q$ to the standard parabolic subgroup Q of G, consider the vector*

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$$v : X^*(Q) \longrightarrow \mathbb{Z},$$
$$\chi \longmapsto \deg \chi_* \mathcal{E}_Q,$$

seen as an element of $X_*(A)_{\mathbb{Q}}$. *Then:*

(i) $v \leq \nu_{\mathcal{E}}$; *and, if this inequality is an equality,*
(ii) $Q \subset P$ *and the canonical reduction* $\mathcal{E}_P \simeq \mathcal{E}_Q \times^Q P$.

2.2 B^+_{dR}-affine Grassmannian

We collect some basic properties of the B^+_{dR}-affine Grassmannian that we will need in what follows. The main references for this section are [SW20] and [FS21]. We consider the B^+_{dR}-affine Grassmannian

$$\mathrm{Gr}_G = \mathrm{Gr}_G^{B^+_{\mathrm{dR}}}$$

as a functor on the category $\mathbf{Perf}/\mathrm{Spd}(F)$ of perfectoid spaces over $\mathbb{F}_q$ lying above $\mathrm{Spd}(F)$. More precisely, Gr_G is the étale sheafification of the functor taking $S = \mathrm{Spa}(R, R^+) \in \mathbf{Perf}/\mathrm{Spd}(F)$, which corresponds to an untilt $(R^\sharp, R^{\sharp +})$ of (R, R^+) over F, to the coset space

$$G(B_{\mathrm{dR}}(R^\sharp))/G(B^+_{\mathrm{dR}}(R^\sharp)).$$

According to [SW20, Proposition 19.1.2],

$$\mathrm{Gr}_G(S) \simeq \left\{ (\mathcal{E}, \iota) \middle| \begin{array}{c} \mathcal{E} \text{ is a } G\text{-bundle on } \mathrm{Spec}(B^+_{\mathrm{dR}}(R^\sharp)), \text{ and} \\ \iota \text{ is a trivialization of } \mathcal{E}|_{\mathrm{Spec}(B_{\mathrm{dR}}(R^\sharp))} \end{array} \right\} / \simeq$$

and Gr_G is a small v-sheaf.

2.2.1 *Schubert varieties.* Let $\bar{F}$ be an algebraic closure of F. We choose a maximal torus $T \subset G_{\bar{F}}$ and a Borel subgroup containing T. Write $\mathrm{Gr}_{G,\bar{F}}$ the restriction of Gr_G to $\mathbf{Perf}/\mathrm{Spd}(\bar{F})$. Let $S = \mathrm{Spa}(C, C^+) \to \mathrm{Spd}(\bar{F})$ be a geometric point of $\mathrm{Spd}(\bar{F})$, given by an untilt $\mathrm{Spa}(C^\sharp, C^{\sharp +})$ of S over $\bar{F}$.

Remark 2.6. The period ring $B^+_{\mathrm{dR}}(C^\sharp)$ is a complete discrete valuation ring with residue field $C^\sharp$. Moreover, as $\bar{F} \subset C^\sharp$, the F-algebra $B^+_{\mathrm{dR}}(C^\sharp)$ is also naturally an $\bar{F}$-algebra. Non-canonically we have $B^+_{\mathrm{dR}}(C^\sharp) \simeq C^\sharp[\![\xi]\!]$ with ξ a uniformizer of $B^+_{\mathrm{dR}}(C^\sharp)$.

By the Cartan decomposition,

$$G(B_{\mathrm{dR}}(C^\sharp)) = \coprod_{\mu \in X_*(T)^+} G(B^+_{\mathrm{dR}}(C^\sharp)) \mu(\xi)^{-1} G(B^+_{\mathrm{dR}}(C^\sharp)).$$

In particular,

$$\begin{aligned} \mathrm{Gr}_G(S) &= G(B_{\mathrm{dR}}(C^\sharp))/G(B^+_{\mathrm{dR}}(C^\sharp)) \\ &= \coprod_{\mu \in X_*(T)^+} G(B^+_{\mathrm{dR}}(C^\sharp)) \mu(\xi)^{-1} G(B^+_{\mathrm{dR}}(C^\sharp))/G(B^+_{\mathrm{dR}}(C^\sharp)). \end{aligned}$$

Here we have the first equality since C is algebraically closed and thus every G-bundle over $\mathrm{Spec}(B^+_{\mathrm{dR}}(C^\sharp))$ is trivial. By abuse of notation, for any geometric conjugacy class $\{\mu\} \in X_*(G)/G(\bar{F})$, we denote by

$$\mathrm{Gr}_{G,\mu} \subset \mathrm{Gr}_{G,\leq \mu} \subset \mathrm{Gr}_{G,\bar{F}}$$

the subfunctors of $\mathrm{Gr}_{G,\bar{F}}$ defined by the condition that a map

$$S' \longrightarrow \mathrm{Gr}_{G,\bar{F}}$$

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with $S' \in \mathbf{Perf}/\mathrm{Spd}(\bar{F})$ factors through $\mathrm{Gr}_{G,\mu}$, or respectively $\mathrm{Gr}_{G,\leq\mu}$, if and only if, for all geometric points $S = \mathrm{Spa}(C, C^+) \to S'$, the corresponding S-valued point of $\mathrm{Gr}_{G,\bar{F}}$ lies in

$$G(B^+_{\mathrm{dR}}(C^\sharp))\mu(\xi)^{-1}G(B^+_{\mathrm{dR}}(C^\sharp))/G(B^+_{\mathrm{dR}}(C^\sharp)),$$

or respectively lies in

$$\coprod_{\mu' \leq \mu} G(B^+_{\mathrm{dR}}(C^\sharp))\mu'(\xi)^{-1}G(B^+_{\mathrm{dR}}(C^\sharp))/G(B^+_{\mathrm{dR}}(C^\sharp)).$$

According to [SW20, Proposition 19.2.3], $\mathrm{Gr}_{G,\leq\mu} \subset \mathrm{Gr}_{G,\bar{F}}$ is a closed subfunctor proper over $\mathrm{Spd}(\bar{F})$, and $\mathrm{Gr}_{G,\mu} \subset \mathrm{Gr}_{G,\leq\mu}$ is an open subfunctor. Furthermore, the small v-sheaf $\mathrm{Gr}_{G,\leq\mu}$ is a spatial diamond [SW20, Theorem 19.2.4].

2.2.2 *Generalized semi-infinite orbits.* We keep the notation in § 2.2.1. Let $P \subset G_{\bar{F}}$ be a standard parabolic subgroup. Let M be the Levi quotient of P, M^{ab} the maximal torus quotient of M and $N \subset P$ the unipotent radical. We have a natural identification of v-sheaves on $\mathbf{Perf}/\mathrm{Spd}(\bar{F})$,

$$\mathrm{Gr}_{M^{ab}} \xrightarrow{\sim} X_*(M^{ab}),$$

which sends $v(\xi)^{-1}M^{ab}(B^+_{\mathrm{dR}}) \in \mathrm{Gr}_{M^{ab}}(C)$ to v for every cocharacter $v \in X_*(M^{ab})$.

Remark 2.7. To be consistent with the definition of Schubert varieties, we insert a minus sign here compared with [FS21].

On the other hand, consider the B^+_{dR}-affine Grassmannian Gr_P associated with the parabolic subgroup P. For $v \in X_*(M^{ab})$, write

$$\mathrm{Gr}_P^v \subset \mathrm{Gr}_{P,\bar{F}},$$

the open and closed subfunctor of $\mathrm{Gr}_{P,\bar{F}}$ (over $\mathbf{Perf}/\mathrm{Spd}(\bar{F})$) obtained as the preimage of $v \in X_*(M^{ab})$ under the following composed map:

$$\mathrm{Gr}_{P,\bar{F}} \longrightarrow \mathrm{Gr}_{M,\bar{F}} \longrightarrow \mathrm{Gr}_{M^{ab},\bar{F}} \xrightarrow{\sim} X_*(M^{ab}).$$

Proposition 2.8 [FS21, Proposition VI 3.1]. *The map*

$$\mathrm{Gr}_P = \coprod_{v \in X_*(M^{ab})} \mathrm{Gr}_P^v \longrightarrow \mathrm{Gr}_{G,\bar{F}}$$

is bijective on geometric points, and it is a locally closed immersion on each Gr_P^v. *Moreover, the image of the union*

$$\bigcup_{v \leq v'} \mathrm{Gr}_P^{v'}$$

is closed in $\mathrm{Gr}_{G,\bar{F}}$. *Here* $X_*(M^{ab}) \subset X_*(T)_{\mathbb{Q}}$ *is equipped with the dominance order, where* $v \leq v'$ *if* $v' - v \in X_*(T)_{\mathbb{Q}}$ *is a sum of positive coroots with* $\mathbb{Z}_{\geq 0}$*-coefficients.*

We also have a finer version of Gr_P^v obtained as follows. Let $\lambda \in X_*(T)$ be an M-dominant cocharacter. We set

$$\mathrm{Gr}_{P,\lambda} \subset \mathrm{Gr}_{P,\bar{F}},$$

the locally closed subfunctor given by the preimage of $\mathrm{Gr}_{M,\lambda} \subset \mathrm{Gr}_M$ through the natural projection

$$\mathrm{pr}_M : \mathrm{Gr}_P \longrightarrow \mathrm{Gr}_M.$$

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Clearly,

$$\mathrm{Gr}_{P,\lambda} \subset \mathrm{Gr}_P^{\bar{\lambda}},$$

where $\bar{\lambda}$ denotes the image of $\lambda \in X_*(M)$ in $X_*(M^{ab})$. So $\mathrm{Gr}_{P,\lambda} \subset \mathrm{Gr}_{G,\bar{F}}$ is locally closed. Moreover, for a geometric point $\mathrm{Spa}(C, C^+) \to \mathrm{Spd}(\bar{F})$ of $\mathrm{Spd}(\bar{F})$ corresponding to an untilt $(C^\sharp, C^{\sharp+})$ over $\bar{F}$, we have

$$\mathrm{Gr}_{P,\lambda}(C, C^+) = N(B_{\mathrm{dR}}(C^\sharp))M(B^+_{\mathrm{dR}}(C^\sharp))\lambda(\xi)^{-1}G(B^+_{\mathrm{dR}}(C^\sharp))/G(B^+_{\mathrm{dR}}(C^\sharp)).$$

Remark 2.9. When the parabolic subgroup P is minimal (i.e., a Borel subgroup), then $\mathrm{Gr}_{P,\lambda} = \mathrm{Gr}_P^\lambda$ is studied by Viehmann (cf. [Vie24, §2.2], see also [FS21, Example VI.3.4]), and is denoted by S_λ in *loc. cit.* It is the analogue of the semi-infinite orbit for $\lambda \in X_*(T)$ of the usual affine Grassmannian.

Motivated by [GHKR06], for a given G-dominant cocharacter μ we write $S_M(\mu; C, C^+)$ for the set of M-dominant cocharacters λ for which the intersection

$$N(B_{\mathrm{dR}}(C^\sharp))\lambda^{-1}(\xi) \cap G(B^+_{\mathrm{dR}}(C^\sharp))\mu^{-1}(\xi)G(B^+_{\mathrm{dR}}(C^\sharp)) \neq \emptyset.$$

We shall see in the next lemma that the set $S_M(\mu; C, C^+)$ does not depend on the choice of the geometric point $\mathrm{Spa}(C, C^+) \to \mathrm{Spd}(\bar{F})$ and hence there will be no confusion to write $S_M(\mu) := S_M(\mu; C, C^+)$.

Lemma 2.10. *We keep the notation above.*

(i) *The set $S_M(\mu) = S_M(\mu; C, C^+)$ does not depend on the choice of the geometric point of $\mathrm{Spd}(\bar{F})$.*
(ii) *Let $\lambda \in X_*(T)$ be M-dominant and $\mu \in X_*(T)$ be G-dominant. Then the following assertions are equivalent:*
  (i) $\mathrm{Gr}_{G,\mu} \cap \mathrm{Gr}_{P,\lambda} \neq \emptyset$;
  (ii) $\mathrm{Gr}_{G,\mu}(C, C^+) \cap \mathrm{Gr}_{P,\lambda}(C, C^+) \neq \emptyset$ *for some geometric point* $\mathrm{Spa}(C, C^+) \to \mathrm{Spd}(\bar{F})$;
  (iii) $\mathrm{Gr}_{G,\mu}(C, C^+) \cap \mathrm{Gr}_{P,\lambda}(C, C^+) \neq \emptyset$ *for every geometric point* $\mathrm{Spa}(C, C^+) \to \mathrm{Spd}(\bar{F})$;
  (iv) $\lambda \in S_M(\mu)$.

Proof. It is enough to prove (i). Recall that non-canonically $B^+_{\mathrm{dR}}(C^\sharp) \simeq C^\sharp[\![\xi]\!]$. Write $\mathcal{G}_G$ for the classical affine Grassmannian, that is, the étale sheafification of the functor on the category $\mathbf{Aff}_{\bar{F}}$ of affine $\bar{F}$-schemes sending $\mathrm{Spec}(R) \in \mathbf{Aff}_{\bar{F}}$ to the coset

$$G(R(\!(\xi)\!))/G(R[\![\xi]\!]).$$

Let $\mathcal{G}_{G,\mu} \subset \mathcal{G}_G$ be the corresponding Schubert variety attached to μ (with our convention on the minus sign), and let $\mathcal{S}_\lambda \subset \mathcal{G}_G$ be the orbit of the $\bar{F}$-point

$$\lambda(\xi)^{-1}G(\bar{F}[\![\xi]\!]) \in \mathcal{G}_G(\bar{F}),$$

under the action of the ind-group scheme

$$\mathcal{N} : \mathbf{Aff}_{\bar{F}} \longrightarrow \mathbf{Grps}, \quad \mathrm{Spec}(R) \longmapsto N(R(\!(\xi)\!)),$$

by left multiplication on $\mathcal{G}_G$. Using the (non-canonical) identification $B^+_{\mathrm{dR}} \simeq C^\sharp[\![\xi]\!]$, we have $\lambda \in S_M(\mu; C, C^+)$ if and only if

$$N(C^\sharp[\![\xi]\!])\lambda^{-1}(\xi) \cap G(C^\sharp[\![\xi]\!])\mu^{-1}(\xi)G(C^\sharp[\![\xi]\!]) \neq \emptyset,$$

or equivalently $(\mathcal{G}_{G,\mu} \cap \mathcal{S}_\lambda)(C^\sharp) \neq \emptyset$. But $\mathcal{G}_{G,\mu} \cap \mathcal{S}_\lambda$ is an $\bar{F}$-scheme of finite type, so it has a point with values in the algebraically closed field $C^\sharp$ if and only if $\mathcal{G}_{G,\mu} \cap \mathcal{S}_\lambda \neq \emptyset$. This shows that the

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set $S_M(\mu; C, C^+)$ is actually independent on the choice of the geometric point $\mathrm{Spa}(C, C^+) \to \mathrm{Spd}(\breve{F})$. □

For the description of $S_M(\mu)$, we have an analogous result as [GHKR06, Lemma 5.4.1].

Lemma 2.11. *Keep the notation above.*

(i) *There are inclusions of finite sets*

$$\Sigma(\mu)_{M\text{-max}} \subseteq S_M(\mu) \subseteq \Sigma(\mu)_{M\text{-dom}},$$

where $\Sigma(\mu)_{M\text{-dom}}$ denotes the set of M-dominant cocharacters $\lambda \in X_*(T)$ such that $(\lambda)_{\mathrm{dom}} \leq \mu$, and $\Sigma(\mu)_{M\text{-max}}$ denotes the set of elements in $\Sigma(\mu)_{M\text{-dom}}$ that are maximal with respect to the partial order $\leq_M$. In particular, when μ is minuscule or $M = T$ is the maximal torus, the inclusions become equalities.

(ii) *For any $\lambda \in S_M(\mu)$, $\kappa(\lambda) = \kappa(\mu)$ in $\pi_1(G)_\Gamma$.*

Proof. The assertion (ii) is obvious.

For (i), let us start with the case where $M = T$ is the maximal torus (and thus $N = U$ is the unipotent radical of the Borel subgroup $B \subset G$), $\Sigma(\mu)_{M\text{-max}} = S_M(\mu) = \Sigma(\mu)_{M\text{-dom}}$. In this case, our lemma says that, for $\lambda \in X_*(T)$,

$$U(B_{\mathrm{dR}}(C^\sharp))\lambda^{-1}(\xi) \cap G(B_{\mathrm{dR}}^+(C^\sharp))\mu^{-1}(\xi)G(B_{\mathrm{dR}}^+(C^\sharp)) \neq \emptyset$$

if and only if $(\lambda)_{\mathrm{dom}} \leq \mu$. Using a non-canonical identification $B_{\mathrm{dR}}^+(C^\sharp) \simeq C^\sharp[\![\xi]\!]$, the desired assertion follows from the geometry of semi-infinite orbits of the usual affine Grassmannian for the group G (cf. [MV07, 3.2 Theorem] or [Zhu17, Lemma 5.3.7 and Theorem 5.3.9]). The proof for the general standard Levi $M \subset G$ is then the same as that of [GHKR06, Lemma 5.4.1]. □

Remark 2.12. Fixing μ, there are only finitely many $v \in X_*(M^{ab})$ such that $\mathrm{Gr}_P^v \cap \mathrm{Gr}_{G,\mu} \neq \emptyset$. Indeed,

$$\coprod_{\substack{\lambda \in X_*(T) \\ \lambda = v \text{ in } \pi_1(M^{ab})}} \mathrm{Gr}_{P,\lambda} \longrightarrow \mathrm{Gr}_P^v$$

induces a bijection on geometric points. By Lemma 2.10, $\mathrm{Gr}_P^v \cap \mathrm{Gr}_{G,\mu} \neq \emptyset$ if and only if v is contained in the image of $S_M(\mu)$ via $\theta : X_*(T) \to X_*(M^{ab})$, if and only if $v \in \theta(W\mu)$ by the previous lemma.

2.2.3 *B_{dR}^+-affine Grassmannian and modifications of G-bundles.* Let S be a perfectoid affinoid space over $\mathbb{F}_q$ lying above $\mathrm{Spd}(\breve{F})$, given by an untilt $S^\sharp$ of S over $\breve{F}$. Let $b \in G(\breve{F})$. Consider the associated G-bundle $\mathcal{E}_b$ on the relative Fargues–Fontaine curve X_S (whose construction is recalled in § 2.1.3). An element $x \in \mathrm{Gr}_G(S)$ gives rise to a modification

$$\mathcal{E}_{b,x}$$

of $\mathcal{E}_b$ à la Beauville and Laszlo whose construction is recalled briefly as follows. Write $S = \mathrm{Spa}(R, R^+)$, and $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp +})$. Let $x \in \mathrm{Gr}_G(S)$ correspond to the G-bundle $\mathcal{F}$ on $B_{\mathrm{dR}}^+(R^\sharp)$ together with a trivialization of G-bundles

$$\iota : G|_{\mathrm{Spec}(B_{\mathrm{dR}}(R^\sharp))} \xrightarrow{\sim} \mathcal{F}|_{\mathrm{Spec}(B_{\mathrm{dR}}(R^\sharp))}.$$

Recall that the completion of X_S along the Cartier divisor $\mathrm{Spec}(R^\sharp) \hookrightarrow X_S$ is given by the de Rham period ring $B_{\mathrm{dR}}^+(R^\sharp)$ (cf. [Far25, 1.33]). Moreover, the G-bundle $\mathcal{E}_b$ is canonically trivialized on $\mathrm{Spec}(B_{\mathrm{dR}}^+(R^\sharp))$ through the natural map $\mathrm{Spec}(B_{\mathrm{dR}}^+(R^\sharp)) \to X_S$ (cf. [CT22, Remark 1.7]).

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So by a theorem of Beauville and Laszlo (cf. [BL95]), we can glue $\mathcal{E}_b$ and the G-bundle $\mathcal{F}$ along the trivialization ι. In this way we obtain the G-bundle $\mathcal{E}_{b,x}$ on X_S.

Proposition 2.13. *Assume $S=\mathrm{Spa}(C,C^+)$. Let $x\in \mathrm{Gr}_{G,\mu}(S)$, and write $\mathcal{E}_{b,x}=\mathcal{E}_{b'}$, with $[b']\in B(G)$. Then $\kappa(b')=\kappa(b)-\mu^{\#}\in \pi_1(G)_\Gamma$.*

Proof. The $b=1$ case is proved in [CS17, Lemma 3.5.5]. For general b, the proof in *loc. cit.* also shows that we may reduce to the case $G=\mathbb{G}_m$ which can be done by explicit computation. □

Recall also the following useful fact.

Proposition 2.14 [CFS21, Lemma 2.6, Vie24, Lemma 3.10, She23, Lemma 6.6]. *Assume G is quasi-split. Let $M\subset G$ be a standard Levi subgroup. Suppose that (b_M,g) is a reduction of b to M, so $b_M\in M(\breve{F})$ and $g\in G(\breve{F})$ such that $b=gb_M\sigma(g)^{-1}$. For a geometric point $x\in \mathrm{Gr}_G(C,C^+)$, also viewed as an element of $\mathrm{Gr}_P(C,C^+)$ via the bijection*

$$\mathrm{Gr}_P(C,C^+)\xrightarrow{\sim}\mathrm{Gr}_G(C,C^+)$$

given by the Iwasawa decomposition, we have a natural isomorphism

$$(\mathcal{E}_{b,g\cdot x})_P\times^P M\simeq(\mathcal{E}_{b_M,x})_P\times^P M\simeq\mathcal{E}_{b_M,\mathrm{pr}_M(x)},$$

where $\mathrm{pr}_M:\mathrm{Gr}_P\to\mathrm{Gr}_M$ is the natural projection.

Proof. Indeed, [CFS21, Lemma 2.6] and [Vie24, Lemma 3.10] deals with the case b basic, while [She23, Lemma 6.6] deals with the general case. For the convenience of the readers, we still give the proof here. Since $b=gb_M\sigma(g)^{-1}$, the element g gives an isomorphism of G-bundles

$$\mathcal{E}_{b_M,x}\xrightarrow{\sim}\mathcal{E}_{b,g\cdot x}.$$

On the other hand, as $b_M\in M(\breve{F})$ the G-bundle $\mathcal{E}_{b_M,x}$ has a natural reduction to the P-bundle $(\mathcal{E}_{b_M,x})_P=\mathcal{E}^P_{b_M,x}$ where we view $x\in\mathrm{Gr}_G(C,C^+)$ as an element of $\mathrm{Gr}_P(C,C^+)$ using the Iwasawa decomposition, yielding a reduction $(\mathcal{E}_{b,g\cdot x})_P$ to P of $\mathcal{E}_{b,g\cdot x}$. By functoriality, $\mathcal{E}^P_{b_M,x}\times^P M\simeq\mathcal{E}_{b_M,\mathrm{pr}_M(x)}$, so we get the required isomorphisms

$$(\mathcal{E}_{b,g\cdot x})_P\times^P M\simeq(\mathcal{E}_{b_M,x})_P\times^P M\simeq\mathcal{E}_{b_M,\mathrm{pr}_M(x)}.$$ □

3. Harder–Narasimhan filtration of normed isocrystals

Let L be a field extension of $\breve{F}$. Recall that a filtered isocrystal over $L/\breve{F}$ consists of an isocrystal D over $\breve{F}|F$ together with a separated exhaustive decreasing filtration by L-subspaces of $D\otimes_{\breve{F}}L$. When $L/\breve{F}$ is of finite degree, Fontaine introduced the notion of weak admissibility for filtered isocrystals, and together with Colmez, they showed in [CF00] that the latter can be used to characterize those filtered isocrystals coming from crystalline representations. Recently, building on their discovery of the fundamental curve (that is, the Fargues–Fontaine curve), Fargues and Fontaine gave an elegant geometric proof of this classical result in [FF18, § 10.5.3]: a key observation is that, as $L/\breve{F}$ is finite, a filtration on $D\otimes_{\breve{F}}L$ corresponds naturally to a $\mathrm{Gal}(\bar{L}/L)$-invariant B^+_{dR}-lattice inside $D\otimes_{\breve{F}}B_{\mathrm{dR}}$, and hence one can apply the powerful theory of modifications of vector bundles on Fargues–Fontaine curves. Here B_{dR} and B^+_{dR} are the de Rham period rings corresponding to C, the p-adic completion of $\bar{L}$. However, such a correspondence between general filtrations and lattices does not exist in general. Hence, this will cause some trouble once we are working with filtered isocrystals in the geometric setting.

To overcome the difficulty mentioned above, a natural idea is that, instead of considering filtrations on $D\otimes_{\breve{F}}C$, we should work directly with the B^+_{dR}-lattices inside $D\otimes_{\breve{F}}B_{\mathrm{dR}}$. When D is semi-stable of slope 0, this point of view has been already adopted in the recent work of

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Cornut and Peche Irissarry [CPI19], Viehmann [Vie24], Nguyen and Viehmann [NV23] and Shen [She23]. In this work, we develop a theory of isocrystals equipped with a B^+_{dR}-lattice without restriction on the slope of the isocrystals. In particular, there exists a natural Harder–Narasimhan formalism for such objects. As the main result of this section, we show that the corresponding Harder–Narasimhan filtration is compatible with tensor products (Theorem 3.23). Later on we will apply this crucial result to construct the Harder–Narasimhan stratification on the B^+_{dR}-affine Grassmannian for a general reductive group.

3.1 The category $\mathbf{NIsoc}^{K}_{\breve{F}|F}$

Let F be a p-adic local field and $\breve{F}$ the completion of a maximal unramified extension of F, with σ the Frobenius F-automorphism of $\breve{F}$. Let K be a henselian discrete valuation field, such that its ring of integers $\mathcal{O}_K$ contains $\breve{F}$. Let $\pi \in \mathcal{O}_K$ be a uniformizer. Recall that a norm α on a finite-dimensional K-vector space $\mathcal{V}$ is said to be *splittable* if there exists a K-basis $\underline{e} = (e_1, \dots, e_r)$ of $\mathcal{V}$ such that

$$\alpha(v) = \max\{|\lambda_i|\alpha(e_i)\} \quad \text{for all } v = \sum \lambda_i e_i \in \mathcal{V}.$$

In this case we say that $\underline{e}$ is an *orthogonal basis* of $(\mathcal{V}, \alpha)$.

Remark 3.1. In [Cor18, § 5.2], Cornut considers normed vector spaces over a general henselian non-archimedean field. Here, for simplicity, we only consider those henselian non-archimedean fields which are discretely valued. In this setting, if K is moreover complete, by [BT84, Proposition 1.5 (i)], every norm on $\mathcal{V}$ is splittable.

Definition 3.2.

(i) A *K-normed isocrystal* over $\breve{F}|F$ is a triple (D, ϕ, α) consisting of an isocrystal $D = (D, \phi)$ over $\breve{F}|F$ equipped with a splittable K-norm α on the K-vector space $D \otimes_{\breve{F}} K$.

(ii) A morphism

$$f : (D_1, \phi_1, \alpha_1) \longrightarrow (D_2, \phi_2, \alpha_2)$$

of K-normed isocrystals is a morphism $f : D_1 \to D_2$ of $\breve{F}|F$-isocrystals such that

$$\alpha_2((f \otimes 1)(v)) \leq \alpha_1(v) \quad \text{for all } v \in D_{1,K} := D_1 \otimes_{\breve{F}} K.$$

Remark 3.3. If there is no possible confusion, a normed isocrystal (D, ϕ, α) is often simply denoted by (D, α), that is, we omit the Frobenius ϕ from the notation.

We denote by

$$\mathbf{NIsoc}^{K}_{\breve{F}|F}$$

the category of K-normed isocrystals over $\breve{F}|F$. Like the category $\mathbf{Norm}_K$ of normed K-vector spaces considered in [Cor18, 5.2], $\mathbf{NIsoc}^{K}_{\breve{F}|F}$ is a quasi-abelian $\otimes$-category. For example, the tensor product of two objects in $\mathbf{NIsoc}^{K}_{\breve{F}|F}$ is given by the formula

$$(D_1, \alpha_1) \otimes (D_2, \alpha_2) := (D_1 \otimes_{\breve{F}} D_2, \alpha_1 \otimes \alpha_2),$$

where $D_1 \otimes_{\breve{F}} D_2$ is the tensor product of isocrystals and, for $v \in (D_1 \otimes D_2)_K \simeq D_{1,K} \otimes_K D_{2,K}$,

$$(\alpha_1 \otimes \alpha_2)(v) := \min\left\{ \max_i\{\alpha_1(v_{1,i})\alpha_2(v_{2,i})\} \;\middle|\; v = \sum_i v_{1,i} \otimes v_{2,i} \in D_{1,K} \otimes D_{2,K} \right\}.$$

This formula indeed defines a splittable K-norm on $(D_1 \otimes D_2)_K$.

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3.2 Harder–Narasimhan filtration for normed isocrystals

There exists a theory of Harder–Narasimhan filtration for normed isocrystals. To see this, recall the following fundamental result.

Theorem 3.4. *Let $\mathcal{V}$ be a finite-dimensional K-vector space. Let α and β be two splittable norms on $\mathcal{V}$.*

(i) *There exists a K-basis $\underline{e}=(e_1,\dots,e_r)$ of $\mathcal{V}$ which is orthogonal for both of the norms α and β.*
(ii) *Write $\alpha(e_i)=|\pi|^{\lambda_i}$ and $\beta(e_i)=|\pi|^{\mu_i}$ for $1\le i\le r$. The quantity*
$$\nu(\alpha,\beta):=\sum_i(\mu_i-\lambda_i)$$
does not depend on the choice of the basis $\underline{e}$.
(iii) *For γ a third splittable norm on $\mathcal{V}$, we have*
$$\nu(\alpha,\gamma)=\nu(\alpha,\beta)+\nu(\beta,\gamma).$$

Proof. In our setting, K is supposed to be discretely valued, so the assertion (i) follows from [BT84, Proposition 1.26]. Moreover, for α a splittable norm on $\mathcal{V}$ and for $\underline{e}=(e_1,\dots,e_r)$ an orthogonal basis of $(\mathcal{V},\alpha)$, by [BT84, Proposition 1.9], the quantity
$$\mathrm{vol}(\alpha):=\mathrm{vol}(\underline{e})-\sum_i\lambda_i$$
does not depend on the choice of the basis $\underline{e}$. Here $\mathrm{vol}(\underline{e})$ is the volume of the lattice generated by the basis $\underline{e}$ of $\mathcal{V}$ relative to some fixed lattice of $\mathcal{V}$. In particular, we have
$$\nu(\alpha,\beta)=\mathrm{vol}(\alpha)-\mathrm{vol}(\beta),$$
from which we get immediately (ii) and (iii). □

Remark 3.5. As indicated in [Cor18, § 5.2.1], the previous theorem also holds for a general henselian non-archimedean base field. For example, (i) is confirmed in [BT84, Appendice, p. 300], while (ii) and (iii) are proved in [Cor20].

Definition 3.6. Let (D,α) be a normed isocrystal.

(i) The *rank* and the *degree* of (D,α) are defined respectively by
$$\mathrm{rank}(D,\alpha)=\dim_{\breve{F}}D,$$
$$\deg(D,\alpha)=\nu(\mathbf{o},\alpha)-\deg(D),$$
with $\mathbf{o}$ the splittable norm on D_K given by
$$\mathbf{o}(v)=\inf\{|\lambda|:\lambda\in K,v\in\lambda(D\otimes_{\breve{F}}\mathcal{O}_K)\subset D_K\},$$
and $\deg(D)$ the degree of the isocrystal $D=(D,\phi)$, that is, the integer $v_{\breve{F}}(\det\phi)$ with $v_{\breve{F}}$ the normalized additive valuation on $\breve{F}$.
(ii) When $D\neq0$, the *slope* $\rho(D,\alpha)$ of (D,α) is defined by the following ratio:
$$\rho(D,\alpha):=\frac{\deg(D,\alpha)}{\mathrm{rank}(D,\alpha)}\in\mathbb{R}.$$
(iii) Assume $D\neq0$ and write $\rho=\rho(D,\alpha)$. We say that the normed isocrystal (D,α) is *semi-stable of slope* ρ if, for any non-zero sub-isocrystal $D'\subset D$, the normed isocrystal $(D',\alpha|_{D'})$ is of slope $\le\rho$.

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Remark 3.7. Let (D,α) be a normed isocrystal over $\breve{F}|F$ with underlying $\breve{F}$-vector space V. Then $\nu(\mathbf{o},\alpha)$ is the degree (defined in [Cor18, § 5.2.4]) of the normed vector space (V,α) obtained from (D,α) by forgetting the Frobenius.

Remark 3.8. As in [Cor18, 5.2.4], the functions rank and deg are additive on short exact sequences and respectively constant and non-decreasing on mono-epis and, for

$$f:(D_1,\alpha_1)\longrightarrow(D_2,\alpha_2)$$

a mono-epis, f is an isomorphism of normed isocrystals if and only if

$$\deg(D_1,\alpha_1)=\deg(D_2,\alpha_2).$$

Consider the following faithful exact F-linear functor

$$\omega:\mathbf{NIsoc}^K_{\breve{F}|F}\longrightarrow\mathbf{Isoc}_{\breve{F}|F},\quad (D,\alpha)\mapsto D.$$

Here $\mathbf{Isoc}_{\breve{F}|F}$ is the category of isocrystals over $\breve{F}|F$. Like the case considered by Cornut in [Cor18], the functions rank and deg above give rise to a Harder–Narasimhan formalism on $\mathbf{NIsoc}^K_{\breve{F}|F}$. More precisely, for (D,α) a normed isocrystal over $\breve{F}|F$, there exists an exhaustive separated decreasing $\mathbb{R}$-filtration

$$\mathcal{F}_{\mathrm{HN}}(D,\alpha)=(\mathrm{Fil}^sD)_{s\in\mathbb{R}}$$

of D by sub-isocrystals such that, for each $s\in\mathbb{R}$,

$$\mathrm{Fil}^sD=\mathrm{Fil}^{s,-}D:=\bigcap_{s'<s}\mathrm{Fil}^{s'}D,$$

and that the isocrystal

$$\mathrm{gr}^s_{\mathcal{F}_{\mathrm{HN}}(D,\alpha)}D:=\mathrm{Fil}^sD/\mathrm{Fil}^{s+}D,$$

equipped with the norm induced from α, is either trivial or semi-stable of slope s. We call $\mathcal{F}_{\mathrm{HN}}(D,\alpha)$ the *Harder–Narasimhan filtration* of the normed isocrystal (D,α). Since the degree function on $\mathbf{NIsoc}^K_{\breve{F}|F}$ is additive on short exact sequences, we have

$$\deg(\mathcal{F}_{\mathrm{HN}}(D,\alpha)):=\sum_{s\in\mathbb{R}}s\dim_{\breve{F}}\mathrm{gr}^s_{\mathcal{F}_{\mathrm{HN}}(D,\alpha)}D=\deg(D,\alpha).$$

The proof of the following properties of Harder–Narasimhan filtration is standard. We refer to Example 3.13 below for the definition of the direct sum of two filtrations.

Proposition 3.9. *Let (D,α) and (D',α') be two normed isocrystals. Then*

$$\mathcal{F}_{\mathrm{HN}}(D\oplus D',\alpha\oplus\alpha')=\mathcal{F}_{\mathrm{HN}}(D,\alpha)\oplus\mathcal{F}_{\mathrm{HN}}(D',\alpha').$$

In other words, the Harder–Narasimhan filtration for normed isocrystals is compatible with direct sums.

3.3 Compatibility of the Harder–Narasimhan filtration with tensor products

Let

$$\mathbf{F}(\mathbf{Isoc}_{\breve{F}|F})$$

be the category of pairs $(D,\mathcal{F})$ with D an isocrystal over $\breve{F}|F$ and $\mathcal{F}=(\mathrm{Fil}^iD)_{i\in\mathbb{R}}$ a finite exhaustive separated decreasing $\mathbb{R}$-filtration of D by sub-isocrystals. The formalism of Harder–Narasimhan filtration for normed isocrystals provides a functor

$$\mathcal{F}_{\mathrm{HN}}:\mathbf{NIsoc}^K_{\breve{F}|F}\longrightarrow\mathbf{F}(\mathbf{Isoc}_{\breve{F}|F}),\quad (D,\alpha)\longmapsto\mathcal{F}_{\mathrm{HN}}(D,\alpha).\tag{3.1}$$

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Note that the category $\mathbf{F}(\mathbf{Isoc}_{\breve{F}|F})$ is naturally a quasi-abelian F-linear rigid $\otimes$-category, so one may wonder if the functor $\mathcal{F}_{\mathrm{HN}}$ is compatible with tensor products. The main result of this section is the following theorem.

Theorem 3.10. *The functor $\mathcal{F}_{\mathrm{HN}}$ in (3.1) is compatible with tensor products. In other words, for (D,α) and (D',α') two normed isocrystals, we have*

$$\mathcal{F}_{\mathrm{HN}}(D\otimes D,\alpha\otimes\alpha')=\mathcal{F}_{\mathrm{HN}}(D,\alpha)\otimes\mathcal{F}_{\mathrm{HN}}(D',\alpha').$$

We refer to Example 3.13 for the definition of the tensor product of two filtrations.

Remark 3.11. It is easy to verify that the assertion of the previous theorem is equivalent to the following: the tensor product of two semi-stable normed isocrystals is still semi-stable.

The analogue of the theorem above for normed vector spaces is due to Cornut (cf. [Cor18, Theorem 5.8]). Our proof of Theorem 3.10 given below is motivated by his proof of *loc. cit.*

3.3.1 *The vectorial Tits building for* $\mathbf{GL}_n$. Let L be a field and V a finite-dimensional L-vector space. Let $\mathbf{F}(V)$ be the set of separated exhaustive decreasing $\mathbb{R}$-filtrations $f=(V_f^a)_{a\in\mathbb{R}}$ of V by sub-spaces so that

$$V_f^a=V_f^{a,-}:=\bigcap_{a'<a}V_f^{a'}\quad\text{for all } a\in\mathbb{R}.$$

For such a filtration f of V we denote as usual by

$$\mathrm{gr}_f V:=\bigoplus_{a\in\mathbb{R}}\mathrm{gr}_f^a V$$

the associated graded L-vector space, where

$$\mathrm{gr}_f^a V:=V_f^a/V_f^{a,+},$$

with $V_f^{a,+}:=\bigcup_{a'>a}V_f^{a'}\subset V_f^a$.

Let $\underline{e}=(e_1,\dots,e_n)$ be an L-basis of V. We say that an $\mathbb{R}$-filtration $f\in\mathbf{F}(V)$ is split by $\underline{e}$ if each subspace V_f^a is spanned by a subset of $\{e_1,\dots,e_n\}$. We have a natural injective map

$$\mathbb{R}^n\longrightarrow\mathbf{F}(V)\tag{3.2}$$

sending $\alpha=(a_1,\dots,a_n)\in\mathbb{R}^n$ to the $\mathbb{R}$-filtration f_α of V given by setting

$$V_{f_\alpha}^a:=\mathrm{Span}\{e_i|a_i\geq a\}\subset V.$$

The image of (3.2) is an *apartment* of $\mathbf{F}(V)$, denoted by $\mathbf{F}(\underline{e})$. Hence, $\mathbf{F}(V)$ is the union of all its apartments. It is known that any two $\mathbb{R}$-filtrations of V can be split by the same L-basis, and thus they are contained in a common apartment.

Example 3.12. Let $f,g\in\mathbf{F}(V)$ and $\lambda\in\mathbb{R}_{>0}$. We can define two new elements $f+g$ and $\lambda\cdot f$ of $\mathbf{F}(V)$ by

$$V_{f+g}^a:=\sum_{b+c=a}V_f^b\bigcap V_g^c\quad\text{and}\quad V_{\lambda\cdot f}^a:=V_f^{\lambda^{-1}a}.$$

Note that the formula defining $f+g$ makes sense since the sets

$$\{V_f^b\mid b\in\mathbb{R}\}\quad\text{and}\quad\{V_g^c\mid c\in\mathbb{R}\}$$

consist of only finitely many subspaces of V. Moreover, if f,g are contained in the apartment $\mathbf{F}(\underline{e})$ for a certain L-basis, and we write $f=f_\alpha$ and $g=f_\beta$ for some $\alpha,\beta\in\mathbb{R}^n$, then $f+g=f_{\alpha+\beta}$ and $\lambda\cdot f=f_{\lambda\alpha}$.

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Example 3.13. Let V, V' be two L-vector spaces. Let f, f' be two $\mathbb{R}$-filtrations of V and V' respectively.

(i) The direct sum of f and g, denoted by $f \oplus g$, is the $\mathbb{R}$-filtration of $V \oplus V'$ given by setting

$$V^a_{f\oplus f'} := V^a_f \oplus V'^a_{f'} \subset V \oplus V', \quad \forall a \in \mathbb{R}.$$

Clearly,

$$\mathrm{gr}^a_{f\oplus f'}(V \oplus V') = \mathrm{gr}^a_f V \oplus \mathrm{gr}^a_{f'} V'.$$

(ii) The tensor product of f and g, denoted by $f \otimes g$, is the $\mathbb{R}$-filtration on $V \otimes V'$ given by setting (one checks easily that the formula below indeed makes sense)

$$V^a_{f\otimes f'} := \sum_{s+t=a} V^s_f \otimes V'^t_{f'} \subset V \otimes V', \quad \forall a \in \mathbb{R}.$$

As for the graded pieces of $f \otimes f'$, we have the following well-known description:

$$\mathrm{gr}^a_{f\otimes f'} = \bigoplus_{s+t=a} \mathrm{gr}^s_f V \otimes \mathrm{gr}^t_{f'} V'.$$

Moreover, if $\underline{e} = (e_1, \ldots, e_n)$ (respectively $\underline{e}' = (e'_1, \ldots, e'_m)$) is an L-basis of V (respectively of V') and $f = f_\alpha \in \mathbf{F}(\underline{e})$ (respectively $f' = f_\beta \in \mathbf{F}(\underline{e}')$) with $\alpha = (a_1, \ldots, a_n) \in \mathbb{R}^n$ (respectively $\beta = (b_1, \ldots, b_m) \in \mathbb{R}^m$), then

$$\underline{e} \otimes \underline{e}' := (e_i \otimes e'_j)_{i,j}$$

is an L-basis of $V \otimes V'$ and $f \otimes f' = f_\alpha \otimes f_\beta = f_{\alpha\otimes\beta} \in \mathbf{F}(\underline{e} \otimes \underline{e}')$, where

$$\alpha \otimes \beta := (a_i + b_j)_{i,j} \in \mathbb{R}^{nm}.$$

The set $\mathbf{F}(V)$ is naturally a metric space [Cor18, § 2.3]: let $f, g \in \mathbf{F}(V)$, and write

$$\langle f, g\rangle := \sum_{s,t\in\mathbb{R}} st \dim \mathrm{gr}^s_f(\mathrm{gr}^t_g V),$$
$$|f| := \sqrt{\langle f, f\rangle},$$
$$d(f, g) := \sqrt{|f|^2 + |g|^2 - 2\langle f, g\rangle}.$$

Here $\mathrm{gr}^s_f(\mathrm{gr}^t_g V)$ is the subquotient of the L-vector space $\mathrm{gr}^t_g V$ associated to the filtration induced from f.

Proposition 3.14. *The function $d(-.-)$ defines a metric on $\mathbf{F}(V)$, and the metric space $(\mathbf{F}(V), d)$ is complete.*

Proof. This can be viewed as the special case of some general results of Cornut. Indeed, as V is a finite-dimensional L-vector space, the set $\mathfrak{X}$ of all L-subspaces, equipped with the partial order $\leq$ given by the inclusion relation of subspaces, is a bounded modular lattice of finite length $\dim_K V$, with

$$W_1 \vee W_2 := W_1 + W_2 \quad \text{and} \quad W_1 \wedge W_2 := W_1 \cap W_2.$$

Moreover, the set $\mathbf{F}(V)$ is naturally identified with the set $\mathbf{F}(\mathfrak{X})$ of all $\mathbb{R}$-filtrations on $\mathfrak{X}$ considered in [Cor18, § 2.2], and the map

$$\mathfrak{X} \longrightarrow \mathbb{R}, \quad W \longmapsto \dim_L W$$

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gives a rank function (in the sense of [Cor18, § 2.1.3]) on $\mathcal{X}$. So our proposition follows from the corresponding general statements for bounded modular lattices of finite length established in [Cor18, Propositions 2.6 and 2.7]. □

Remark 3.15. Let $\underline{e}$ be an L-basis of V. Restricting to the apartment $\mathbf{F}(\underline{e})$, identified with $\mathbb{R}^n$ via (3.2), the pairing above is just the standard scalar product $\langle -, - \rangle$ in $\mathbb{R}^n$, i.e.,

$$\langle f_\alpha, f_\beta \rangle = \langle \alpha, \beta \rangle, \quad \forall \alpha, \beta \in \mathbb{R}^n.$$

In particular, for $f, g, h \in \mathbf{F}(\underline{e})$, we have

$$\langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle.$$

In general, the pairing $\langle -, - \rangle$ is only concave [Cor18, Lemma 2.5]: we have

$$\langle f, g + h \rangle \geq \langle f, g \rangle + \langle f, h \rangle, \quad \forall f, g, h \in \mathbf{F}(V). \tag{3.3}$$

Let $C \subset \mathbf{F}(V)$ be a closed subset. Assume moreover that it is *convex*, i.e., for any $f, g \in C$,

$$\lambda \cdot f + (1 - \lambda) \cdot g \in C, \quad \forall \lambda \in [0, 1].$$

Then, for every $f \in \mathbf{F}(V)$, there exists a unique $p_C(f) \in C$, called the *convex projection* of f to C, such that

$$d(f, p_C(f)) = \min\{d(f, g) \mid g \in C\}.$$

Lemma 3.16. *Let $f \in \mathbf{F}(V)$, with convex projection $p_C(f) \in C$. We have*

$$|p_C(f)| \leq |f|.$$

Moreover, if $g \in \mathbf{F}(V)$ so that $p_C(f) + t \cdot g \in C$ for sufficiently small positive real t, we have

$$\langle p_C(f), g \rangle \geq \langle f, g \rangle.$$

Proof. The inequality $|p_C(f)| \leq |f|$ follows from the fact that the convex projection p_C is non-expanding [BH99, II.2.4]. The second assertion is also well known and let us recall the proof for the sake of completeness. Note that, for $t \in \mathbb{R}_{>0}$ with $p_C(f) + tg \in C$,

$$\begin{aligned}
d(f, p_C(f))^2 &\leq d(f, p_C(f) + tg)^2 \\
&= |f|^2 + |p_C(f) + tg|^2 - 2\langle f, p_C(f) + tg \rangle \\
&= |f|^2 + |p_C(f)|^2 + |g|^2 t^2 + 2\langle p_C(f), g \rangle t - 2\langle f, p_C(f) + tg \rangle \\
&\leq |f|^2 + |p_C(f)|^2 + |g|^2 t^2 + 2\langle p_C(f), g \rangle t - 2\langle f, p_C(f) \rangle - 2\langle f, tg \rangle \\
&= |f|^2 + |p_C(f)|^2 + |g|^2 t^2 + 2\langle p_C(f), g \rangle t - 2\langle f, p_C(f) \rangle - 2t\langle f, g \rangle \\
&= |g|^2 t^2 + 2\left(\langle p_C(f), g \rangle - \langle f, g \rangle\right) t + |f|^2 + |p_C(f)|^2 - 2\langle f, p_C(f) \rangle \\
&= |g|^2 t^2 + 2\left(\langle p_C(f), g \rangle - \langle f, g \rangle\right) t + d(f, p_C(f))^2.
\end{aligned}$$

Here the second inequality follows from the fact that the pairing $\langle -, - \rangle$ is concave (cf. (3.3)). Since these inequalities hold for any sufficiently small positive real number t, it follows that the polynomial

$$|g|^2 t^2 + 2\left(\langle p_C(f), g \rangle - \langle f, g \rangle\right) t,$$

attains its minimum at some $t_0 \leq 0$. As a result, we find $\langle p_C(f), g \rangle \geq \langle f, g \rangle$, as required. □

3.3.2 *Vectorial Tits building and Harder–Narasimhan filtration.* We use the notation fixed at the beginning of § 3.1. Let (D, α) be a normed isocrystal over K/F, with V the underlying $\breve{F}$-vector space. Let $\mathbf{F}(D) \subset \mathbf{F}(V)$ be the subset of $\mathbb{R}$-filtrations of D by sub-isocrystals. As in the

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proof of Proposition 3.14, the set of all sub-isocrystals of D, equipped with the partial order $\leq$ given by the inclusion relation, is naturally a bounded modular lattice of finite length $\leq \dim_{\breve{F}} D$. So, by [Cor18, Proposition 2.7], the subset $\mathbf{F}(D)$, equipped with the distance function induced from that on $\mathbf{F}(V)$ is complete, and thus is closed in $\mathbf{F}(D)$. The norm α on $D \otimes_{\breve{F}} K$ defines a map

$$\langle \alpha, - \rangle : \mathbf{F}(D) \longrightarrow \mathbb{R}, \quad f \longmapsto \langle \alpha, f \rangle = \sum_{s \in \mathbb{R}} s \cdot \deg(\mathrm{gr}_f^s D, \alpha).$$

Using the vectorial Tits building, the Harder–Narasimhan filtration for the normed isocrystal (D, α) has the following characterization.

Proposition 3.17 [Cor18, Proposition 2.12]. *The Harder–Narasimhan filtration for (D, α) is the unique filtration $\mathcal{F} \in \mathbf{F}(D)$ such that*

$$|\mathcal{F}|^2 - 2\langle \alpha, \mathcal{F} \rangle \leq |f|^2 - 2\langle \alpha, f \rangle$$

for any $f \in \mathbf{F}(D)$.

On can slightly strengthen the above inequality.

Corollary 3.18. *Let (D, α) be a normed isocrystal, with $\mathcal{F} = \mathcal{F}_{\mathrm{HN}}(D, \alpha)$ its Harder–Narasimhan filtration. Then, for any $f \in \mathbf{F}(D)$,*

$$|\mathcal{F}|^2 - 2\langle \alpha, \mathcal{F} \rangle \leq |f|^2 - 2\langle \alpha, f \rangle - \frac{(\deg(f) - \deg(D, \alpha))^2}{\dim_{\breve{F}} D}.$$

Here $\deg(f)$ denotes the degree of the filtration f, i.e.,

$$\deg(f) := \sum_{a \in \mathbb{R}} a \cdot \dim_{\breve{F}} \mathrm{gr}_f^a D.$$

Proof. For $r \in \mathbb{R}$, let f_r be the $\mathbb{R}$-filtration on D with

$$D_{f_r}^a := D_f^{a-r} \quad \text{for all } a \in \mathbb{R}.$$

Then

$$\begin{aligned} |\mathcal{F}|^2 - 2\langle \alpha, \mathcal{F} \rangle &\leq |f_r|^2 - 2\langle \alpha, f_r \rangle \\ &= \sum_{a \in \mathbb{R}} (a + r)^2 \dim_{\breve{F}} \mathrm{gr}_f^a D - 2 \sum_{a \in \mathbb{R}} (a + r) \deg(\mathrm{gr}_f^a D, \alpha). \\ &= r^2 \cdot \dim_{\breve{F}} D + 2r \cdot (\deg(f) - \deg(D, \alpha)) + |f|^2 - 2\langle \alpha, f \rangle. \end{aligned}$$

Since this holds for any r, taking $r = -(\deg(f) - \deg(D, \alpha))/\dim_{\breve{F}} D$, we obtain the desired inequality. □

3.3.3 *Compatibility of Harder–Narasimhan filtration for normed isocrystals with tensor products.* Let (D, α) and (D', α') be two normed isocrystals on K/F, with underlying $\breve{F}$-vector spaces V and V', respectively. Consider the isocrystal

$$E = D \oplus D' \oplus (D \otimes D')$$

equipped with the norm β on E_K induced from α and α'. We have the following map:

$$\mathbf{F}(D) \times \mathbf{F}(D') \longrightarrow \mathbf{F}(E), \quad (f, f') \mapsto f \oplus f' \oplus (f \otimes f').$$

Lemma 3.19. *The above map is injective and identifies $\mathbf{F}(D) \times \mathbf{F}(D')$ as a closed convex subset of $\mathbf{F}(E)$.*

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Proof. The injectivity of the above map is clear. So we can view $\mathbf{F}(D)\times\mathbf{F}(D')$ as a subset of $\mathbf{F}(E)$. To check that $\mathbf{F}(D)\times\mathbf{F}(D')$ is convex in $\mathbf{F}(E)$, we claim that, for (f,f') and (g,g') in $\mathbf{F}(D)\times\mathbf{F}(D')$,

$$(f+g)\otimes(f'+g')=f\otimes f'+g\otimes g'\in\mathbf{F}(D\otimes D').$$

For this, let $\underline{e}=(e_1,\dots,e_n)$ (respectively $\underline{e}'=(e'_1,\dots,e'_m)$) be an $\breve{F}$-basis that split f and g (respectively f' and g') at the same time, with the corresponding n-tuples (respectively m-tuples) $\alpha=(a_1,\dots,a_n)$ and $\beta=(b_1,\dots,b_n)$ (respectively $\alpha'=(a'_1,\dots,a'_m)$ and $\beta'=(b'_1,\dots,b'_m)$). Then the $\breve{F}$-basis $(e_i\otimes e'_j)_{i,j}$ splits both the filtrations $(f+g)\otimes(f'+g')$ and $f\otimes f'+g\otimes g'$, and the corresponding mn-tuples are the same, given by

$$((a_i+b_i)+(a'_j+b'_j))_{i,j}=((a_i+a'_j)+(b_i+b'_j))_{i,j},$$

giving our claim.

It remains to check that $\mathbf{F}(D)\times\mathbf{F}(D')\subset\mathbf{F}(E)$ is closed. Let (f,f') and (g,g') be two elements in $\mathbf{F}(D)\times\mathbf{F}(D')$. Viewed as two elements in $\mathbf{F}(E)$, their distance is

$$\sqrt{d(f,g)^2+d(f',g')^2+d(f\otimes f',g\otimes g')^2}.$$

On the other hand, consider the $\breve{F}$-bases $\underline{e}$ and $\underline{e}'$, and the tuples α,α',β and β' as above, so we then have

$$\begin{aligned}d(f\otimes f',g\otimes g')^2&=\sum_{i=1}^{n}\sum_{j=1}^{m}(a_i+a'_j-b_i-b'_j)^2\\&\le\sum_{i=1}^{n}\sum_{j=1}^{m}2\left((a_i-b_i)^2+(a'_j-b'_j)^2\right)\\&=2m\cdot d(f,g)^2+2n\cdot d(f',g'),\end{aligned}$$

with $m=\dim_{\breve{F}}D'$ and $n=\dim_{\breve{F}}D$. Consequently,

$$\begin{aligned}\sqrt{d(f,g)^2+d(f',g')^2}&\le\sqrt{d(f,g)^2+d(f',g')^2+d(f\otimes f',g\otimes g')^2}\\&\le\sqrt{(2m+1)d(f,g)^2+(2n+1)d(f',g')^2}.\end{aligned}$$

In particular, the distance function on $\mathbf{F}(D)\times\mathbf{F}(D')$ induced from the distance function on $\mathbf{F}(E)$ is equivalent to the product of the distance functions on $\mathbf{F}(D)$ and on $\mathbf{F}(D')$. Since $(\mathbf{F}(D),d)$ and $(\mathbf{F}(D'),d)$ are complete, it follows that, as a subset of $\mathbf{F}(E)$, $\mathbf{F}(D)\times\mathbf{F}(D')$ is also complete, and thus is closed, as required. □

As a result, we can consider the convex projection of $\mathbf{F}(E)$ to the closed convex subset $\mathbf{F}(D)\times\mathbf{F}(D')$, denoted by

$$p:\mathbf{F}(E)\longrightarrow\mathbf{F}(D)\times\mathbf{F}(D')$$

in what follows. Recall that the normed isocrystals have underlying $\breve{F}$-vector spaces V and V', respectively. Let

$$W=V\oplus V'\oplus(V\otimes V')$$

be the underlying $\breve{F}$-vector space of the isocrystal E. A similar argument shows that the natural map

$$\mathbf{F}(V)\times\mathbf{F}(V')\longrightarrow F(W),\quad(f,f')\longmapsto f\oplus f'\oplus(f\otimes f')\tag{3.4}$$

identifies $\mathbf{F}(V)\times\mathbf{F}(V')$ as a closed convex subset of $\mathbf{F}(W)$. Let us denote the corresponding convex projection by q.

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Lemma 3.20. *The following square is commutative.*

$$\begin{array}{ccc} \mathbf{F}(E) & \xrightarrow{p} & \mathbf{F}(D)\times\mathbf{F}(D') \\ \downarrow & & \downarrow \\ \mathbf{F}(W) & \xrightarrow{q} & \mathbf{F}(V)\times\mathbf{F}(V') \end{array}$$

Proof. By the uniqueness of convex projection, it suffices to check that, for $f\in\mathbf{F}(E)$ viewed as an element in $\mathbf{F}(W)$, its convex projection $q(f)\in\mathbf{F}(V)\times\mathbf{F}(W)$ is an $\mathbb{R}$-filtration by sub-isocrystals. To show this, by abuse of notation, we denote by ϕ the Frobenius on the related isocrystals in the proof. So for $\mathcal{D}\in\{D,D',E\}$, with underlying $\breve{F}$-vector space $\mathcal{V}$, we have an isomorphism of $\breve{F}$-spaces

$$\phi\otimes 1:\mathcal{V}\otimes_{\breve{F},\sigma}\breve{F}\longrightarrow\mathcal{V}.$$

In particular, for $g\in\mathbf{F}(\mathcal{V})$, we obtain a filtration $\phi(g)$ of $\mathcal{V}$ by setting

$$\mathcal{V}^a_{\phi(g)}=(\phi\otimes 1)(\mathcal{V}^a_g)\subset\mathcal{V}.$$

As the Frobenius map ϕ is bijective, we obtain the bijection

$$\phi:\mathbf{F}(\mathcal{V})\to\mathbf{F}(\mathcal{V})$$

and $\mathbf{F}(\mathcal{D})\subset\mathbf{F}(\mathcal{V})$ is precisely the subset of elements fixed by ϕ. For g' a second filtration of $\mathbf{F}(\mathcal{V})$, we have

$$\langle g,g'\rangle=\langle\phi(g),\phi(g')\rangle,$$

and thus $d(\phi(g),\phi(g'))=d(g,g')$. Moreover, the map (3.4) is equivariant with respect to the bijection ϕ on both sides. It follows that, for $f\in\mathbf{F}(E)\subset\mathbf{F}(W)$, $q(f)$ and $\phi(q(f))$ are two elements in $\mathbf{F}(V)\times\mathbf{F}(V')$ minimizing the function

$$\mathbf{F}(V)\times\mathbf{F}(V')\ni g\longmapsto d(f,g).$$

As a result, $\phi(q(f))=q(f)\in\mathbf{F}(V)\times\mathbf{F}(V')$ and thus $q(f)\in\mathbf{F}(D)\times\mathbf{F}(D')$, as required. □

We are now ready to prove the main result of this section.

Proof of Theorem 3.10. Recall that (D,α) and (D',α') are two normed isocrystals on K/F, and

$$(E,\beta):=(D,\alpha)\oplus(D',\alpha')\oplus((D,\alpha)\otimes(D',\alpha')).$$

Let

$$\mathcal{F}=(\mathcal{F}_{\mathrm{HN}}(D,\alpha),\mathcal{F}_{\mathrm{HN}}(D',\alpha'))\in\mathbf{F}(D)\times\mathbf{F}(D'),$$

and view it as the element

$$\mathcal{F}_{\mathrm{HN}}(D,\alpha)\oplus\mathcal{F}_{\mathrm{HN}}(D',\alpha')\oplus(\mathcal{F}_{\mathrm{HN}}(D,\alpha)\otimes\mathcal{F}_{\mathrm{HN}}(D',\alpha'))$$

in $\mathbf{F}(E)$. We want to show that this coincides with the Harder–Narasimhan filtration $\mathcal{F}_{\mathrm{HN}}(E,\beta)$ for (E,β). For this, it suffices to check that $\mathcal{F}$ satisfies the inequality characterizing the Harder–Narasimhan filtration $\mathcal{F}_{\mathrm{HN}}(E,\beta)$:

$$|\mathcal{F}|^2-2\langle\beta,\mathcal{F}\rangle\le|f|^2-2\langle\beta,f\rangle,\quad\forall f\in\mathbf{F}(E).\tag{3.5}$$

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For the moment, let us check the inequality (3.5) for $f=(g,g')\in\mathbf{F}(D)\times\mathbf{F}(D')\subset\mathbf{F}(E)$. Firstly,

$$\begin{aligned}|f|^2&=\sum_{a\in\mathbb{R}}a^2\dim\mathrm{gr}_f^a(D\oplus D'\oplus D\otimes D')\\&=\sum_{a\in\mathbb{R}}a^2(\dim\mathrm{gr}_g^a(D)\oplus\mathrm{gr}_{g'}^a(D')\oplus\mathrm{gr}_{g\otimes g'}^a(D\otimes D')))\\&=|g|^2+|g'|^2+\sum_a a^2\dim\Big(\bigoplus_{r+s=a}\mathrm{gr}_g^rD\otimes\mathrm{gr}_{g'}^sD'\Big)\\&=|g|^2+|g'|^2+\sum_{r,s\in\mathbb{R}}(r+s)^2\dim(\mathrm{gr}_g^rD)\cdot\dim(\mathrm{gr}_{g'}^sD')\\&=(\dim D'+1)|g|^2+(\dim D+1)|g'|^2+2\deg(g)\deg(g').\end{aligned}$$

Secondly,

$$\begin{aligned}\langle\alpha\otimes\alpha',g\otimes g'\rangle&=\sum_{a\in\mathbb{R}}a\cdot\deg(\mathrm{gr}_{g\otimes g'}^a,\alpha\otimes\alpha')\\&=\sum_{r,s\in\mathbb{R}}(r+s)\deg(\mathrm{gr}_g^rD\otimes\mathrm{gr}_{g'}^sD',\alpha\otimes\alpha')\\&=\sum_{r,s\in\mathbb{R}}(r+s)(\deg(\mathrm{gr}_g^rD,\alpha)\cdot\dim(\mathrm{gr}_{g'}^sD')+\deg(\mathrm{gr}_{g'}^sD',\alpha')\cdot\dim(\mathrm{gr}_g^rD))\\&=\langle\alpha,g\rangle\dim D'+\langle\alpha',g'\rangle\dim D+\deg(D,\alpha)\deg(g')+\deg(D',\alpha')\deg(g),\end{aligned}$$

yielding

$$\begin{aligned}\langle\beta,f\rangle&=\langle\alpha,g\rangle+\langle\alpha',g'\rangle+\langle\alpha\otimes\alpha',g\otimes g'\rangle\\&=\langle\alpha,g\rangle(\dim D'+1)+\langle\alpha',g'\rangle(\dim D+1)+\deg(D,\alpha)\deg(g')+\deg(D',\alpha')\deg(g).\end{aligned}$$

So

$$\begin{aligned}|f|^2-2\langle\beta,f\rangle=&(\dim D'+1)(|g|^2-2\langle\alpha,g\rangle)+(\dim D+1)(|g'|^2-2\langle\alpha',g'\rangle)\\&+2\deg(g)\deg(g')-2\deg(D,\alpha)\deg(g')-2\deg(D',\alpha')\deg(g).\end{aligned}$$

Applying this equality to $f=\mathcal{F}$, and noticing that

$$\deg(\mathcal{F}_{\mathrm{HN}}(D,\alpha))=\deg(D,\alpha)\quad\text{and}\quad\deg(\mathcal{F}_{\mathrm{HN}}(D',\alpha'))=\deg(D',\alpha'),$$

we find that

$$\begin{aligned}|\mathcal{F}|^2-2\langle\beta,\mathcal{F}\rangle=&(\dim D'+1)(|\mathcal{F}_{\mathrm{HN}}(D,\alpha)|^2-2\langle\alpha,\mathcal{F}_{\mathrm{HN}}(D,\alpha)\rangle)\\&+(\dim D+1)(|\mathcal{F}_{\mathrm{HN}}(D',\alpha')|^2-2\langle\alpha',\mathcal{F}_{\mathrm{HN}}(D',\alpha')\rangle)\\&-2\deg(D,\alpha)\deg(D',\alpha').\end{aligned}$$

Using Corollary 3.18, we deduce

$$\begin{aligned}&|\mathcal{F}|^2-2\langle\beta,\mathcal{F}\rangle-(|f|^2-2\langle\beta,f\rangle)\\&\quad\le-\left(\frac{\dim D'+1}{\dim D}(\deg(g)-\deg(D,\alpha))^2+\frac{\dim D+1}{\dim D'}(\deg(g')-\deg(D',\alpha'))^2\right)\\&\qquad-2(\deg(g)-\deg(D,\alpha))(\deg(g')-\deg(D',\alpha'))\end{aligned}$$

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$$\begin{aligned}&\leq -\left(\frac{\dim D'}{\dim D}(\deg(g)-\deg(D,\alpha))^2+\frac{\dim D}{\dim D'}(\deg(g')-\deg(D',\alpha'))^2\right)\\&\quad -2(\deg(g)-\deg(D,\alpha))(\deg(g')-\deg(D',\alpha'))\\&\leq 0.\end{aligned}$$

Here, for the last inequality, we use the classical inequality

$$a^2+b^2\geq 2ab,\quad \forall a,b\in\mathbb{R}.$$

As a result, we see that the inequality (3.5) holds once $f\in\mathbf{F}(D)\times\mathbf{F}(D')\subset\mathbf{F}(E)$.

For general $f\in\mathbf{F}(E)$, consider its convex projection $p(f)$ to $\mathbf{F}(D)\times\mathbf{F}(D')$. According to Lemma 3.16, we have

$$|p(f)|^2\leq|f|^2.$$

On the other hand, viewing f as an $\mathbb{R}$-filtration of the underlying $\breve{F}$-vector space W of E, we have by Remark 3.7 the following equality

$$\deg(\mathrm{gr}_f^a E,\beta)=\deg(\mathrm{gr}_f^a W,\beta)-\deg(\mathrm{gr}_f^a E),$$

where the first term on the right-hand side is the degree of the normed vector space $\mathrm{gr}_f^a W$, equipped with the norm induced from β, and the second term $\deg(\mathrm{gr}_f^a E)$ is the degree of the isocrystsal $\mathrm{gr}_f^a W$. Hence

$$\begin{aligned}\langle\beta,f\rangle&=\sum_{a\in\mathbb{R}}a\cdot\deg(\mathrm{gr}_f^a E,\beta)\\&=\sum_{a\in\mathbb{R}}a\cdot\deg(\mathrm{gr}_f^a W,\beta)-\sum_a a\cdot\deg(\mathrm{gr}_f^a E).\end{aligned}$$

Now the first sum

$$\langle\beta,f\rangle':=\sum_a a\cdot\deg(\mathrm{gr}_f^a W,\beta)$$

has been already considered by Cornut in [Cor18, § 5.2]. As $p(f)$ is also the convex projection of $f\in\mathbf{F}(E)\subset\mathbf{F}(W)$ to $\mathbf{F}(V)\times\mathbf{F}(V')\subset\mathbf{F}(W)$ according to Lemma 3.20, by [Cor18, Proposition 5.6 and § 5.2.13], we have

$$\langle\beta,f\rangle'\leq\langle\beta,p(f)\rangle'.\tag{3.6}$$

Furthermore, if we write $\mathcal{F}_{\mathrm{N}}\in\mathbf{F}(E)$ as the Newton filtration on the isocrystal E and $\mathcal{F}_{\mathrm{N}}^*$ its opposite (defined from the slope decomposition), it follows that

$$\sum_a a\cdot\deg(\mathrm{gr}_f^a E)=\langle\mathcal{F}_{\mathrm{N}},f\rangle=-\langle\mathcal{F}_{\mathrm{N}}^*,f\rangle.$$

Clearly, $\mathcal{F}_{\mathrm{N}},\mathcal{F}_{\mathrm{N}}^*\in\mathbf{F}(D)\times\mathbf{F}(D')\subset\mathbf{F}(E)$, and both $t\cdot\mathcal{F}_N+p(f)$ and $t\cdot\mathcal{F}_{\mathrm{N}}^*+p(f)$ are contained in $\mathbf{F}(D)\times\mathbf{F}(D')$ for any $t\in\mathbb{R}_{>0}$ sufficiently close to 0. By Lemma 3.16 we have

$$\langle\mathcal{F}_{\mathrm{N}},f\rangle\leq\langle\mathcal{F}_{\mathrm{N}},p(f)\rangle\quad\text{and}\quad\langle\mathcal{F}_{\mathrm{N}}^*,f\rangle\leq\langle\mathcal{F}_{\mathrm{N}}^*,p(f)\rangle,$$

and thus an equality

$$\langle\mathcal{F}_{\mathrm{N}},f\rangle=\langle\mathcal{F}_{\mathrm{N}},p(f)\rangle.$$

Combining with the inequality (3.6), we find

$$\langle\beta,f\rangle=\langle\beta,f\rangle'-\langle\mathcal{F}_{\mathrm{N}},f\rangle\leq\langle\beta,p(f)\rangle'-\langle\mathcal{F}_{\mathrm{N}},p(f)\rangle=\langle\beta,p(f)\rangle,$$

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and thus

$$\begin{aligned}|\mathcal{F}|^2 - 2\langle\beta, \mathcal{F}\rangle &\leq |p(f)|^2 - 2\langle\beta, p(f)\rangle \\ &\leq |f|^2 - 2\langle\beta, f\rangle\end{aligned}$$

for any $f \in \mathbf{F}(E)$, where the first inequality follows from the previous step. Therefore, we find

$$\mathcal{F}_{\mathrm{HN}}(D,\alpha) \oplus \mathcal{F}_{\mathrm{HN}}(D',\alpha') \oplus (\mathcal{F}_{\mathrm{HN}}(D,\alpha) \otimes \mathcal{F}_{\mathrm{HN}}(D',\alpha')) = \mathcal{F}_{\mathrm{HN}}(E,\beta),$$

while

$$\mathcal{F}_{\mathrm{HN}}(E,\beta) = \mathcal{F}_{\mathrm{HN}}(D,\alpha) \oplus \mathcal{F}_{\mathrm{HN}}(D',\alpha') \oplus \mathcal{F}_{\mathrm{HN}}(D \otimes D', \alpha \otimes \alpha')$$

by Proposition 3.9. In other words,

$$\mathcal{F}_{\mathrm{HN}}(D,\alpha) \otimes \mathcal{F}_{\mathrm{HN}}(D',\alpha') = \mathcal{F}_{\mathrm{HN}}(D \otimes D', \alpha \otimes \alpha'),$$

and the functor (3.1) is compatible with tensor products. □

3.4 A variant

Let

$$\mathbf{BunIsoc}^K_{\breve{F}|F}$$

be the quasi-abelian F-linear $\otimes$-category whose objects are pairs (D, Ξ), where D is an isocrystal over $\breve{F}|F$ and $\Xi \subset D \otimes_{\breve{F}} K$ is a lattice. A morphism

$$f : (D_1, \Xi_1) \longrightarrow (D_2, \Xi_2)$$

in $\mathbf{BunIsoc}^K_{\breve{F}|F}$ consists of a morphism $f : D_1 \to D_2$ of isocrystals with

$$(f \otimes 1)(\Xi_1) \subseteq \Xi_2.$$

As in [Cor18, 5.2.3], there is a natural F-linear exact $\otimes$-functor

$$\mathbf{BunIsoc}^K_{\breve{F}|F} \longrightarrow \mathbf{NIsoc}^K_{\breve{F}|F}, \quad (D,\Xi) \mapsto (D, \alpha_\Xi). \tag{3.7}$$

Here α_Ξ denotes the *gauge norm* of the lattice $\Xi \subset D \otimes_{\breve{F}} K$: for $v \in D \otimes_{\breve{F}} K$,

$$\alpha_\Xi(v) := \inf\{|\lambda| : v \in \lambda\Xi\}.$$

For example, for the lattice $\Xi_0 := D \otimes_{\breve{F}} \mathcal{O}_K$, the gauge norm α_{Ξ_0} is the norm $\mathbf{o}$ used in the definition of the degree of a normed isocrystal (Definition 3.6).

Remark 3.21. The functor (3.7) above identifies $\mathbf{BunIsoc}^K_{\breve{F}|F}$ with a full subcategory of $\mathbf{NIsoc}^K_{\breve{F}|F}$, made of those (D,α) such that $\alpha(D \otimes_{\breve{F}} K) \subset |K|$, which is stable under strict sub-objects and quotients.

For (D,Ξ) an object in $\mathbf{BunIsoc}^K_{\breve{F}|F}$, define

$$\mathrm{rank}(D,\Xi) := \dim_{\breve{F}} D \quad \text{and} \quad \deg(D,\Xi) := \deg(D,\alpha_\Xi).$$

Remark 3.22. Let (D,Ξ) be an object in $\mathbf{BunIsoc}^K_{\breve{F}|F}$. Let $\mathcal{F}_\Xi$ be the *residue filtration* on $D \otimes_{\breve{F}} C$ defined by the lattice Ξ, with C the residue field of K: for $i \in \mathbb{Z}$,

$$\mathcal{F}^i_\Xi(D \otimes C) = \frac{(\pi^i\Xi) \cap \Xi_0 + \pi\Xi_0}{\pi\Xi_0} \hookrightarrow D \otimes_{\breve{F}} C.$$

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Here $\Xi_0 = D \otimes_{\breve{F}} \mathcal{O}_K$. In particular, we obtain a filtered isocrystal $(D, \mathcal{F}_\Xi)$, and this construction yields a functor

$$\mathbf{BunIsoc}^K_{\breve{F}|F} \longrightarrow \mathbf{FilIsoc}^C_{\breve{F}|F}, \quad (D, \Xi) \longmapsto (D, \mathcal{F}_\Xi).$$

By the adapted basis theorem, there exists a basis $\underline{e} = (e_1, \dots, e_r)$ of the free $\mathcal{O}_K$-module $\Xi_0 = D \otimes_{\breve{F}} \mathcal{O}_K$, and elements $\lambda_1, \dots, \lambda_r \in \mathbb{Z}$ such that $(\pi^{\lambda_1} e_1, \dots, \pi^{\lambda_r} e_r)$ forms an $\mathcal{O}_K$-basis of Ξ. Then $\underline{e}$ is an orthogonal basis for α_Ξ and for $\mathbf{o} = \alpha_{\Xi_0}$ at the same time. Moreover,

$$\alpha_\Xi(e_i) = |\pi|^{-\lambda_i} \quad \text{and} \quad \alpha_\Xi(e_i) = 1, \quad i \in \mathbb{Z}.$$

So

$$\deg(D, \Xi) = -\sum_i \lambda_i - \dim(D) = \deg(\mathcal{F}_\Xi) - \dim(D) = \deg(D, \mathcal{F}_\Xi).$$

In other words, $\deg(D, \Xi)$ is the same as the degree of the filtered isocrystal $(D, \mathcal{F}_\Xi)$. However, we caution the readers that, for $D' \subset D$ a sub-isocrystal, setting $\mathcal{F}_\Xi|_{D'_C}$ to be the filtration on $D'_C = D' \otimes_{\breve{F}} C$ induced from $\mathcal{F}_\Xi$, we then have in general

$$\deg(D', \Xi \cap (D' \otimes_{\breve{F}} K)) \neq \deg(D', \mathcal{F}_\Xi|_{D'_C}).$$

See [She23, Example 2.1] for an explicit example of Viehmann. As a result, the Harder–Narasimhan filtration on D defined by the lattice Ξ as we consider here differs from the one defined by the residue filtration $\mathcal{F}_\Xi$. We will come back to this point later in § 5.4.

Theorem 3.10 combined with Remark 3.21 give the following result.

Theorem 3.23. *The functions* rank *and* deg *above on* $\mathbf{BunIsoc}^K_{\breve{F}/F}$ *induce a Harder–Narasimhan formalism on* $\mathbf{BunIsoc}^K_{\breve{F}|F}$, *whose Harder–Narasimhan filtration*

$$\mathcal{F}_{\mathrm{HN}} : \mathbf{BunIsoc}^K_{\breve{F}|F} \longrightarrow \mathbf{F}(\mathrm{Isoc}_{\breve{F}|F}) \tag{3.8}$$

is induced from the Harder–Narasimhan filtration $\mathcal{F}_{\mathrm{HN}}$ *on* $\mathbf{NIsoc}^K_{\breve{F}|F}$ *in (3.1). Moreover,* $\mathcal{F}_{\mathrm{HN}}$ *is compatible with tensor products, or, equivalently, the tensor product of two semi-stable objects in* $\mathbf{BunIsoc}^K_{\breve{F}|F}$ *is still semi-stable.*

Remark 3.24. The similar result of Theorem 3.23 for filtered isocrystals was proved by Faltings in [Fal94] and by Totaro in [Tot94]. Both proofs work by reducing the problem of σ-linear algebra to a different problem of pure linear algebra. Motivated by the idea of Ramanan and Ramanathan from geometric invariant theory, Totaro gave in [Tot96] a different elementary proof which avoids the reduction from filtered isocrystals to filtered vector spaces. The method of Cornut in [Cor18] can be viewed as an axiomatized version of Totaro's overall strategy in [Tot96] in which the GIT tools are replaced by tools from convex metric geometry.

In the next section, we will apply Theorem 3.23 in the case where $K = B_{\mathrm{dR}}$ to define Harder–Narasimhan stratification for a general reductive group. Here B_{dR} is the de Rham period ring associated with an algebraically closed perfectoid field of characteristic 0.

4. Harder–Narasimhan stratification on B_{dR}^+-affine Grassmannian

The goal of this section is to define the Harder–Narasimhan stratification on the B_{dR}^+-affine Grassmannian of a reductive group G. For $G = \mathrm{GL}_n$, it would be enough to use the Harder–Narasimhan formalism, i.e., the existence of Harder–Narasimhan filtration for isocrystals with

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lattices. In order to pass to more general reductive groups, we use Tannakian duality, and hence the compatibility of Harder–Narasimhan filtration with tensor products (Theorem 3.23) becomes crucial. Note that the special case of the Harder–Narasimhan stratification for $b=1$ with general reductive groups has already been worked out by Nguyen and Viehmann [NV23] and by Shen [She23].

4.1 HN-vectors

Let G be a reductive group over F. Let $S=\mathrm{Spa}(C,C^+)\to \mathrm{Spd}(\breve{F})$ be a geometric point of $\mathrm{Spd}(\breve{F})$, given by an untilt $S^\sharp=\mathrm{Spa}(C^\sharp,C^{\sharp+})$ over $\breve{F}$. Write

$$B_{\mathrm{dR}}=B_{\mathrm{dR}}(C^\sharp)\quad\text{and}\quad B_{\mathrm{dR}}^+=B_{\mathrm{dR}}^+(C^\sharp)$$

the corresponding de Rham period rings (cf. § 2.1.1). Let $b\in G(\breve{F})$. Consider the following faithful exact $\otimes$-functor:

$$\omega_{G,b}:\mathbf{Rep}(G)\longrightarrow \mathbf{Isoc}_{\breve{F}|F},\quad (V,\rho)\longmapsto \big(V\otimes_F \breve{F},\rho(b)\circ(1\otimes\sigma)\big). \tag{4.1}$$

Let $x\in \mathrm{Gr}_G(S)=G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+)$. For an F-representation $\rho=(V,\rho)$ of G, it induces an element

$$\rho(x)\in \mathrm{Gr}_{\mathrm{GL}_V}(S)=\mathrm{GL}(V\otimes_F B_{\mathrm{dR}})/\mathrm{GL}(V\otimes_F B_{\mathrm{dR}}^+),$$

or, equivalently, a B_{dR}^+-lattice $\Xi_{x,\rho}$ in $V\otimes_F B_{\mathrm{dR}}$:

$$\Xi_{x,\rho}=\rho(x)(V\otimes_F B_{\mathrm{dR}}^+)\subseteq V\otimes_F B_{\mathrm{dR}}.$$

The pair (b,x) then gives rise to a $\otimes$-functor

$$\begin{aligned}\omega_{G,b,x}:\mathbf{Rep}(G)&\longrightarrow \mathbf{BunIsoc}_{\breve{F}|F}^{B_{\mathrm{dR}}},\\ (V,\rho)&\longmapsto (\omega_{G,b}(V,\rho),\Xi_{x,\rho}).\end{aligned} \tag{4.2}$$

Composing it with the functor $\mathcal{F}_{\mathrm{HN}}$ in (3.8), we obtain

$$\mathcal{F}_{\mathrm{HN}}(b,x):\mathbf{Rep}(G)\longrightarrow \mathbf{BunIsoc}_{\breve{F}|F}^{B_{\mathrm{dR}}}\xrightarrow{\mathcal{F}_{\mathrm{HN}}}\mathbf{F}(\mathbf{Isoc}_{\breve{F}|F}),$$

which is exact and compatible with tensor products by Corollary 3.23. Since G is reductive, there exists some rational cocharacter

$$\nu:\mathbb{D}_{\breve{F}}\longrightarrow G_{\breve{F}} \tag{4.3}$$

that splits $\mathcal{F}_{\mathrm{HN}}(b,x)$, i.e., the composed functor

$$\mathbf{Rep}(G)\xrightarrow{\mathcal{F}_{\mathrm{HN}}(b,x)}\mathbf{F}(\mathbf{Isoc}_{\breve{F}|F})\longrightarrow \mathbf{F}(\mathbf{Vect}_{\breve{F}})$$

of $\mathcal{F}_{\mathrm{HN}}(b,x)$ with the forgetful functor $\mathbf{F}(\mathbf{Isoc}_{\breve{F}|F})\to\mathbf{F}(\mathbf{Vect}_{\breve{F}})$ admits the following factorization.

$$\begin{array}{ccc}\mathbf{Rep}(G) & \longrightarrow & \mathbf{F}(\mathbf{Vect}_{\breve{F}})\\ \downarrow & & \uparrow\\ \mathbf{Rep}(\mathbb{D}_{\breve{F}}) & = & \mathbb{Q}\text{-}\mathbf{Vect}_{\breve{F}}\end{array}$$

Here the left-vertical functor is induced by the rational cocharacter ν, while the right-vertical functor takes a $\mathbb{Q}$-graded $\breve{F}$-vector space $V=\bigoplus_{\alpha\in\mathbb{Q}}V_\alpha$ to the filtered $\breve{F}$-space $(V,\mathrm{Fil}^\bullet V)$ with

$$\mathrm{Fil}^\alpha V=\bigoplus_{\beta\geq\alpha}V_\beta,\quad \forall\alpha\in\mathbb{R}.$$

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Remark 4.1. Let $P \subset G_{\breve{F}}$ be the parabolic subgroup of elements of $G_{\breve{F}}$ that preserves the Harder–Narasimhan filtration. In particular,

$$P(\breve{F}) = \{g \in G(\breve{F}) \mid \rho(g) \text{ preserves } \mathcal{F}_{\mathrm{HN}}(b,x)(V,\rho), \forall (V,\rho) \in \mathbf{Rep}(G)\}.$$

Then P is also the parabolic subgroup attached to the rational cocharacter v.

Consider as in (4.3) a rational cocharacter v that splits the Harder–Narasimhan filtration $\mathcal{F}_{\mathrm{HN}}(b,x)$. Since the Tannakian category $\mathbf{Isoc}_{\breve{F}|F}$ has a fiber functor over F^{un}, the maximal unramified extension (inside $\bar{F}$) of F, the rational cocharacter v is $G(\breve{F})$-conjugate to a rational cocharacter $v' \in \mathrm{Hom}_{F^{un}}(\mathbb{D}, G)$, whose $G(F^{un})$-conjugacy class depends only on the $G(\breve{F})$-conjugacy class of v: for $v'' \in \mathrm{Hom}_{F^{un}}(\mathbb{D}, G)$ a second choice, the two rational F^{un}-cocharacters v' and v'' are $G(\breve{F})$-conjugate, so our assertions follows from the fact that the set of $c \in G(F^{un})$ with $\mathrm{Int}(c) \circ v' = v''$ is the set of F^{un}-points of a torsor under the centralizer of v' which itself is a connected reductive subgroup of $G_{F^{un}}$, and the famous result of Steinberg saying that every torsor under a smooth connected linear algebraic group over F^{un} has a rational point (cf. [Ste65, Theorem 1.9] and [Ser94, Chapter II, § 3.3 (c)]). Let us denote the resulting $G(\bar{F})$-conjugacy class of v' by

$$v_{b,x} \in X_*(G)_{\mathbb{Q}}/G(\bar{F}).$$

Lemma 4.2. *We have* $v_{b,x} \in \mathcal{N}(G) = (X_*(G)_{\mathbb{Q}}/G(\bar{F}))^{\Gamma}$.

In the following, we call the element $v_{b,x} \in \mathcal{N}(G)$ the *HN-vector* of the pair (b,x). When G is quasi-split, we also consider $v_{b,x}$ as an element in $X_*(A)^+_{\mathbb{Q}}$ via the identification $\mathcal{N}(G) \simeq X_*(A)^+_{\mathbb{Q}}$.

Proof of Lemma 4.2. Consider the rational cocharacter v in (4.3). So it suffices to check that the class $[v] \in \mathrm{Hom}_{\breve{F}}(\mathbb{D}, G)/G(\breve{F})$ is invariant under the action of the Frobenius σ, i.e., $[v^\sigma] = [v]$ with v^σ being the base change of $v : \mathbb{D}_{\breve{F}} \to G_{\breve{F}}$ induced by the Frobenius σ. This is a consequence of the fact that the Harder–Narasimhan filtration is a filtration by *sub-isocrystals*. More precisely, recall that v is a splitting of the composed functor

$$\mathcal{F}_{\mathrm{HN}}(b,x)' : \mathbf{Rep}(G) \xrightarrow{\mathcal{F}_{\mathrm{HN}}(b,x)} \mathbf{F}(\mathbf{Isoc}_{\breve{F}|F}) \xrightarrow{\text{forgetful}} \mathbf{F}(\mathbf{Vect}_{\breve{F}}).$$

By functoriality, its twist v^σ by Frobenius splits

$$\mathcal{F}_{\mathrm{HN}}(b,x)'' : \mathbf{Rep}(G) \xrightarrow{\mathcal{F}_{\mathrm{HN}}(b,x)} \mathbf{F}(\mathbf{Vect}_{\breve{F}}) \xrightarrow{-\otimes_{\breve{F},\sigma}\breve{F}} \mathbf{F}(\mathbf{Vect}_{\breve{F}}).$$

On the other hand, for (V,ρ) a representation of G and for $W \subset (V_{\breve{F}}, \rho(b)\sigma)$ a sub-isocrystal, we have $(\rho(b)\sigma)(W) = W$. In particular, the isomorphism

$$\rho(b)^{-1} : V_{\breve{F}} \longrightarrow V_{\breve{F}}$$

maps W onto $\sigma(W)$. Moreover, under the natural identification $V_{\breve{F}} \otimes_{\breve{F},\sigma} \breve{F} \simeq V \otimes_F \breve{F}$, we have that $W \otimes_{\breve{F},\sigma} \breve{F}$ corresponds to $\sigma(W) \subset V_{\breve{F}}$. As a result, the bijection $\rho(b)^{-1}$ above sends the Harder–Narasimhan filtration of $(\omega_{G,b}(V,\rho), \Xi_{x,\rho})$ to its twist by Frobenius. In this way, we obtain a natural transformation of functors

$$b^{-1} : \mathcal{F}_{\mathrm{HN}}(b,x)' \xrightarrow{\sim} \mathcal{F}_{\mathrm{HN}}(b,x)''.$$

So $\mathrm{Int}(b^{-1}) \circ v$ also splits $\mathcal{F}_{\mathrm{HN}}(b,x)''$. Hence, the cocharacter v^σ is $G(\breve{F})$-conjugate to $\mathrm{Int}(b^{-1}) \circ v$, and the class $v_{b,x}$ of v in $X_*(G)_{\mathbb{Q}}/G(\bar{F})$ is Galois invariant. □

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4.2 Compatibility with inner-twists

Let H be an inner-twist of G, given by a basic element $b_0 \in G(\breve{F})$. In particular, $H = J_{b_0}$ and for any $\breve{F}$-algebra R, we have

$$H(R) = \{g \in G(\breve{F} \otimes_F R) \mid gb_0\sigma(g)^{-1} = b_0\} \xrightarrow{\sim} G(R).$$

Here the morphism is induced by the natural map $\breve{F} \otimes_F R \to R$. Therefore, we have an identification of algebraic groups over $\breve{F}$:

$$\iota : H_{\breve{F}} \xrightarrow{\sim} G_{\breve{F}}.$$

Using the element b_0 we can deduce a well-defined bijection

$$\tilde{\iota} : B(H) \longrightarrow B(G), \quad [h] \longmapsto [\iota(h)b_0].$$

Indeed, identify $H_{\breve{F}}$ with $G_{\breve{F}}$ and thus $H(\breve{F})$ with $G(\breve{F})$ through the isomorphism ι, so the Frobenius automorphism σ_H on $H(\breve{F})$ is given by the composed map $\mathrm{Int}(b_0) \circ \sigma_G$ on $H(\breve{F}) = G(\breve{F})$. Therefore, for any $h, h_1 \in H(\breve{F}) = G(\breve{F})$,

$$h_1(hb_0)\sigma_G(h_1)^{-1} = h_1 h b_0 \sigma_G(h_1)^{-1} b_0^{-1} b_0 = (h_1 h \sigma_H(h_1)^{-1})b_0,$$

and the map $h \mapsto hb_0$ sends a σ-conjugacy class in $H(\breve{F})$ to a σ-conjugacy class in $G(\breve{F})$. Thus the map $\tilde{\iota}$ above is well defined and it is clear that $\tilde{\iota}$ is bijective. On the other hand, the isomorphism ι also induces an isomorphism between the B^+_{dR}-affine Grassmannians, still denoted by ι in what follows:

$$\mathrm{Gr}_{H,\breve{F}} \longrightarrow \mathrm{Gr}_{G,\breve{F}}.$$

Proposition 4.3. *Keep the notation above. Let $h \in H(\breve{F})$ and $x \in \mathrm{Gr}_{H,\breve{F}}$ be a geometric point. Then $v_{\iota(h)b_0,\iota(x)} = v_{h,x} - \nu_{b_0} \in \mathcal{N}(G) = \mathcal{N}(H)$.*

Proof. The proof is inspired by [Kot97, 3.4], where a similar result for the Newton vectors can be found. Take a faithful representation (V, ρ) of G. Together with the Frobenius induced by b_0, the base change $V \otimes_F \breve{F}$ becomes an isocrystal over F, denoted by $\breve{V}$. One can also use the Frobenius structure induced by $\iota(h)b_0$ on $V \otimes_F \breve{F}$, and we denote the resulting isocrystal by $\breve{V}'$. Let $U = \mathrm{End}_{\sigma,\breve{F}}(\breve{V})$, and more generally, for any F-algebra R,

$$U \otimes_F R = \mathrm{End}_{\sigma,\breve{F}\otimes R}(\breve{V} \otimes_F R).$$

Then $H(R) \subset U(R)$, so U gives a faithful representation of H by left multiplication. In this way, $\breve{U} := U \otimes_F \breve{F}$ becomes an isocrystal whose Frobenius structure is given by h. If moreover R is an $\breve{F}$-algebra, we have naturally an identification induced by the natural morphism $\breve{F} \otimes_F R \to R$ of F-algebras (recall that $b_0 \in G(\breve{F})$ is basic),

$$U \otimes_F R \xrightarrow{\sim} \mathrm{End}_R(V \otimes_F R).$$

Under this identification, the action of $H_{\breve{F}}$ on $\breve{U} := U \otimes_F \breve{F}$ is the same as the action of $G_{\breve{F}}$ on $\mathrm{End}_F(V) \otimes_F \breve{F}$ by left multiplication, and hence the isocrystal $\breve{U}$ can be identified with the isocrystal

$$\mathrm{End}_F(V) \otimes_F \breve{F} \simeq \mathrm{Hom}_{\breve{F}}(V \otimes_F \breve{F}, V \otimes_F \breve{F}),$$

where the Frobenius is induced by the left multiplication given by $\rho(\iota(h)) \in \mathrm{GL}(V \otimes_F \breve{F})$. Consequently, we get a morphism of isocrystals

$$\breve{U} \otimes \breve{V} \longrightarrow \breve{V}'. \tag{4.4}$$

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Furthermore, once we put the norm structure on $\breve{U}\otimes_{\breve{F}} B_{\mathrm{dR}}$ given by $x\in \mathrm{Gr}_{H,\breve{F}}$, or, equivalently, the norm structure on

$$\mathrm{End}_F(V)\otimes_F B_{\mathrm{dR}} = \mathrm{Hom}_{B_{\mathrm{dR}}}(V\otimes_F B_{\mathrm{dR}}, V\otimes_F B_{\mathrm{dR}})$$

by the gauge norm induced by the lattice

$$\mathrm{Hom}_{B^+_{\mathrm{dR}}}(V\otimes_F B^+_{\mathrm{dR}}, \Xi_{\iota(x),\rho})\subset \mathrm{End}_F(V)\otimes_F B_{\mathrm{dR}},$$

and the norm on $\breve{V}\otimes_{\breve{F}} B_{\mathrm{dR}}$ (respectively on $\breve{V}'\otimes_{\breve{F}} B_{\mathrm{dR}} = V\otimes_F B_{\mathrm{dR}}$) by the trivial B^+_{dR}-lattice $\breve{V}\otimes B^+_{\mathrm{dR}}$ (respectively $\Xi_{\iota(x),\rho}$), the map (4.4) above is even a surjective map between the normed isocrystals. Now take $v\in X_*(H)_{\mathbb{Q}}$ (respectively $v_0\in X_*(G)_{\mathbb{Q}}$) to be a rational cocharacter that splits the Harder–Narasimhan filtration $\mathcal{F}_{\mathrm{HN}}(h,x)$ (respectively corresponding to the slope decomposition of the isocrystal $\breve{V}$). In particular, $-v_0\in X_*(G)_{\mathbb{Q}}$ gives a splitting of the Harder–Narasimhan filtration $\mathcal{F}_{\mathrm{HN}}(b_0,x_0)$, where $x_0\in \mathrm{Gr}_{G,\breve{F}}$ is given by the identity element of G. As the Harder–Narasimhan filtration is compatible with tensor products (Theorem 3.23), using the surjective map (4.4), we deduce that the rational cocharacter

$$\rho\circ\iota\circ v-\rho\circ v_0$$

splits the Harder–Narasimhan filtration of the normed isocrystal $\breve{V}'$. As V is a faithful representation of G, it follows that the rational cocharacter $\iota\circ v - v_0$ splits $\mathcal{F}_{\mathrm{HN}}(\iota(h)b_0,\iota(x))$. Therefore, we obtain the required equality $v_{h,x}-\nu_{b_0}=v_{\iota(h)b_0,\iota(x)}\in\mathcal{N}(H)=\mathcal{N}(G)$. □

4.3 HN-vectors in the quasi-split case

We keep the notation in the previous subsection. In particular, $S=\mathrm{Spa}(C,C^+)\to\mathrm{Spd}(\breve{F})$ is a geometric point of $\mathrm{Spd}(\breve{F})$, given by an untilt $S^\sharp=\mathrm{Spa}(C^\sharp,C^{\sharp+})$ over $\breve{F}$, with B_{dR} and B^+_{dR} the corresponding de Rham period rings. Suppose that G is quasi-split. Abusing the notation, we identify $v_{b,x}\in\mathcal{N}(G)$ with its dominant representative in $X_*(A)^+_{\mathbb{Q}}$ in the following via the natural identification $X_*(A)^+_{\mathbb{Q}}\simeq\mathcal{N}(G)$. Let $P\subset G_{\breve{F}}$ be the parabolic subgroup corresponding to the Harder–Narasimhan filtration of $\omega_{G,b,x}$ (Remark 4.1). Since the Harder–Narasimhan filtration on a normed isocrystal is a filtration by sub-isocrystals, the intersection of the parabolic subgroup P with the centralizer of the Newton vector ν_b descends to a subgroup of J_b defined over F.

Lemma 4.4. *There exist a standard parabolic $Q\subset G$ over F with standard Levi M, and a reduction (b_Q,g) of b to Q satisfying the following properties.*

- *The subgroup Q is the parabolic subgroup of G defined by $v_{b,x}\in X_*(A)^+_{\mathbb{Q}}$, and $gQ_{\breve{F}}g^{-1}=P$. In particular, $v_{b,x}$ is dominant Q-regular.*
- *Write $b_M\in M(\breve{F})$ the image of $b_Q\in Q(\breve{F})$, and recall the natural projection $\mathrm{pr}_M:\mathrm{Gr}_Q\to\mathrm{Gr}_M$. Then the HN-vector of the pair $(b_M,\mathrm{pr}_M(g^{-1}x))$ is $v_{b,x}$, and thus the functor $\omega_{M,b_M,\mathrm{pr}_M(g^{-1}x)}$ (defined in equation (4.2)) is semi-stable in the sense that the HN-vector of $(b_M,\mathrm{pr}_M(g^{-1}x))$ is central in M.*

Proof. Let $v\in X_*(G)_{\mathbb{Q}}$ be a rational cocharacter defined over $\breve{F}$ that splits the Harder–Narasimhan filtration of $\omega_{G,b,x}$, so $[v]=[v_{b,x}]\in\mathcal{N}(G)$. Since G is quasi-split, v is $G(\breve{F})$-conjugate to $v_{b,x}\in X_*(A)^+_{\mathbb{Q}}$. Let $g\in G(\breve{F})$ with $\mathrm{Int}(g)\circ v_{b,x}=v$. Consequently,

$$gQ_{\breve{F}}g^{-1}=P\subset G_{\breve{F}},$$

with $Q\subset G$ the standard parabolic subgroup defined by $v_{b,x}\in X_*(A)^+_{\mathbb{Q}}$. In particular, $v_{b,x}$ is dominant Q-regular.

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On the other hand, since the Harder–Narasimhan filtration of $\omega_{G,b,x}$ is a filtration by sub-isocrystals, $b\sigma(P)b^{-1}=P$. It follows that

$$b\sigma(g)Q_{\breve{F}}\sigma(g)^{-1}b^{-1}=b\sigma(g)\sigma(Q_{\breve{F}})\sigma(g)^{-1}b^{-1}=gQ_{\breve{F}}g^{-1},$$

and thus

$$g^{-1}b\sigma(g)Q_{\breve{F}}\sigma(g)^{-1}b^{-1}g=Q_{\breve{F}}.$$

As a result, $b_Q:=g^{-1}b\sigma(g)\in G(\breve{F})$ normalizes $Q_{\breve{F}}$ and hence is contained in $Q(\breve{F})$. So $b=gb_Q\sigma(g)^{-1}$ has a reduction (b_Q,g) to Q. Furthermore, $g\in G(\breve{F})$ gives an isomorphism from $\omega_{G,b_Q,g^{-1}x}$ to $\omega_{G,b,x}$, and thus an isomorphism of functors

$$\mathcal{F}_{\mathrm{HN}}(b_Q,g^{-1}x)\xrightarrow{\sim}\mathcal{F}_{\mathrm{HN}}(b,x).$$

In particular, the Harder–Narasimhan filtration of $\omega_{G,b_Q,g^{-1}x}$ has a splitting given by $v_{b,x}=\mathrm{Int}(g^{-1})\circ v$. By considering a faithful representation of G (and thus a faithful representation of Q through the embedding $Q\hookrightarrow G$), we check that $v_{b,x}\in X_*(A)_{\mathbb{Q}}$ splits the Harder–Narasimhan filtration of $\omega_{Q,b_Q,g^{-1}x}$, and hence also splits the Harder–Narasimhan filtration of the functor $\omega_{M,b_M,\mathrm{pr}_M(g^{-1}x)}$. Since $v_{b,x}$ is central in M, it coincides with the HN-vector of the pair $(b_M,\mathrm{pr}_M(g^{-1}x))$ in $X_*(A)^{M,+}_{\mathbb{Q}}$, and the functor $\omega_{M,b_M,\mathrm{pr}_M(g^{-1}x)}$ is semi-stable. □

To go further, it is convenient to use the notion of (modifications of) G-bundles on the Fargues–Fontaine curve $X=X_{(C,C^+)}$. We first introduce the notion of slope vector of a G-bundle for any G. Later on, we will use the slope vector of an M-bundle where M is a Levi subgroup of G. The faithful $\otimes$-functor $\omega_{G,b}$ of (4.1) corresponds naturally to the G-bundle $\mathcal{E}_b$ on X recalled in § 2.1.3. The map

$$\begin{aligned}X^*(G)&\longrightarrow\mathbb{Z},\\ \chi&\longmapsto\deg(\chi_*\mathcal{E}_b)\end{aligned}$$

is Galois invariant and can thus be viewed as an element $v(\mathcal{E}_b)\in X_*(Z_G)^{\Gamma}_{\mathbb{Q}}$, where Z_G denotes the center of G and $\chi_*(\mathcal{E}_b)$ is the pushout of the G-bundle $\mathcal{E}_b$ by the character χ, and thus is a line bundle on $X_{\bar{F}}$, the base change to $\bar{F}$ of the Fargues–Fontaine curve X/F.[1] In other words,

$$\langle v(\mathcal{E}_b),\chi\rangle=\deg(\chi_*\mathcal{E}_b),\quad\forall\chi\in X^*(G).$$

In fact, $v(\mathcal{E}_b)=-\mathrm{Av}_G(\nu_b)$, where Av_G denote the average over the action of the Weyl group W_G of (G,T). The element $v(\mathcal{E}_b)\in X_*(Z_G)^{\Gamma}_{\mathbb{Q}}$ is called the *slope vector* of $\mathcal{E}_b$.

Let $(b_{Q'},g')$ be a reduction of b to a standard parabolic subgroup $Q'\subseteq G$. Together with the G-bundle $\mathcal{E}_{b,x}$ on X, the pair $(b_{Q'},g')$ gives rise to a reduction $(\mathcal{E}_{b,x})_{b_{Q'}}$ to Q' so that

$$(\mathcal{E}_{b,x})_{b_{Q'}}\simeq\mathcal{E}^{Q'}_{b_{Q'},g'^{-1}x}.$$

The slope vector of the M'-bundle $(\mathcal{E}_{b,x})_{b_{Q'}}\times^{Q'}M'$ gives an element (by abuse of notation) still called the *slope vector* for the pair $(b_{Q'},g')$:

$$v_{(b_{Q'},g')}\in X_*(Z_{M'})^{\Gamma}_{\mathbb{Q}}\subset X_*(T)^{\Gamma}_{\mathbb{Q}}=X_*(A)_{\mathbb{Q}},$$

where $Z_{M'}$ is the center of the standard Levi M' corresponding to Q'.

The following result is an analogue of [Sch15, Theorem 4.1].

Proposition 4.5. *Let Q_1 and Q_2 be two standard parabolic subgroups of G, with M_1 and M_2 their corresponding standard Levi subgroups. For $i=1$ or 2, let (b_{Q_i},g_i) be a reduction of b to Q_i,*

[1] Over $X_{\bar{F}}$ every vector bundle is a direct sum of line bundles of rational slopes [FF18, Théorème 8.5.1].

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and write b_{M_i} (respectively x_{M_i}) the image of $b_{Q_i} \in Q_i(\breve{F})$ (respectively of $g_i^{-1}x \in \mathrm{Gr}_G(C, C^+) = \mathrm{Gr}_{Q_i}(C, C^+)$) in $M_i(\breve{F})$ (respectively $\mathrm{Gr}_{M_i}(C, C^+)$) via the natural projection $Q_i \to M_i$. Assume that:

(i) *the M_1-bundle $\mathcal{E}^{M_1}_{b_{M_1}, x_{M_1}}$ is HN-semi-stable, in the sense that the HN-vector of the pair (b_{M_1}, x_{M_1}) is central in M_1;*
(ii) *the slope vector $v_{(b_{Q_1}, g_1)}$ is dominant Q_1-regular.*

Then the following assertions hold:

(i) $v_{(b_{Q_2}, g_2)} \leq v_{(b_{Q_1}, g_1)}$;
(ii) *we have $Q_1 = Q$, the standard parabolic subgroup in Lemma 4.4, and $v_{(b_{Q_1}, g_1)} = v_{b,x}$;*
(iii) *if the inequality in (i) is an equality, then $Q_2 \subset Q_1$, and (b_{Q_2}, g_2), viewed as a reduction of b to Q_1, is equivalent to (b_{Q_1}, g_1).*

Proof. To simplify notation, write $v_i = v_{(b_{Q_i}, g_i)}$. Write also $v_1' \in X_*(A)_{\mathbb{Q}}$ as the HN-vector of the pair (b_{M_1}, x_{M_1}). We first check that, as v_1' is central in M_1 by assumption (i) above, v_1' is equal to the slope vector v_1 of the M_1-bundle $\mathcal{E}_{b_{M_1}, x_{M_1}}$, or, equivalently, of the Q_1-bundle $(\mathcal{E}_{b,x})_{b_{Q_1}} = \mathcal{E}^{Q_1}_{b_{Q_1}, g^{-1}x}$. More precisely, for every $\lambda \in X^*(Q_1)^\Gamma = X^*(M_1)^\Gamma$, we want to show that

$$\langle v_1', \lambda \rangle = \deg(\lambda_*(\mathcal{E}_{b,x})_{b_{Q_1}}). \tag{4.5}$$

By the definition of the HN-vector, $\lambda \circ v_1'$ splits the Harder–Narasimhan filtration of the one-dimensional isocrystal $(\breve{F}, \lambda(b_Q))$ equipped with the norm (or, equivalently, the B^+_{dR}-lattice) defined by $\lambda(g^{-1}x) \in \mathrm{Gr}_{\mathbb{G}_m}(C, C^+)$. Therefore, $\lambda \circ v_1' = \langle v_1', \lambda \rangle \in \mathbb{Q}$ is the degree of this one-dimensional normed isocrystal, i.e.,

$$\deg(\lambda_*(\mathcal{E}_{b,x})_{b_{Q_1}}) = \deg\big(\mathcal{E}^{\mathbb{G}_m}_{\lambda(b_{Q_1}), \lambda(g_1^{-1}x)}\big) = \langle v_1', \lambda \rangle,$$

as desired by (4.5). So $v_1 = v_1' \in X_*(A)_{\mathbb{Q}}$ since v_1' is already central in M_1.

To check the inequality $v_2 \leq v_1$, observe firstly that the difference

$$v_1 - v_2 \in \mathbb{Q}\Phi_0^\vee,$$

where $\Phi_0^\vee$ is the set of coroots for (G, A). Indeed, for any $\lambda \in X^*(G) \subset X^*(T)_{\mathbb{Q}}$, we have

$$\langle v_1, \lambda \rangle = \langle v_2, \lambda \rangle = \deg \lambda_* \mathcal{E}_{b,x}.$$

Hence $v_1 - v_2 \in X_*(T)_{\mathbb{Q}}$ lies in the orthogonal complementary of $X^*(Z_G)_{\mathbb{Q}} \subset X^*(T)$ under the natural pairing

$$X_*(T)_{\mathbb{Q}} \times X^*(T)_{\mathbb{Q}} \longrightarrow \mathbb{Q}.$$

Consequently $v_1 - v_2 \in \mathbb{Q}\Phi^\vee$, the $\mathbb{Q}$-subspace of $X_*(T)_{\mathbb{Q}}$ generated by the set $\Phi^\vee$ of coroots for (G, T). As $v_1 - v_2$ is Galois-invariant, so $v_1 - v_2 \in \mathbb{Q}\Phi_0^\vee$, as wanted.

(i) To complete the proof, we need to check that, for any dominant $\lambda \in X^*(A)^+_{\mathbb{Q}}$,

$$\langle v_1, \lambda \rangle \geq \langle v_2, \lambda \rangle.$$

Let $\rho : G \to \mathbf{GL}_V$ be a finite-dimensional representation of highest weight λ, and let

$$V = \bigoplus_{\mu \in X^*(A)} V[\mu]$$

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be its weight decomposition. For any $q \in \mathbb{Q}$, consider the linear subspaces

$$V_{1,q} = \bigoplus_{\langle v_1, \mu \rangle \geq q} V[\mu] \subset V.$$

Then $V_{1,q'} \subset V_{1,q}$ once $q \leq q'$, and we obtain a $\mathbb{Q}$-filtration $V_{1,\bullet} = (V_{1,q})_{q \in \mathbb{Q}}$ of V. Since v_1 is Q_1-dominant, for any character μ occurring in the weight decomposition above we have $\mu \leq \lambda$ and thus

$$\langle v_1, \mu \rangle \leq \langle v_1, \lambda \rangle.$$

Consequently, the maximal rational number q_1 occurring among the jumps of the $\mathbb{Q}$-filtration $V_{1,\bullet}$ is equal to $\langle v_1, \lambda \rangle$. Moreover, as in [Sch15, Lemma 5.1], the filtration $V_{1,\bullet}$ is a filtration of V by Q_1-subrepresentations, and the Q_1-action on each quotient $\mathrm{gr}_q V_{1,\bullet}$ descends to an action of the Levi M_1.

We claim that the normed isocrystal

$$(\mathrm{gr}_q V_{1,\bullet} \otimes_F \breve{F}, b_{M_1} \circ (1 \otimes \sigma), \alpha_{x_{M_1}}) \tag{4.6}$$

is semi-stable of slope q. Indeed, as a representation of A, we have weight decomposition

$$\mathrm{gr}_q V_{1,\bullet} \simeq \bigoplus_{\langle v_1, \mu \rangle = q} V[\mu].$$

Since the M_1-bundle $\mathcal{E}^{M_1}_{b_{M_1}, x_{M_1}}$ is HN-semi-stable, $v_1 = v_1'$ is the HN-vector of (b_{M_1}, x_{M_1}) and thus splits the Harder–Narasimhan filtration of the normed isocrystal (4.6) above. As $\langle v_1, \mu \rangle = q$ for all characters μ occurring in the weight decomposition of $\mathrm{gr}_q V_{1,\bullet}$, it follows that the normed isocrystal (4.6) has only one jump q in its Harder–Narasimhan filtration, and hence is semi-stable of slope q as claimed. As a result, the $\mathbf{Q}$-filtration $V_{1,\bullet} \otimes \breve{F}$ is precisely the Harder–Narasimhan filtration of the normed isocrystal $(V \otimes_F \breve{F}, b_Q \circ (1 \otimes \sigma), \alpha_{g^{-1}x})$. In particular, the maximal destabilized sub-normed isocrystal of

$$(V \otimes_F \breve{F}, b_Q \circ (1 \otimes \sigma), \alpha_{g^{-1}x}) \xrightarrow{\sim} (V \otimes_F \breve{F}, b \circ (1 \otimes \sigma), \alpha_x)$$

is of slope $q_1 = \langle v_1, \lambda \rangle$. On the other hand, for the standard parabolic Q_2, we can define a subspace

$$V_2' = \bigoplus_{\mu \in \lambda + \mathbb{Z}\Phi_{M_2,0}} V[\mu]$$

of V. As in [Sch15, Lemma 3.5] we know that V_2' is a Q_2-subrepresentation of V, so it gives a sub-normed isocrystal $(V_2' \otimes_F \breve{F}, b_{Q_2} \circ (1 \otimes \sigma), \alpha_{g^{-1}x})$ of slope $\langle v_2, \lambda \rangle$ of

$$(V \otimes_F \breve{F}, b_{Q_2} \circ (1 \otimes \sigma), \alpha_{g^{-1}x}) \xrightarrow{\sim} (V \otimes_F \breve{F}, b \circ (1 \otimes \sigma) \cdot \alpha_x).$$

So $\langle v_1, \lambda \rangle \geq \langle v_2, \lambda \rangle$. As this inequality holds for every dominant weight, we deduce

$$v_2 = v_{(b_{Q_2}, g_2)} \leq v_1 = v_{(b_{Q_1}, g_1)}$$

as asserted by (i).

(ii) We use the notation of Lemma 4.4. As $v_{b,x} \in X_*(A)^+_{\mathbb{Q}}$ is the HN-vector of the pair $(b_M, \mathrm{pr}_M(g^{-1}x))$ and is central in M, it coincides with the slope vector of $\mathcal{E}^M_{b_M, \mathrm{pr}_M(g^{-1}x)}$. Thus $v_{(b_Q, g)} = v_{b,x}$. Moreover, $v_{b,x} \in X_*(A)^+_{\mathbb{Q}}$ is dominant Q-regular. Consequently, by (i) we must have

$$v_{(b_{Q_1}, g_1)} = v_{(b_Q, g)} = v_{b,x}$$

and thus also $Q_1 = Q$.

(iii) Finally suppose $v_1 = v_2$. Since v_1 is dominant Q_1-regular, necessarily $Q_2 \subset Q_1 = Q$. It remains to check that the pair (b_{Q_2}, g_2), viewed as a reduction of b to $Q_1 = Q$, is equivalent to

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(b_{Q_1}, g_1). Replacing Q_2 by $Q_1 = Q$ and b_{Q_2} by its image in Q if needed, we assume moreover that $Q_1 = Q_2 = Q$. Write $b_i = b_{Q_i}$ for $i = 1, 2$. So we have two reduction (b_i, g_i) of b to Q and hence $b_2 = (g_2^{-1} g_1) b_1 \sigma(g_2^{-1} g_1)^{-1}$. To conclude the result we need to check that

$$g_2^{-1} g_1 \in Q(\breve{F}).$$

For each $i = 1, 2$, the element $g_i \in G(\breve{F})$ induces an isomorphism of functors

$$\omega_{G, b_i, g_i^{-1} x} \xrightarrow{\sim} \omega_{G,b,x}. \tag{4.7}$$

We claim that the parabolic subgroup $Q_{\breve{F}} \subset G_{\breve{F}}$ preserves the Harder–Narasimhan filtration on $\omega_{G, b_i, g_i^{-1} x}$. To see this, it is enough to check the corresponding assertion after evaluating $\omega_{G, b_i, g_i^{-1} x}$ at an algebraic representation (V, ρ) of G. Clearly, we can assume moreover that (V, ρ) is a highest weight representation with highest weight $\lambda \in X_*(A)^+_{\mathbb{Q}}$. Using the fact that v_1 is dominant Q-regular, the construction in (i) provides a filtration $V_\bullet = (V_q)_{q \in \mathbb{Q}}$ of V, which is stable under the action of Q. Moreover, as observed above, $V_{\bullet, \breve{F}} = (V_q \otimes \breve{F})_{q \in \mathbb{Q}}$ gives the Harder–Narasimhan filtration of

$$\omega_{G, b_1, g_1^{-1} x}(V, \rho) = (V \otimes \breve{F}, \rho(b_1)(1 \otimes \sigma), \Xi_{\rho, g_1^{-1} x}).$$

Hence this verifies our claim when $i = 1$. On the other hand, we have two $\mathbb{R}$-filtrations on the normed isocrystal $(D, \alpha) := \omega_{G, b_2, g_2^{-1} x}(V, \rho)$: the first one is the Harder–Narasimhan filtration $D_\bullet = (D_q)_{q \in \mathbb{R}}$ of (D, α), and since $v_1 = v_2$, the same filtration $V_{\bullet, \breve{F}}$ of $V \otimes \breve{F}$ as above yields a second filtration by sub-isocrystals of (D, α). Using the isomorphism (4.7) for $i = 1$ and 2, we deduce that both filtrations have the same numerical data, i.e.,

$$\deg(\mathrm{gr}^q_{D_\bullet} D, \alpha) = \deg(\mathrm{gr}^q_{V_{\bullet, \breve{F}}} D, \alpha) = q \quad \text{and} \quad \dim(\mathrm{gr}^q_{D_\bullet} D) = \dim(\mathrm{gr}^q_{V_{\bullet, \breve{F}}} D), \quad \forall q \in \mathbb{R}.$$

Therefore, according to the lemma below, $V_{\bullet, \breve{F}}$ coincides with the Harder–Narasimhan filtration $D_\bullet$ of $(D, \alpha) = \omega_{G, b_2, g_2^{-1} x}(V, \rho)$, showing our claim when $i = 2$. In particular, it follows that $Q_{\breve{F}} \subset g_i^{-1} P g_i$ and thus $Q_{\breve{F}} = g_i^{-1} P g_i$ since $Q_{\breve{F}}$ is a $G(\breve{F})$-conjugate of P. As a result, we get

$$g_1 Q_{\breve{F}} g_1^{-1} = P = g_2 Q_{\breve{F}} g_2^{-1},$$

and hence $g_2^{-1} g_1 \in Q(\breve{F})$ as desired. □

Lemma 4.6. *Let (D, α) be a normed isocrystal. Let $D_\bullet = (D_q)_{q \in \mathbb{R}}$ and $D'_\bullet = (D'_q)_{q \in \mathbb{R}}$ be two $\mathbb{R}$-filtrations of D by sub-isocrystals. Assume the following two conditions:*

- *$D_\bullet$ is the Harder–Narasimhan filtration of (D, α);*
- *$D'_\bullet$ has the same numerical data as $D_\bullet$, i.e.,*

$$\deg(\mathrm{gr}^q_{D_\bullet} D, \alpha) = \deg(\mathrm{gr}^q_{D'_\bullet} D, \alpha) \quad \text{and} \quad \dim(\mathrm{gr}^q_{D_\bullet} D) = \dim(\mathrm{gr}^q_{D'_\bullet} D).$$

Then $D_\bullet = D'_\bullet$.

Proof. Write r (respectively r') to be the maximal real number occurring among the jumps of the $\mathbb{R}$-filtration $D_\bullet$ (respectively of $D'_\bullet$). Then (D_r, α) is the maximal destabilized sub-normed isocrystal of (D, α) and it is of slope r. Since $D'_\bullet$ has the same numerical data as $D_\bullet$, it follows that $r = r'$ and that (D'_r, α) is a sub-normed isocrystal of (D, α) of slope r. Therefore, (D'_r, α) must be semi-stable, and hence coincides with (D_r, α) as $\dim_{\breve{F}} D_r = \dim_{\breve{F}} D'_r$. Next, replacing D by $D/D_r = D/D_{r'}$, we check by induction that $D_\bullet = D'_\bullet$. □

Combining Lemma 4.4 and Proposition 4.5 we obtain the following.

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Corollary 4.7 (Canonical reduction for the Harder–Narasimhan filtration). *Assume G quasi-split. Let $b \in G$ and $x \in \mathrm{Gr}_G(C, C^+)$ be a geometric point. Then*

$$v_{b,x} = \max \left\{ v_{(b_{Q'}, g')} \middle| \begin{array}{c} Q' \text{ is a standard parabolic of } G, \text{ and} \\ (b_{Q'}, g') \text{ is a reduction of } b \text{ to } Q' \end{array} \right\}.$$

Moreover, there exist a unique standard parabolic subgroup Q with Levi component M and a unique class of reduction (b_Q, g) of b to Q satisfying the following properties:

(i) *write $b_M \in M(\breve{F})$ as the image of b_Q and recall the projection $\mathrm{pr}_M : \mathrm{Gr}_P \to \mathrm{Gr}_M$; then the M-bundle $\mathcal{E}_{b_M, \mathrm{pr}_M(g^{-1}x)}$ is HN-semi-stable in the sense that the HN-vector of (b_M, x_M) is central in M;*
(ii) *the slope vector of the M-bundle in (i) is dominant Q-regular.*

Finally, the slope vector of the M-bundle in (i) is $v_{b,x}$.

Corollary 4.8. *Let G be a reductive group over F, which is not necessarily quasi-split. Let $b \in G(\breve{F})$ and $x \in \mathrm{Gr}_{G,\mu}$ be a geometric point, such that $\mathcal{E}_{b,x} = \mathcal{E}_{b'}$ with $[b'] \in B(G)$. Then there exists a unique element $[b''] \in B(G)$ such that $\nu_{b''} = -w_0 v_{b,x}$ and $\kappa(b'') = \kappa(b') = \kappa(b) - \mu^\sharp$. Moreover, $\nu_{b''} \le \nu_{b'}$.*

In the following, we shall denote by $\mathrm{HN}(b, x)$ this unique element $[b''] \in B(G)$.

Proof. The uniqueness is clear as any element in $B(G)$ is uniquely determined by Newton invariant and Kottwitz invariant (2.1). It suffices to prove the existence of b''.

Assume first that G is quasi-split. Let Q be the standard parabolic subgroup of G as given in Lemma 4.4, with M its corresponding standard Levi which is the centralizer of $v_{b,x}$. Let $b_Q = (b_Q, g)$ be a reduction of b to Q such that $v_{(b_Q, g)} = v_{b,x}$. Let $b_M \in M(\breve{F})$ be the image of b_Q via the natural projection. So

$$(\mathcal{E}_{b,x})_{b_Q} \times^Q M \simeq \mathcal{E}_{b_M, x'},$$

with $x' \in \mathrm{Gr}_{M,\lambda}$ a geometric point for some $\lambda \in S_M(\mu)$. Its M-equivariant first Chern class $c_M^1(\mathcal{E}_{b_M,x'}) \in \pi_1(M)_\Gamma$ gives a basic element $[\tilde{b}''_M] \in B(M)$ with

$$\kappa_M(\tilde{b}''_M) = c_M^1(\mathcal{E}_{b_M,x'}) = \lambda^\sharp - \kappa_M(b_M) \quad \text{in } \pi_1(M)_\Gamma.$$

Then $\nu_{\tilde{b}''_M} = v_{b,x}$. Let $[\tilde{b}''] \in B(G)$ be its image via the natural map $B(M) \to B(G)$. Recall that, by [CT22, Corollary 2.9], the map

$$\begin{aligned} X_*(A)^+_{\mathbb{Q}} &\simeq X_*(A)^+_{\mathbb{Q}}, \\ v &\mapsto v^* := -w_0 v \end{aligned}$$

induces a bijection between generalized Kottwitz sets

$$B(G, \epsilon, \delta) \simeq B(G, -\epsilon, -w_0\delta).$$

Let $[b''] \in B(G)$ be the image of $[\tilde{b}'']$ under this bijection. Then it is clear that

$$\kappa(b'') = -\kappa(\tilde{b}'') = -\mu^\sharp + \kappa(b) = \kappa(b')$$

and

$$\nu_{b''} = -w_0 \nu_{\tilde{b}''} = -w_0 v_{b,x}.$$

The inequality $\nu_{b''} \le \nu_{b'}$, or equivalently $v_{b,x} \le \nu_{\mathcal{E}_{b,x}}$ (where the HN polygon $\nu_{\mathcal{E}_{b,x}}$ of $\mathcal{E}_{b,x}$ is defined in § 2.1.4), follows from the previous result, as $v_{\mathcal{E}_{b,x}}$ can be characterized similarly as the

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maximum of the slope vectors of all possible reductions of $\mathcal{E}_{b,x}$ to standard parabolic subgroups of G (Proposition 2.5).

Now consider the general case where G is not necessarily quasi-split. Let H be the unique quasi-split inner form over F of G, and write $H = J_{b_0}$ for some basic $b_0 \in G(\breve{F})$. In particular, we have a natural identification $\iota : H_{\breve{F}} = J_{b_0,\breve{F}} \xrightarrow{\sim} G_{\breve{F}}$. Write $b_H = \iota^{-1}(bb_0^{-1}) \in H(\breve{F})$, $x_H = \iota^{-1}(x) \in \mathrm{Gr}_H$ and $\mathcal{E}_{b_H,x_H} = \mathcal{E}_{b'_H}$. Then, under the natural identification $\mathcal{N}(H) = \mathcal{N}(G)$, we have $v_{b,x} = v_{b_H,x_H} - \nu_{b_0}$ by Proposition 4.3. After applying our corollary to the quasi-split group H for the triple (b_H, x_H, b'_H), there exists a unique $[b''_H] \in B(H)$ such that

$$\nu_{b''_H} = -w_0 v_{b_H,x_H} \quad \text{and} \quad \kappa(b''_H) = \kappa(b'_H) = \kappa(b_H) - \mu^\sharp.$$

Let $[b''] \in B(G)$ be the image of $[b''_H] \in B(H)$ via the identification $B(H) \simeq B(G)$. In fact $[b''] = [\iota(b''_H)b_0] \in B(G)$. So

$$-w_0 v_{b,x} = -w_0 v_{b_H,x_H} + \nu_{b_0} = \nu_{b''_H} + \nu_{b_0} = \nu_{b''}$$

and

$$\kappa_G(b'') = \kappa_H(b''_H) + \kappa_G(b_0) = \kappa_H(b'_H) + \kappa_G(b_0) = \kappa_G(b'). \qquad \square$$

4.4 Harder–Narasimhan stratification on B^+_{dR}-affine Grassmannian

Let $b \in G(\breve{F})$. Then we have the following simple observation.

Lemma 4.9. *Let $\mathrm{Spa}(C_2, C_2^+) \to \mathrm{Spa}(C_1, C_1^+)$ be a morphism in $\mathbf{Perf}/\mathrm{Spd}(F)$ with C_1, C_2 two algebraically closed perfectoid fields with $C_1^+ \subset C_1$ and $C_2^+ \subset C_2$ bounded valuation subrings. For $x_1 \in \mathrm{Gr}_G(C_1, C_1^+)$ with image $x_2 \in \mathrm{Gr}_G(C_2, C_2^+)$, we have*

$$\mathrm{HN}(b, x_1) = \mathrm{HN}(b, x_2).$$

Proof. By Tannakian duality, we reduce to the case $G = \mathrm{GL}_V$ with V a finite-dimensional F-vector space. Write $(C_i^\sharp, C_i^{\sharp +})$ for the corresponding untilt of (C_i, C_i^+), $B_i^+ := B^+_{\mathrm{dR}}(C_i^\sharp)$ and $B_i = B_{\mathrm{dR}}(C_i^\sharp)$ for $i = 1, 2$. Then the natural map $B_1^+ \to B_2^+$ is an unramified extension of DVRs whose residue field extension is just $C_1^\sharp \subset C_2^\sharp$. Let $\Xi_1 \subset V \otimes_F B_1$ be a B_1^+-lattice and $\Xi_2 = \Xi_1 \otimes_{B_1^+} B_2^+$. Then the Harder–Narasimhan filtration for the normed isocrystal $(V \otimes \breve{F}, b, \alpha_{\Xi_1})$ coincides with that for $(V \otimes \breve{F}, b, \alpha_{\Xi_2})$, from where our lemma follows. $\square$

Consequently, we dispose of a map

$$\mathrm{HN}_b : |\mathrm{Gr}_G| \longrightarrow B(G), \quad x \longmapsto \mathrm{HN}(b, x). \tag{4.8}$$

Let $[b''] \in B(G)$. We write

$$|\mathrm{Gr}_G|^{\mathrm{HN}_b \geq [b'']} \quad (\text{respectively } |\mathrm{Gr}_G|^{\mathrm{HN}_b = [b'']}),$$

the subset of $|\mathrm{Gr}_G|$ consisting of points x such that

$$\mathrm{HN}(b, x) \geq [b''] \quad (\text{respectively } \mathrm{HN}(b, x) = [b'']).$$

Write also

$$\mathrm{Gr}_{G,b}^{\mathrm{HN}=[b'']} := \mathrm{Gr}_G \times_{|\mathrm{Gr}_G|} |\mathrm{Gr}_G|^{\mathrm{HN}_b \geq [b'']} \quad \big(\text{respectively } \mathrm{Gr}_{G,b}^{\mathrm{HN} \geq [b'']} := \mathrm{Gr}_G \times_{|\mathrm{Gr}_G|} |\mathrm{Gr}_G|^{\mathrm{HN}_b = [b'']}\big),$$

that is, the v-subsheaf of Gr_G, such that a morphism $S \to \mathrm{Gr}_G$ from a perfectoid space $S \in \mathbf{Perf}/\mathrm{Spd}(F)$ factors through $\mathrm{Gr}_{G,b}^{\mathrm{HN}=[b'']}$ (respectively $\mathrm{Gr}_{G,b}^{\mathrm{HN} \geq [b'']}$) if and only if $\mathrm{HN}(b, x) = [b'']$ (respectively $\mathrm{HN}(b, x) \geq [b'']$) for every geometric point x of S. On the other hand, by the local

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constancy of the Kottwitz invariant [FS21, Theorem III.2.7], the following composed map is locally constant:

$$\kappa_b : |\mathrm{Gr}_G| \xrightarrow{BL_b} |\mathrm{Bun}_G| \xrightarrow{\kappa} \pi_1(G)_\Gamma.$$

Here the first map is the Beauville–Laszlo morphism defined by using the element $b \in G(\breve{F})$. The preimage of an element $\alpha \in \pi_1(G)_\Gamma$ is then the underlying topological space of an open and closed v-subsheaf, denoted by

$$\mathrm{Gr}_G^{\kappa_b=\alpha},$$

of Gr_G, and we deduce a decomposition in open and closed subsheaves

$$\mathrm{Gr}_G = \coprod_{\alpha \in \pi_1(G)_\Gamma} \mathrm{Gr}_G^{\kappa_b=\alpha}.$$

For a geometric conjugacy class $\{\mu\}$ of cocharacters of G, write also

$$\mathrm{Gr}_{G,\leq\mu}^{\kappa_b=\alpha} = \mathrm{Gr}_{G,\bar{F}}^{\kappa_b=\alpha} \cap \mathrm{Gr}_{G,\leq\mu}.$$

Proposition 4.10. *Let $[b''] \in B(G)$.*

(i) *Suppose moreover G quasi-split. Let $S \to \mathrm{Spd}(\bar{F})$ be a geometric point of $\mathrm{Spd}(\bar{F})$. We have*

$$\mathrm{Gr}_{G,b}^{\mathrm{HN}\geq[b'']}(S) = \left(\bigcup_{\substack{\text{reduction } (b_P,g) \text{ of } b \text{ to} \\ \text{a standard parabolic } P}} g \cdot \left(\bigcup_{\substack{u \in X_*(M^{ab}) \text{ such that} \\ -\mathrm{Av}_M(\nu_{b_M}) + u^\diamond \geq -w_0\nu_{b''}}} \mathrm{Gr}_P^u(S) \right) \right) \bigcap \mathrm{Gr}_G^{\kappa_b=\kappa(b'')}(S),$$

where:

- *Gr_P^u is defined in § 2.2.2;*
- *$[b_M]$ is the image of $[b_P]$ via the natural map $B(P) \to B(M)$;*
- *in the inequality, $u^\diamond$ is the Galois average of u and is considered to be an element in $X_*(A)_\mathbb{Q}$ via the natural inclusion $X_*(M^{ab})_\mathbb{Q}^\Gamma \subset X_*(A)_\mathbb{Q}$.*

(ii) *For general reductive group G and for a geometric conjugacy class $\{\mu\}$ of cocharacters of G, the subfunctor*

$$\mathrm{Gr}_{G,b,\bar{F}}^{\mathrm{HN}\geq[b'']} \cap \mathrm{Gr}_{G,\leq\mu} \quad \left(\text{respectively } \mathrm{Gr}_{G,b,\bar{F}}^{\mathrm{HN}=[b'']} \cap \mathrm{Gr}_{G,\leq\mu}\right)$$

is closed (respectively locally closed) in $\mathrm{Gr}_{G,\leq\mu}$, and thus a spatial diamond (respectively a locally spatial diamond). In particular, the restriction to $|\mathrm{Gr}_{G,\leq\mu}|$ of the map HN_b in (4.8) is upper semi-continuous.

Proof. (i) Take $x \in \mathrm{Gr}_{G,b}^{\mathrm{HN}\geq[b'']}(S)$. Then $v_{b,x} \geq -w_0\nu_{b''}$ and $\kappa_b(x) = \kappa(b'')$ by Corollary 4.8. Let $M \subset G$ be the centralizer of $v_{b,x}$, with $P \subset G$ the corresponding standard parabolic subgroup. In particular, b has a reduction (b_P, g) to P such that

$$\mathcal{E}_{b,x} \simeq \mathcal{E}_{b_P,g^{-1}x}$$

with $g^{-1}x \in \mathrm{Gr}_P^u$ for some $u \in X_*(M^{ab})$. Moreover, the HN-vector $v_{b,x}$ is the slope vector of the M-bundle $\mathcal{E}_{b_M,\mathrm{pr}_M(g^{-1}x)}$, which is nothing but $-\mathrm{Av}_M(\nu_{b_M}) + u^\diamond$. To see the last assertion, write $x_M := \mathrm{pr}_M(g^{-1}x) \in \mathrm{Gr}_M^u$. Note that, for any $\chi \in X^*(M)^\Gamma$, we have

$$\langle v(\mathcal{E}_{b_M,x_M}), \chi\rangle = \deg(\chi_*(\mathcal{E}_{b_M,x_M})) = \deg(\mathcal{E}_{\chi(b_M),\chi(x_M)}) = \langle -\nu_{b_M}, \chi\rangle + \langle u, \chi\rangle = \langle -\nu_{b_M} + u^\diamond, \chi\rangle.$$

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Here the last equality holds because χ is Γ-invariant, and therefore $\langle u, \chi\rangle = \langle u^\diamond, \chi\rangle$. As a result

$$v\left(\mathcal{E}_{b_M, x_M}\right) = \mathrm{Av}_M(-\nu_{b_M} + u^\diamond) = -\mathrm{Av}_M(\nu_{b_M}) + u^\diamond,$$

as asserted. It follows that

$$-\mathrm{Av}_M(\nu_{b_M}) + u^\diamond \geq -w_0\nu_{b''}.$$

Consequently, $x \in \mathrm{Gr}_{G,b}^{\mathrm{HN}\geq[b'']}(S)$ is contained in the right-hand side of the equality in (i).

Conversely, suppose that b has a reduction (b_P, g) and $x \in g\mathrm{Gr}_P^u(S)$ for some $u \in X_*(M^{ab})$ with $-\mathrm{Av}_M(\nu_{b_P}) + u^\diamond \geq -w_0\nu_{b''}$. The slope vector of the M-bundle $\mathcal{E}_{b_M, \mathrm{pr}(g^{-1}x)}$ is precisely

$$-\mathrm{Av}_M(\nu_{b_M}) + u^\diamond.$$

Therefore,

$$v_{b,x} \geq -\mathrm{Av}_M(\nu_{b_P}) + u^\diamond \geq -w_0\nu_{b''}.$$

If moreover $\kappa_b(x) = \kappa(b'')$, we then obtain $\mathrm{HN}(b, x) \geq [b'']$, as desired.

(ii) Clearly we may assume that G quasi-split and it is enough to show that $\mathrm{Gr}_{G,b}^{\mathrm{HN}\geq[b'']} \cap \mathrm{Gr}_{G,\leq\mu}$ is closed in $\mathrm{Gr}_{G,\leq\mu}$, i.e., for every totally disconnected perfectoid space S mapping to $\mathrm{Gr}_{G,\leq\mu}$, the pullback

$$(\mathrm{Gr}_{G,b}^{\mathrm{HN}\geq[b'']} \cap \mathrm{Gr}_{G,\leq\mu}) \times_{\mathrm{Gr}_{G,\leq\mu}} S$$

is representable by a closed immersion to S (cf. [Sch17, Definition 10.7 (ii)]). Since

$$\mathrm{Gr}_{G,b}^{\mathrm{HN}\geq[b'']} \cap \mathrm{Gr}_{G,\leq\mu} = \mathrm{Gr}_{G,\leq\mu} \times_{|\mathrm{Gr}_{G,\leq\mu}|} |\mathrm{Gr}_{G,b}^{\mathrm{HN}\geq[b'']} \cap \mathrm{Gr}_{G,\leq\mu}|$$

according to [Sch17, Lemma 7.6 and Definition 5.6], we only need to show that the subset

$$|\mathrm{Gr}_{G,b}^{\mathrm{HN}\geq[b'']} \cap \mathrm{Gr}_{G,\leq\mu}| \subset |\mathrm{Gr}_{G,\leq\mu}|$$

is closed and generalizing.

For a standard parabolic P and $\lambda := -w_0\nu_{b''}$, consider the following v-subsheaves

$$\mathcal{S}_{b_P,\lambda} := \left(\bigcup_{\substack{u \in X_*(M^{ab}) \text{ such that} \\ -\mathrm{Av}_M(\nu_{b_M}) + u^\diamond \geq \lambda}} \mathrm{Gr}_P^u \right) \cap \mathrm{Gr}_{G,\leq\mu}^{\kappa_b = \kappa(b'')}$$

and

$$\mathcal{S} := \bigcup_{\substack{\text{reduction } (b_P, g) \text{ of } b \text{ to} \\ \text{a standard parabolic } P}} g \cdot \mathcal{S}_{b_P,\lambda}.$$

Clearly we have inclusion

$$\mathcal{S} \subset \mathrm{Gr}_{G,b}^{\mathrm{HN}\geq[b'']} \cap \mathrm{Gr}_{G,\leq\mu}$$

of subsheaves of $\mathrm{Gr}_{G,\leq\mu}$ and they have the same geometric points by (i). So

$$|\mathcal{S}| = |\mathrm{Gr}_{G,b}^{\mathrm{HN}\geq[b'']} \cap \mathrm{Gr}_{G,\leq\mu}| \subset |\mathrm{Gr}_{G,\leq\mu}|.$$

Moreover, for $g \in P(\breve{F})$, $g\mathrm{Gr}_P^u = \mathrm{Gr}_P^u$, so the subfunctor $\mathcal{S}_{b_P,\lambda}$ is preserved by the left multiplication by an element of $P(\breve{F})$.

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We first claim that $|\mathcal{S}_{b_P,\lambda}|$ is closed in $|\mathrm{Gr}_{G,\leq\mu}|$. Indeed, for $u, u' \in X_*(M^{ab})$ with $u' \geq u$, we have also $u'^{\diamond} \geq u^{\diamond}$. It follows that

$$\mathcal{S}_{b_P,\lambda} = \left(\bigcup_{\substack{u\in X_*(M^{ab}) \text{ such that} \\ -\mathrm{Av}_M(\nu_{b_M})+u^{\diamond}\geq\lambda}} \mathrm{Gr}_P^u \right) \cap \mathrm{Gr}_{G,\leq\mu}^{\kappa_b=\kappa(b'')}$$
$$= \bigcup_{\substack{u\in X_*(M^{ab}) \text{ such that} \\ -\mathrm{Av}_M(\nu_{b_M})+u^{\diamond}\geq\lambda}} \left(\left(\bigcup_{\substack{u'\in X_*(M^{ab}) \\ \text{with } u'\geq u}} \mathrm{Gr}_P^{u'} \right) \cap \mathrm{Gr}_{G,\leq\mu}^{\kappa_b=\kappa(b'')} \right).$$

On the other hand, the union

$$\bigcup_{\substack{u'\in X_*(M^{ab}) \\ \text{with } u'\geq u}} \mathrm{Gr}_P^{u'}$$

is closed in Gr_G by Proposition 2.8, and there exist only finitely many $u \in X_*(M^{ab})$ with $\mathrm{Gr}_P^u \cap \mathrm{Gr}_{G,\leq\mu} \neq \emptyset$ by Remark 2.12. So $\mathcal{S}_{b_P,\lambda}$ is a union of finitely many closed subfunctors of $\mathrm{Gr}_{G,\leq\mu}$. Consequently

$$|\mathcal{S}_{b_P,\lambda}| \subset |\mathrm{Gr}_{G,\leq\mu}|$$

is closed.

Next we show that $|\mathcal{S}| \subset |\mathrm{Gr}_{G,\leq\mu}|$ is also closed. By Lemma 4.11 below, $[b] \in B(G)$ has a finite preimage via the natural map $B(M) \to B(G)$. We denote by $[b_i] \in B(M)$ $(1 \leq i \leq n)$ all these preimages. Suppose $g_i b_i \sigma(g_i)^{-1} = b$ with $g_i \in G(\breve{F})$, and write $J_i := J_{b_i}$ where b_i is viewed as an element of $G(\breve{F})$. Let (b_P, g) be a reduction of b to P. By Lemma 2.2, there exist some $p \in P(\breve{F})$ and $1 \leq i \leq n$ such that $b_P = p b_i \sigma(p)^{-1}$, and thus

$$b = g b_P \sigma(g)^{-1} = (gp) b_i \sigma(gp)^{-1} = g_i b_i \sigma(g_i)^{-1}.$$

Therefore, $g_i^{-1} gp \in J_i(F)$. Hence all the reductions of b to P are of the form

$$(p b_i \sigma(p)^{-1}, g) \quad \text{with } p \in P(\breve{F}) \text{ and } g \in g_i J_i(F) p^{-1},$$

for some $1 \leq i \leq n$. Moreover, as $[b_M] = [b_i] \in B(M)$, it follows that $\mathcal{S}_{b_P,\lambda} = \mathcal{S}_{b_i,\lambda}$, and

$$\mathcal{S} = \bigcup_{\text{std parabolic } P} \left(\bigcup_{\substack{\text{reduction } (b_P,g) \\ \text{of } b \text{ to } P}} g \cdot \mathcal{S}_{b_P,\lambda} \right) = \bigcup_{\text{std parabolic } P} \left(\bigcup_{i=1}^{n} g_i \cdot \left(\bigcup_{g\in J_i(F)} g \cdot \mathcal{S}_{b_i,\lambda} \right) \right).$$

Since, for a fixed standard parabolic P, the closed subfunctor $\mathcal{S}_{b_i,\lambda}$ is preserved by the action of an element of $P(\breve{F})$, the union

$$\bigcup_{g\in J_i(F)} g \cdot |\mathcal{S}_{b_i,\lambda}|$$

is closed in $|\mathrm{Gr}_{G,\leq\mu}|$ if we can show that the quotient $J_i(F)/J_i(F) \cap P(\breve{F})$ is compact. But $J_i(F) \cap P(\breve{F})$ is the set of F-points of a parabolic subgroup $P_i \subset J_i$, so the compactness of

$$J_i(F)/J_i(F) \cap P(\breve{F}) = J_i(F)/P_i(F) \simeq (J_i/P_i)(F)$$

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follows from the fact that J_i/P_i is projective over locally compact field F (see [Spr10, 15.1.4 Corollary] for the last identification above). As a result, being a finite union of closed subsets,

$$|\mathcal{S}| = \bigcup_{\text{std parabolic } P} \left(\bigcup_{i=1}^{n} g_i \cdot \left(\bigcup_{g \in J_i(F)} g \cdot |\mathcal{S}_{b_i,\lambda}| \right) \right)$$

is closed in $|\mathrm{Gr}_{G,\leq\mu}|$.

Finally we show that the closed subset $|\mathcal{S}| \subset |\mathrm{Gr}_{G,\leq\mu}|$ is generalizing. To check this, let

$$f : T \longrightarrow \mathrm{Gr}_{G,\leq\mu}$$

be a surjective map from a perfectoid space (such a map exists since $\mathrm{Gr}_{G,\leq\mu}$ is diamond). Because the induced map $|f| : |T| \to |\mathrm{Gr}_{G,\leq\mu}|$ is generalizing [Sch17, Proposition 11.18] and surjective, it is enough to check that the locus of points $x \in T$ whose image by $|f|$ is contained in $|\mathcal{S}|$ is generalizing. Since T is an analytic adic space (and hence every specialization in T is vertical), we are reduced to showing that for any perfectoid field C with an open and bounded valuation subring $C^+ \subset C$, and every morphism $\mathrm{Spa}(C, C^+) \to \mathrm{Gr}_{G,\leq\mu}$, the induced map

$$|\mathrm{Spa}(C, C^+)| \longrightarrow |\mathrm{Gr}_{G,\leq\mu}|$$

factors through $|\mathcal{S}|$ if and only if $|\mathcal{S}|$ contains the image of the closed point of $\mathrm{Spa}(C, C^+)$. But this is clear since $\mathrm{Gr}_G(C, C^+)$, $\mathrm{Gr}^u_P(C, C^+)$ and also the decomposition

$$\mathrm{Gr}_G(C, C^+) = \coprod_{u \in X_*(M^{ab})} \mathrm{Gr}^u_P(C, C^+)$$

do not depend on the choice of C^+. □

Lemma 4.11. *Let G be a quasi-split group and $M \subset G$ a standard Levi. Then every element in $B(G)$ has at most finite preimages through the map $B(M) \to B(G)$.*

Proof. Recall that, by [RR96, 4.13], the combination of the Newton map and the Kottwitz map give an injective map

$$(\nu, \kappa) : B(G) \longrightarrow \mathcal{N}(G) \times \pi_1(G)_\Gamma.$$

Moreover, up to torsion, the Kottwitz map κ is determined by the Newton map ν. As the subgroup of torsion elements in $\pi_1(G)_\Gamma$ is finite, to prove our lemma it suffices to check that the natural map

$$\mathcal{N}(M) \longrightarrow \mathcal{N}(G)$$

is finite-to-one. But this is clear since the later can be embedded into

$$X_*(A)^{M,+}_{\mathbb{Q}} \longrightarrow X_*(A)^{+}_{\mathbb{Q}}, \quad \lambda \longmapsto \lambda_{\mathrm{dom}},$$

which is finite-to-one. □

Definition 4.12. Fix a triple (G, μ, b) and $[b'] \in B(G)$.

(i) Let

$$\mathrm{Gr}^{\mathrm{HN}=[b']}_{G,\mu,b} := \mathrm{Gr}^{\mathrm{HN}=[b']}_{G,b} \cap \mathrm{Gr}_{G,\mu},$$

$$\mathrm{Gr}^{\mathrm{HN}\geq[b']}_{G,\mu,b} := \mathrm{Gr}^{\mathrm{HN}\geq[b']}_{G,b} \cap \mathrm{Gr}_{G,\mu}.$$

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The subfunctor $\mathrm{Gr}_{G,\mu,b}^{\mathrm{HN}=[b']}$ (respectively $\mathrm{Gr}_{G,\mu,b}^{\mathrm{HN}\geq[b']}$) is locally closed (respectively closed) in $\mathrm{Gr}_{G,\mu}$. This defines the Harder–Narasimhan stratification on $\mathrm{Gr}_{G,\mu}$. We will also call it Harder–Narasimhan stratification for simplicity.

(ii) If b' is basic, then the Harder–Narasimhan stratum $\mathrm{Gr}_{G,\mu,b}^{\mathrm{HN}=[b']}$ is said to be *semi-stable* or *weakly admissible*. There is at most one non-empty semi-stable locus and is denoted by $\mathrm{Gr}_{G,\mu,b}^{wa}$.

Remark 4.13.

(i) Harder–Narasimhan stratification on the B_{dR}^+-affine Grassmannian has been studied by many people. When μ is minuscule, this has been studied by Dat, Orlik and Rapoport [DOR10, IX.6]. Indeed, they study the Harder–Narasimhan stratification on the flag variety $\mathcal{F}(G,\mu)$ for arbitrary μ. When μ is minuscule, the flag variety $\mathcal{F}(G,\mu)$ is isomorphic to $\mathrm{Gr}_{G,\mu}$ via the Bialynicki–Birula map. When $G=\mathrm{GL}_n$ or $b=1$ (or more generally b is basic), the Harder–Narasimhan stratification has been studied by Nguyen and Viehmann [NV23] and Shen [She23] based on a result about the compatibility of Harder–Narasimhan filtration with tensor product by Cornut and Peche Irissarry [CPI19], which is Theorem 3.10 for the case $b=1$.

(ii) Here the definition of the weakly admissible locus $\mathrm{Gr}_{G,\mu,b}^{wa}$ is the same as the definition in [Vie24, NV23] and is more general than the notion in some of the literature such as [RZ96, CFS21, She23], where the weakly admissible locus is referred to $\mathrm{Gr}_{G,\mu,b}^{\mathrm{HN}=[b']}$ with $[b']=[1]$. Therefore, in the literature, the condition $[b]\in B(G,\mu)$ is usually imposed. Under this condition, our definition of weakly admissible locus coincides with theirs in [RZ96, CFS21, She23].

5. Compatibility of Harder–Narasimhan stratification

In the rest of the article, we change the notation. Let C be a complete algebraically closed non-archimedean field over $\bar{F}$ (unlike in the previous sections, C usually stands for a complete algebraically closed non-archimedean field of characteristic p). Let $C^+\subset C$ be an open valuation subring. To simplify notation, we write $\mathrm{Gr}_G(C,C^+)$ (respectively $\mathrm{Gr}_{G,\mu}(C,C^+)$, respectively $\mathrm{Gr}_{G,\leq\mu}(C,C^+)$) for $\mathrm{Gr}_G(C^\flat,C^{\flat,+})$ (respectively $\mathrm{Gr}_{G,\mu}(C^\flat,C^{\flat,+})$, respectively $\mathrm{Gr}_{G,\leq\mu}(C^\flat,C^{\flat,+})$). As $\mathrm{Gr}_G(C,C^+)$ only depends on C but not on C^+, we also write $\mathrm{Gr}_G(C)$ for $\mathrm{Gr}_G(C,C^+)$. The notation $\mathrm{Gr}_{G,\mu}(C)$ and $\mathrm{Gr}_{G,\leq\mu}(C)$ are similar.

In this section, we want to compare the Harder–Narasimhan stratification for different groups (inner twists and adjoint) and also compare it with the Newton stratification on $\mathrm{Gr}_{G,\mu}$ and the Harder–Narasimhan stratification on flag varieties.

5.1 Compatibility with inner twists

We take the notation as in §4.2. Let $H=J_{b_0}$ be an inner twist of G with $b_0\in G(\breve{F})$ basic. Then the inner twisting $\iota: H_{\breve{F}}\to G_{\breve{F}}$ induces an isomorphism $\mathrm{Gr}_{H,\breve{F}}\xrightarrow{\sim}\mathrm{Gr}_{G,\breve{F}}$ on which the Harder–Narasimhan stratifications on both sides are compatible by Proposition 4.3, i.e.

$$\mathrm{Gr}_{H,\mu,b}^{\mathrm{HN}=[b']}\xrightarrow{\sim}\mathrm{Gr}_{G,\iota(\mu),\iota(b)b_0}^{\mathrm{HN}=[\iota(b')b_0]}.$$

Therefore, when we study the properties of Harder–Narasimhan strata, we may always assume that G is quasi-split.

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5.2 Compatibility with adjoint groups

Consider the natural homomorphism

$$(-)^{ad} : G \longrightarrow G^{ad}.$$

This induces a morphism

$$\pi : \mathrm{Gr}_G \longrightarrow \mathrm{Gr}_{G^{ad}},$$

which is compatible with Harder–Narasimhan stratification on both sides in the following sense.

Proposition 5.1. *Suppose* $\mathrm{Gr}^{\mathrm{HN}=[\tilde{b'}]}_{G^{ad},\mu^{ad},b^{ad}}$ *is non-empty, then*

$$\pi^{-1}(\mathrm{Gr}^{\mathrm{HN}=[\tilde{b'}]}_{G^{ad},\mu^{ad},b^{ad}}) = \mathrm{Gr}^{\mathrm{HN}=[b']}_{G,\mu,b},$$

where $[b'] \in B(G)$ *is the unique element which is mapped to* $[\tilde{b'}]$ *via* $B(G) \to B(G^{ad})$ *and to* $\kappa_G(b') = \kappa_G(b) - \mu^\sharp$ *in* $\pi_1(G)_\Gamma$.

Proof. Here the existence and uniqueness of $[b']$ is due to Kottwitz [Kot97, Corollary 4.11].

By § 5.1, we may assume that G is quasi-split. Note that, for any $x \in \mathrm{Gr}_G(C)$,

$$\mathcal{E}_{b^{ad},\pi(x)} = \mathcal{E}_{b,x} \times^G G^{ad}.$$

In addition, there is a canonical bijection

$$\{\text{standard parabolic subgroups of } G\} \longrightarrow \{\text{standard parabolic subgroups of } G^{ad}\},$$
$$Q \longmapsto Q^{ad}.$$

Moreover, the natural map

$$\{\text{reduction of } b \text{ to } Q\} \longrightarrow \{\text{reduction of } b^{ad} \text{ to } Q^{ad}\},$$
$$(b_Q, g) \longmapsto (b_Q^{ad}, g^{ad})$$

is surjective with slope vectors

$$v^{ad}_{(b_Q,g)} = v_{(b_Q^{ad},g^{ad})},$$

where the slope vectors are defined in § 4.3. Then the result follows from Proposition 4.5. □

5.3 Compatibility with Newton stratification

On the B^+_{dR}-affine Grassmannian $\mathrm{Gr}_{G,\mu}$, we have another stratification called the Newton stratification. Let us first briefly recall its definition. For each geometric point $x \in \mathrm{Gr}_{G,\mu}(C, C^+)$, the isomorphism class of the modification $\mathcal{E}_{b,x}$ of the G-bundle $\mathcal{E}_b$ corresponds to a unique element $\mathrm{New}(b, x) \in B(G)$ by a theorem of Fargues ([Far20], see also § 2.1.3). So we obtain a map

$$\mathrm{Gr}_{G,\mu}(C, C^+) \longrightarrow B(G).$$

Letting (C, C^+) vary, we deduce a map

$$\mathrm{Newt}_b : |\mathrm{Gr}_{G,\mu}| \longrightarrow B(G), \quad x \longmapsto \mathrm{New}(b, x). \tag{5.1}$$

For $[b'] \in B(G)$, the corresponding Newton strata $\mathrm{Gr}^{\mathrm{New}=[b']}_{G,\mu,b}$ is defined to be the preimage of $[b'] \in B(G)$ via the map New_b; so for any complete algebraically closed field extension C of $\breve{F}$ and for any valuation subring $C^+ \subset C$,

$$\mathrm{Gr}^{\mathrm{New}=[b']}_{G,\mu,b}(C, C^+) = \{x \in \mathrm{Gr}_{G,\mu}(C, C^+) | \mathcal{E}_{b,x} \simeq \mathcal{E}_{b'}\}.$$

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In particular, we have a decomposition

$$\mathrm{Gr}_{G,\mu} = \coprod_{[b']\in B(G)} \mathrm{Gr}_{G,\mu,b}^{\mathrm{New}=[b']} \tag{5.2}$$

of the B_{dR}^{+}-affine Grassmannian $\mathrm{Gr}_{G,\mu}$. Write also

$$\mathrm{Gr}_{G,\mu,b}^{\mathrm{New}\geq[b']} = \bigcup_{B(G)\ni[b'']\geq[b']} \mathrm{Gr}_{G,\mu,b}^{\mathrm{New}=[b'']}.$$

The non-empty Newton stratum $\mathrm{Gr}_{G,\mu,b}^{\mathrm{New}=[b']}$ with $[b']\in B(G)$ basic is called the admissible locus and is denoted by $\mathrm{Gr}_{G,\mu,b}^{a}$.

Remark 5.2. The definition of the admissible locus is more general to the one in some literature which is parallel to the situation of the weakly admissible locus (cf. Remark 4.13(ii)). It is also known that each Newton stratum $\mathrm{Gr}_{G,\mu,b}^{\mathrm{New}=[b'']}$ is locally closed and $\mathrm{Gr}_{G,\mu,b}^{\mathrm{New}\geq[b'']}$ is closed in $\mathrm{Gr}_{G,\mu}$ [KL15, Theorem 7.4.5, CS17, Corollary 3.5.9, Vie24, § 3.1].

Proposition 5.3. *For any $x\in \mathrm{Gr}_{G,\mu}(C)$, we have*

$$\mathrm{HN}(b,x)\leq \mathrm{New}(b,x),$$

which implies

$$\mathrm{Gr}_{G,\mu,b}^{\mathrm{HN}=[b']} \subseteq \mathrm{Gr}_{G,\mu,b}^{\mathrm{New}\geq[b']}.$$

In particular, when $[b]\in B(G,\mu)$, we have $\mathrm{Gr}_{G,\mu,b}^{a}\subseteq \mathrm{Gr}_{G,\mu,b}^{wa}$.

Proof. The inequality $\mathrm{HN}(b,x)\leq \mathrm{New}(b,x)$ follows from Corollary 4.8, from which the second assertion follows. □

Remark 5.4. When $b=1$, this result is known from Nguyen and Viehmann [NV23, Lemma 6.1] and Shen [She23, Proposition 3.4].

5.4 Compatibility with Harder–Narasimhan stratification on flag varieties

Dat, Orlik and Rapoport defined in [DOR10, IX.6] the Harder–Narasimhan formalism for the flag variety associated to the triple (G,μ,b). We recall briefly its construction which is parallel to ours.

Let $K|\breve{F}$ be a field extension. Denote by

$$\mathbf{FilIsoc}_{\breve{F}|F}^{K}$$

the category of filtered isocrystals consisting of triples $(N,\varphi,\mathrm{Fil}^{\bullet}N_K)$, where:

- (N,φ) is an isocrystal over $\breve{F}|F$;
- $\mathrm{Fil}^{\bullet}N_K$ is a separated exhaustive decreasing $\mathbb{Z}$-filtration of K-subspaces on $N_K := N\otimes_{\breve{F}} K$.

For an object $(N,\varphi,\mathrm{Fil}^{\bullet}N_K)$ in $\mathbf{FilIsoc}_{K|\breve{F}}$, define the slope

$$\mathrm{slope}(N,\varphi,\mathrm{Fil}^{\bullet}N_K) := \frac{\deg(N,\varphi,\mathrm{Fil}^{\bullet}N_K)}{\mathrm{rank}(N,\varphi,\mathrm{Fil}^{\bullet}N_K)},$$

where

$$\mathrm{rank}(N,\varphi,\mathrm{Fil}^{\bullet}N_K) := \mathrm{rank}_{\breve{F}}N,$$
$$\deg(N,\varphi,\mathrm{Fil}^{\bullet}N_K) := \deg(\mathrm{Fil}^{\bullet}N_K) - \deg(N,\varphi),$$

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with $\deg(\mathrm{Fil}^\bullet N_K) := \sum_{i\in\mathbb{Z}} i \dim_K(\mathrm{Fil}^i N_K/\mathrm{Fil}^{i+1}N_K)$, and $\deg(N,\varphi) = v_F(\det(\varphi))$ is the p-adic valuation of the determinant of φ where v_F denotes the normalized valuation on F. This slope function leads to a Harder–Narasimhan formalism on the category $\mathbf{FilIsoc}_{K|\breve{F}}$. In particular, we have the Harder–Narasimhan filtration for each object which defines the functor

$$\tilde{\mathcal{F}}_{\mathrm{HN}} : \mathbf{FilIsoc}^K_{\breve{F}|F} \longrightarrow \mathbf{F}(\mathrm{Isoc}_{\breve{F}|F}).$$

It is known that $\tilde{\mathcal{F}}_{\mathrm{HN}}$ is compatible with tensor products [Fal94, Tot94].

Now we apply a similar argument as in § 4.3 to construct the HN-vector associated to a pair (b,x) with $x \in \mathcal{F}(G,\mu)(K)$. We have the functor

$$\begin{array}{rccc} \tilde{\mathcal{F}}_{HN}(b,x): & \mathrm{Rep}_F(G) & \longrightarrow & \mathbf{FilIsoc}^K_{\breve{F}|F} \xrightarrow{\tilde{\mathcal{F}}_{HN}} \mathbf{F}(\mathrm{Isoc}_{\breve{F}|F}), \\ & (V,\rho) & \longmapsto & (V_{\breve{F}}, \rho(b)\sigma, \mathrm{Fil}^\bullet_{\rho(x)} V_K). \end{array}$$

There exists some rational cocharacter of G which splits $\tilde{\mathcal{F}}_{HN}(b,x)$. Its conjugacy class does not depend on the choice of the splitting and is denoted by $\tilde{v}_{b,x}$. Similar arguments as Corollary 4.8 show that there exists a unique element $\mathrm{HN}(b,x) \in B(G)$ called the Harder–Narasimhan vector for the pair (b,x), such that

$$\nu_{\mathrm{HN}(b,x)} = -\tilde{v}_{b,x} \quad \text{and} \quad \kappa(\mathrm{HN}(b,x)) = \kappa(b') = \kappa(b) - \mu^\sharp.$$

The parametrization by the HN-vector gives the Harder–Narasimhan stratification on the flag variety $\mathcal{F}(G,\mu)$:

$$\mathcal{F}(G,\mu) = \coprod_{[b']\in B(G)} \mathcal{F}(G,\mu,b)^{\mathrm{HN}=[b']},$$

where, for each geometric point $x \in \mathcal{F}(G,\mu)(C,C^+)$,

$$x \in \mathcal{F}(G,\mu,b)^{\mathrm{HN}=[b']}(C,C^+) \Longleftrightarrow \mathrm{HN}(b,x) = [b'].$$

Define also

$$\mathcal{F}(G,\mu,b)^{\mathrm{HN}\geq[b']} := \coprod_{[b'']\geq[b']} \mathcal{F}(G,\mu,b)^{\mathrm{HN}=[b'']}.$$

Remark 5.5. In [DOR10], the index for the each Harder–Narasimhan stratum is given by $\tilde{v}_{b,x}$. Here we take the index to be $\mathrm{HN}(b,x)$ in order to be compatible with the index of the Harder–Narasimhan strata and Newton strata on the B^+_{dR}-Grassmannian.

Proposition 5.6. *Via the Bialynicki–Birula map* $\mathrm{BB} : \mathrm{Gr}_{G,\mu} \to \mathcal{F}(G,\mu)$ *we have*

$$\mathrm{HN}(b,x) \leq \mathrm{HN}(b,\mathrm{BB}(x)).$$

In particular,

$$\mathrm{BB}^{-1}(\mathcal{F}(G,\mu,b)^{\mathrm{HN}=[b']}) \subseteq \coprod_{[b'']\leq[b']\in B(G),} \mathrm{Gr}^{\mathrm{HN}=[b'']}_{G,\mu,b}$$

and $\mathrm{Gr}^{wa}_{G,\mu,b} \supseteq \mathrm{BB}^{-1}(\mathcal{F}(G,\mu,b)^{ss})$.

Proof. By the Tannakian formalism, we reduce to the GL_n case. In this case, the BB map gives a functor still denoted by BB:

$$\begin{aligned} \mathrm{BB} : \mathbf{BunIsoc}^{B_{\mathrm{dR}}}_{\breve{F}|F} &\longrightarrow \mathbf{FilIsoc}^C_{\breve{F}|F}, \\ (D,\Xi) &\longmapsto (D,\mathcal{F}_\Xi), \end{aligned}$$

where $\mathcal{F}_\Xi$ is defined in Remark 3.22.

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We want to compare the HN vectors on both sides. First note that the BB map preserves the rank and degree functions, i.e., $\deg(D,\Xi)=\deg(D,\mathcal{F}_\Xi)$ and $\operatorname{rank}(D,\Xi)=\operatorname{rank}(D,\mathcal{F}_\Xi)$. Let D_1 be a sub-isocrystal of D. Let (D_1,Ξ_1) (respectively $(D_1,\mathcal{F}_{\Xi,1})$) be the corresponding strict sub-object of (D,Ξ) (respectively $(D,\mathcal{F}_\Xi)$) in $\mathrm{BunIsoc}^{B_{\mathrm{dR}}}_{\breve{F}|F}$ (respectively $\mathrm{FilIsoc}^{C}_{\breve{F}|F}$), so

$$\Xi_1=(D_1\otimes_{\breve{F}}B_{\mathrm{dR}})\cap\Xi,$$

and $\mathcal{F}_{\Xi,1}$ is the intersection of $\mathcal{F}_\Xi$ with $D_1\otimes_{\breve{F}}C$. In general, the BB map does *not* preserve strict sub-objects, i.e., $\mathcal{F}_{\Xi,1}\neq\mathcal{F}_{\Xi_1}$. More precisely note that, for $i\in\mathbb{Z}$,

$$\mathcal{F}^i_{\Xi_1}=\frac{\pi^i(D_1\otimes B_{\mathrm{dR}}\cap\Xi)\cap\Xi_{1,0}+\pi\Xi_{1,0}}{\pi\Xi_{1,0}}\subset D_1\otimes_{\breve{F}}C,$$
$$\mathcal{F}^i_{\Xi,1}=\frac{\pi^i\Xi\cap\Xi_0+\pi\Xi_0}{\pi\Xi_0}\cap(D_1\otimes_{\breve{F}}C),$$

where $\Xi_0=D\otimes B^+_{\mathrm{dR}}$ and $\Xi_{1,0}=D_1\otimes B^+_{\mathrm{dR}}$. We can check that the identity map on $D_1\otimes_{\breve{F}}C$ induces a morphism of filtered C-vector spaces

$$(D_1\otimes C,\mathcal{F}_{\Xi_1})\longrightarrow(D_1\otimes C,\mathcal{F}_{\Xi,1}),$$

which implies that

$$\deg\mathcal{F}_{\Xi_1}\leq\deg\mathcal{F}_{\Xi,1},$$

and hence

$$\deg(D_1,\Xi_1)=\deg(D_1,\mathcal{F}_{\Xi_1})\leq\deg(D_1,\mathcal{F}_{\Xi,1}),$$

where the first equality holds since the BB map preserves degree. On the other hand,

$$\operatorname{rank}(D_1,\Xi_1)=\operatorname{rank}_{\breve{F}}D_1=\operatorname{rank}(D_1,\mathcal{F}_{\Xi,1}),$$

and it follows that $\mathrm{HN}(b,x)\leq\mathrm{HN}(b,\mathrm{BB}(x))$. □

Remark 5.7.

(i) Shen mentioned in [She23, § 2.4] that the BB map

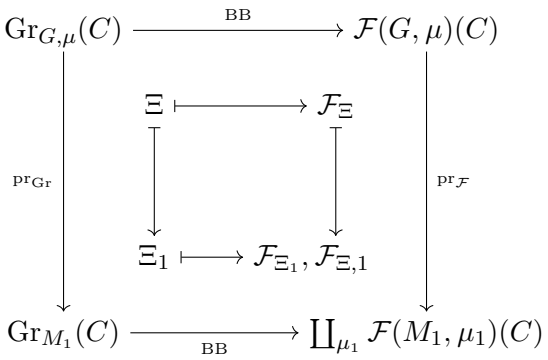
$$\mathrm{BB}:\mathbf{BunIsoc}^{B_{\mathrm{dR}}}_{\breve{F}|F}\longrightarrow\mathbf{FilIsoc}^{C}_{\breve{F}|F}$$

does not preserve strict sub-objects. Viehmann gave an example in [Vie24, Example 4.10] to illustrate that the semi-stable locus of the two sides does not correspond to each other via the BB map.

(ii) We keep the notation as in the proof of the previous proposition, so the lattice Ξ could be considered as an element in $\mathrm{Gr}_{G,\mu}(C)$ for some μ, where $G=\mathrm{GL}(D)$. Let $M_1=\mathrm{GL}(D_1)$ and let $P=\mathrm{stab}(D_1)\subset G$ be a parabolic subgroup with Levi component $\mathrm{GL}(D_1)\times\mathrm{GL}(D/D_1)$. Then there is a natural projection $\mathrm{pr}_{M_1}:P\to M_1$.

The fact that the BB map is not compatible with sub-objects could be reformulated by the fact that the diagram

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is not commutative, where

- $\mathrm{pr}_{\mathrm{Gr}}(gG(B_{\mathrm{dR}}^+)) := \mathrm{pr}_{M_1}(p)M_1(B_{\mathrm{dR}}^+)$, if $gG(B_{\mathrm{dR}}^+) = pG(B_{\mathrm{dR}}^+)$ for some $p \in P(B_{\mathrm{dR}})$ by Iwasawa decomposition;
- $\mathrm{pr}_{\mathcal{F}}(gP_\mu(C)) := \mathrm{pr}_{M_1}(\tilde{p}) \in \mathcal{F}(M_1,\mu_1)(C)$, where μ_1 is the projection of $w\mu$ to the M_1-component, if $g \in \tilde{p}wP_\mu(C)$ for some $\tilde{p} \in P(C)$ by Bruhuat decomposition;
- $\mathrm{pr}_{\mathrm{Gr}}(\Xi) = \Xi_1$ and $\mathrm{pr}_{\mathcal{F}}(\mathcal{F}_\Xi) = \mathcal{F}_{\Xi,1}$.

The non-commutativity of the diagram follows from the fact that the Iwasawa decomposition and Bruhat decomposition on $G(B_{\mathrm{dR}})$ are in general *not* compatible.

On the other hand, for $\Xi \in \mathrm{Gr}_{G,\mu}$, if

$$\Xi \in P(B_{\mathrm{dR}}^+)w\mu^{-1}(\xi)G(B_{\mathrm{dR}}^+)/G(B_{\mathrm{dR}}^+)(C), \tag{5.3}$$

for some $w \in W_G$, then we can read off simultaneously the Iwasawa decomposition and the Bruhat decomposition for Ξ, and it follows by direct verification that the previous diagram is commutative for that Ξ. For example, when μ is minuscule or $\Xi \in G(C)\mu^{-1}(\xi)G(B_{\mathrm{dR}}^+)/G(B_{\mathrm{dR}}^+)$ for some section $C \to B_{\mathrm{dR}}^+$, then Ξ satisfies (5.3) (compare also [NV23, Theorem 5.1]).

5.5 Classical points

Recall that $\{\mu\}$ is a geometric conjugacy class of cocharacters of the F-reductive group G, with E its reflex field inside a fixed algebraic closure $\bar{F}$ of F. Write $\breve{E} = E \cdot \breve{F}$, the p-adic completion of the maximal unramified extension of E. Then the B_{dR}^+-affine Grassmannian $\mathrm{Gr}_{G,\mu}$ is defined over E, giving in particular a functor on $\mathbf{Perf}/\mathrm{Spd}(\breve{E})$. By abusing the notation, for a finite field extension $K/\breve{E}$, we write

$$\mathrm{Gr}_{G,\mu}(K) := \mathrm{Hom}\left(\mathrm{Spd}(K), \mathrm{Gr}_{G,\mu}\right)$$

and call it the set of K-valued classical points of $\mathrm{Gr}_{G,\mu}$. The same notation applies if we replace $\mathrm{Gr}_{G,\mu}$ by $\mathcal{F}(G,\mu)^\diamond$ or more generally by any v-sheaf on $\mathbf{Perf}/\mathrm{Spd}(\breve{E})$.

Lemma 5.8. *Let $K/\breve{E}$ be a finite field extension and denote by C the p-adic completion of an algebraic closure of K.*

(i) *The natural maps*

$$\mathcal{F}(G,\mu)^\diamond(K) \longrightarrow \mathcal{F}(G,\mu)^\diamond(C) = \mathcal{F}(G,\mu)(C) \quad \text{and} \quad \mathrm{Gr}_{G,\mu}(K) \longrightarrow \mathrm{Gr}_{G,\mu}(C)$$

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are injective, and induce bijections

$$\mathcal{F}(G,\mu)^\diamond(K) \xrightarrow{\sim} \mathcal{F}(G,\mu)(C)^{\mathrm{Gal}(C/K)} \quad \text{and} \quad \mathrm{Gr}_{G,\mu}(K) \xrightarrow{\sim} \mathrm{Gr}_{G,\mu}(C)^{\mathrm{Gal}(C/K)}.$$

(ii) *[FF18, Proposition 10.4.4, Vie24, Proposition 5.1]. The Bialynicki–Birula map induces a bijection* $\mathrm{Gr}_{G,\mu}(K) \xrightarrow{\sim} \mathcal{F}(G,\mu)^\diamond(K)$.

Proof. The first part of this lemma is well known and the second part of the lemma is proved by [FF18] and [Vie24]. However, we still give the proof for the convenience of the readers.

The morphism $\mathrm{Spd}(C) \to \mathrm{Spd}(K)$ of pro-étale sheaves on $\mathbf{Perf}/\mathrm{Spd}(\breve{E})$ is surjective. It follows that the natural map

$$\mathcal{F}(G,\mu)^\diamond(K) \longrightarrow \mathcal{F}(G,\mu)^\diamond(C)$$

is injective, whose image is obviously contained in $\mathcal{F}(G,\mu)(C)^{\mathrm{Gal}(C/K)}$. On the other hand, as $C^{\mathrm{Gal}(C/K)} = K$, every element $x \in \mathcal{F}(G,\mu)(C)^{\mathrm{Gal}(C/K)}$ is a K-point of $\mathcal{F}(G,\mu)$. The latter defines a morphism $\mathrm{Spd}(K) \to \mathcal{F}(G,\mu)^\diamond$ by mapping any affinoid perfectoid space $S = \mathrm{Spa}(R, R^+)$ over K to the R-point of $\mathcal{F}(G,\mu)$ induced by x. Therefore, $\mathcal{F}(G,\mu)^\diamond(K) \xrightarrow{\sim} \mathcal{F}(G,\mu)^\diamond(C)^{\mathrm{Gal}(C/K)}$.

The remaining part of the proof is adapted from the proof of [Vie24, Proposition 5.1]. Recall that, in the GL_n-case, $\mathrm{Gr}_{\mathrm{GL}_n}(C)^{\mathrm{Gal}(C/K)}$ classifies the $B_{\mathrm{dR}}^+(C)$-lattices in $B_{\mathrm{dR}}(C)^n$ that are invariant under the action of $\mathrm{Gal}(C/K)$. Therefore, by [FF18, Proposition 10.4.4], the Bialynicki–Birula map induces a bijection

$$\mathrm{Gr}_{\mathrm{GL}_n,\mu}(C)^{\mathrm{Gal}(C/K)} \xrightarrow{\sim} \mathcal{F}\left(\mathrm{GL}_n,\mu\right)(K).$$

Now, for a general reductive group G, we take a closed embedding $G \hookrightarrow \mathrm{GL}_n$ over F. The latter induces a commutative diagram as follows.

$$\begin{array}{ccccccc}
\mathrm{Gr}_{G,\mu}(K) & \longrightarrow & \mathrm{Gr}_{G,\mu}(C)^{\mathrm{Gal}(C/K)} & \xrightarrow{\mathrm{BB}} & \mathcal{F}(G,\mu)^\diamond(C)^{\mathrm{Gal}(C/K)} & = & \mathcal{F}(G,\mu)(K) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Gr}_{\mathrm{GL}_n,\mu}(K) & \longrightarrow & \mathrm{Gr}_{\mathrm{GL}_n,\mu}(C)^{\mathrm{Gal}(C/K)} & \xrightarrow{\mathrm{BB}} & \mathcal{F}(\mathrm{GL}_n,\mu)^\diamond(C)^{\mathrm{Gal}(C/K)} & = & \mathcal{F}(\mathrm{GL}_n,\mu)(K)
\end{array}$$

Here the vertical maps are all induced from the closed immersion $G \hookrightarrow \mathrm{GL}_n$ and thus are injective: for example, the injectivity of the first two maps follows from [SW20, Lemma 19.1.5]. As the Bialynicki–Birula map for GL_n in the diagram above is bijective, together with the fact that the second vertical map (from left) is injective, we deduce that the Bialynicki–Birula map for general G induces an injective map

$$\mathrm{Gr}_{G,\mu}(K) \longrightarrow \mathcal{F}(G,\mu)^\diamond(K) = \mathcal{F}(G,\mu)(K). \tag{5.4}$$

To complete the proof of our lemma, it remains to show that the map (5.4) is also surjective (which implies also $\mathrm{Gr}_{G,\mu}(K) \xrightarrow{\sim} \mathrm{Gr}_{G,\mu}(C)^{\mathrm{Gal}(C/K)}$). Take $x \in \mathcal{F}(G,\mu)(K)$. Since the field K is strictly henselian with algebraically closed residue field, x lifts to an element in $G(K)$ still denoted by x. On the other hand, since the ξ-adic complete ring $B_{\mathrm{dR}}^+(C)$ is naturally an $\breve{F}$-algebra whose residue field $B_{\mathrm{dR}}^+(C)/\xi B_{\mathrm{dR}}^+(C) \simeq C$ contains the finite separable extension $K/\breve{F}$, it follows by Hensel's Lemma that $B_{\mathrm{dR}}^+(C)$ is also naturally an algebra over K. Hence the element $x \in G(K)$ can be viewed as an element of $G(B_{\mathrm{dR}}^+(C))$, yielding an element

$$x\mu(\xi)^{-1}G(B_{\mathrm{dR}}^+(C)) \in \mathrm{Gr}_{G,\mu}(C)$$

whose image via the Bialynicki–Birula map is $x \in \mathcal{F}(G,\mu)(C)$. Furthermore, x also defines a morphism

$$\alpha : \mathrm{Spd}(K) \longrightarrow \mathrm{Gr}_{G,\mu}$$

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by sending every affinoid perfectoid $\mathrm{Spa}(R, R^+)$ over K to $x\mu(\xi)^{-1}B^+_{\mathrm{dR}}(R) \in \mathrm{Gr}_{G,\mu}(R, R^+)$. Here, as R is an K-algebra, the period ring $B^+_{\mathrm{dR}}(R)$ contains naturally K as a subring by the similar argument using Hensel's Lemma mentioned above. Finally we check that the image by (5.4) of α is exactly x, showing the desired surjectivity. □

Remark 5.9 (Cf. [NV23, Remark 5.4]). We keep the notation of Lemma 5.8. By the proof of the lemma above, a K-valued classical point, viewed as an element in $\mathrm{Gr}_{G,\mu}(C)$, always has the form $x\mu(\xi)^{-1}G(B^+_{\mathrm{dR}})$ for some $x \in G(K)$, where K is identified with $B^{\mathrm{Gal}(C|K)}_{\mathrm{dR}}$.

The main result for classical points is the following theorem.

Theorem 5.10. *Let x be a classical point of $\mathrm{Gr}_{G,\mu}$ with value in some finite field extension of $\breve{E}$, and let $[b'] \in B(G)$. Then the following statements are equivalent:*

(i) $x \in \mathrm{Gr}^{\mathrm{New}=[b']}_{G,\mu,b}$;

(ii) $x \in \mathrm{Gr}^{\mathrm{HN}=[b']}_{G,\mu,b}$;

(iii) $\mathrm{BB}(x) \in \mathcal{F}(G,\mu,b)^{\mathrm{HN}=[b']}$.

Remark 5.11. When b is basic, this theorem is proved in [Vie24, Theorem 5.2] for $[b'] = 1$ and in [NV23, Proposition 5.5] for general b'.

The rest of this section is devoted to the proof of Theorem 5.10. We first prove some weaker results.

Proposition 5.12. *Let K be a finite extension of $\breve{E}$ and $[b'] \in B(G)$, so*

$$\mathrm{BB}\big(\mathrm{Gr}^{\mathrm{HN}=[b']}_{G,\mu,b}(K)\big) = \mathcal{F}(G,\mu,b)^{\mathrm{HN}=[b']}(K).$$

Proof. It suffices to show that $\mathrm{HN}(b,x) = \mathrm{HN}(b,\mathrm{BB}(x))$ for any $x \in \mathrm{Gr}_{G,\mu}(K)$. When b is basic, this is done in [NV23, Theorem 5.1]. Their proof could be adapted to this setup. For the convenience of the readers, we still give the proof here.

Using the Tannakian formalism, we may first reduce to the $G = \mathrm{GL}_n$ case. Recall that, in general, the Bialynicki–Birula map

$$\begin{aligned} \mathrm{BB} : \mathrm{BunIsoc}^{B_{\mathrm{dR}}}_{\breve{F}|F} &\longrightarrow \mathrm{FilIsoc}^{C}_{\breve{F}|F}, \\ (D,\Xi) &\longmapsto (D,\mathcal{F}_\Xi) \end{aligned}$$

is *not* compatible with sub-objects. Here, we want to show that BB is compatible with sub-objects for the particular object $(D_b,\Xi_x) \in \mathrm{BunIsoc}^{B_{\mathrm{dR}}}_{\breve{F}|F}$, where D_b is the isocrystal corresponding to $[b]$ and Ξ_x is the lattice corresponding to x. For simplicity, we will write $D = D_b$ and $\Xi = \Xi_x$. Let D_1 be a sub-isocrystal of D. Let (D_1,Ξ_1) (respectively $(D_1,\mathcal{F}_{\Xi,1})$) be the corresponding sub-object of (D,Ξ) (respectively $(D,\mathcal{F}_\Xi)$) in $\mathrm{BunIsoc}^{B_{\mathrm{dR}}}_{\breve{F}|F}$ (respectively $\mathrm{FilIsoc}^{C}_{\breve{F}|F}$). We want to show that $\deg(D_1,\Xi) = \deg(D_1,\mathcal{F}_{\Xi,1})$. It suffices to show that $\mathcal{F}_{\Xi_1} = \mathcal{F}_{\Xi,1}$. By Remark 5.9 and Remark 5.7(ii), $x \in \mathrm{Gr}_{G,\mu}(K)$ satisfies (5.3). It follows that the diagram in Remark 5.7(ii) is commutative for the particular element Ξ and hence BB map is compatible with sub-objects for the particular object (D,Ξ). In particular,

$$\mathrm{BB}(D_1,\Xi_1) = (D_1,\mathcal{F}_{\Xi,1}),$$

and hence $\mathcal{F}_{\Xi_1} = \mathcal{F}_{\Xi,1}$. □

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The following result shows that weakly admissible locus can always be considered as an algebraic approximation of the admissible locus.

Proposition 5.13. *Let x be a classical point of $\mathrm{Gr}_{G,\mu}$, so*

$$x \in \mathrm{Gr}^{a}_{G,\mu,b} \Longleftrightarrow x \in \mathrm{Gr}^{wa}_{G,\mu,b},$$

Remark 5.14. When μ is minuscule and $G = \mathrm{GL}_n$, this theorem is proved by Colmez and Fontaine [CF00, Theorem A]. When b is basic, this theorem is proved by Viehmann [Vie24, Theorem 5.2].

Proof. The proof is the same as [Vie24, Theorem 5.2]. Therefore, we only outline a sketch.

By Proposition 5.12,

$$x \in \mathrm{Gr}^{wa}_{G,\mu,b} \Longleftrightarrow \mathrm{BB}(x) \in \mathcal{F}(G,\mu,b)^{ss}.$$

Therefore, it suffices to show

$$x \in \mathrm{Gr}^{a}_{G,\mu,b} \Longleftrightarrow \mathrm{BB}(x) \in \mathcal{F}(G,\mu,b)^{ss}.$$

When $G = \mathrm{GL}_n$, this is proved in [FF18, Proposition 10.5.6]. The general case is reduced to the GL_n by choosing some suitable faithful representation of G as done in the proof of [Vie24, Theorem 5.2]. □

Now we are ready to prove the main result about classical points.

Proof of Theorem 5.10. The strategy is the same as [NV23, Proposition 5.5]. By Proposition 5.12, we have that (ii) is equivalent to (iii). It remains to show that (i) is equivalent to (ii). Suppose that $[b'] = \mathrm{HN}(b,x)$; it remains to show that $\mathcal{E}_{b,x} \simeq \mathcal{E}_{b'}$. Let M be the centralizer of $w_0\nu_{b'}$ and P the corresponding parabolic subgroup of G. Let b'_M be the reduction of b' to M such that $\nu_{b'_M} = w_0\nu_{b'}$. By Proposition 4.7, there exists a canonical reduction $(\mathcal{E}_{b,x})_P$ of $\mathcal{E}_{b,x}$ such that $\mathrm{pr}_M(x)$ is semi-stable and $\mathrm{HN}(b_M, \mathrm{pr}_M(x)) = [b'_M]$. As $\mathrm{pr}_M(x) \in \mathrm{Gr}_M$ is again a classical point, by Proposition 5.13 it follows that $(\mathcal{E}_{b,x}) \times^P M \simeq \mathcal{E}_{b'_M}$. As the slope vector of $\mathcal{E}_{b'_M}$ is G-dominant, by [Che23, Theorem 2.7, Corollary 2.9], $\mathcal{E}_{b,x} \simeq \mathcal{E}_{b'}$. □

6. Basic properties of Harder–Narasimhan strata

In this section, we will discuss some basic properties of a Harder–Narasimhan stratum, such as non-emptiness, the dimension formula and classical points. By § 5.1, a Harder–Narasimhan stratum associated to (G,μ,b) is always isomorphic to a Harder–Narasimhan stratum associated to some (H,μ^H,b^H) with H an inner twist of G. Therefore, without loss of generality, in this section we will always assume that G is quasi-split.

For a given Harder–Narasimhan stratum $\mathrm{Gr}^{\mathrm{HN}=[b']}_{G,\mu,b}$, let M be the centralizer of $w_0\nu_{b'}$ and P be the associated standard parabolic subgroup of G. Let b'_M be the reduction of b' to M such that $\nu_{b'_M} = w_0\nu_{b'}$. By passing to the graded quotient of the canonical filtration, for any algebraically closed perfectoid field C containing $\breve{F}$ with a bounded valuation subring C^+, we have the following inclusion

$$\mathrm{Gr}^{\mathrm{HN}=[b']}_{G,\mu,b}(C) \subset \bigcup_{\substack{\text{reduction } (b_M,g)\\ \text{of } b \text{ to } M}} \bigcup_{\substack{\lambda \in S_M(\mu) \text{ with}\\ \lambda = \kappa_M(b_M) - \kappa_M(b'_M) \text{ in } \pi_1(M)_\Gamma}} g \cdot \left(\mathrm{Gr}_{G,\mu} \cap \mathrm{Gr}_{P,\lambda}\right)(C),$$

where $\mathrm{Gr}_{P,\lambda}$ and $S_M(\mu)$ are defined in § 2.2.2 and the conditions on λ follow from Lemma 2.10 and the fact that

$$\mathrm{Gr}^{\mathrm{HN}=[b'_M]}_{M,b_M,\lambda} \neq \emptyset \Rightarrow \kappa_M(b_M) - \lambda = \kappa_M(b'_M) \text{ in } \pi_1(M)_\Gamma.$$

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On the other hand, we have a projection map

$$\mathrm{Gr}_{P,\lambda} \longrightarrow \mathrm{Gr}_{M,\lambda},$$

and we denote by $\mathrm{Gr}_{P,\lambda,b_M}^{ss}$ the preimage of the HN-semi-stable locus $\mathrm{Gr}_{M,\lambda,b_M}^{wa} \subset \mathrm{Gr}_{M,\lambda}$ through the above projection.

Proposition 6.1. *Keep the above notation.*

(i) *Let C be an algebraically closed perfectoid field containing $\bar{F}$ with a bounded valuation subring C^+. We have*

$$\mathrm{Gr}_{G,\mu,b}^{\mathrm{HN}=[b']}(C) = \bigcup_{\substack{\text{reduction } (b_M,g)\\ \text{of } b \text{ to } M}} \bigcup_{\substack{\lambda \in S_M(\mu) \text{ with}\\ \lambda = \kappa_M(b_M) - \kappa_M(b'_M) \text{ in } \pi_1(M)_\Gamma}} g \cdot (\mathrm{Gr}_{G,\mu} \cap \mathrm{Gr}_{P,\lambda,b_M}^{ss})(C).$$

(ii) *Assume μ minuscule. Then the map*

$$\mathrm{Gr}_{G,\mu} \cap \mathrm{Gr}_{P,\lambda,b_M}^{ss} \longrightarrow \mathrm{Gr}_{M,\lambda,b_M}^{wa}$$

is (the diamond attached to) an affine fibration with fiber $N/N \cap P_\lambda$, where N is the unipotent radical of P.

Proof. Assertion (i) follows from Proposition 4.5. For (ii), we need to study the map

$$\mathrm{Gr}_{G,\mu} \cap \mathrm{Gr}_{P,\lambda} \longrightarrow \mathrm{Gr}_{M,\lambda}$$

for any $\lambda \in S_M(\mu)$, provided that μ is minuscule. Recall that, since μ is minuscule, the Bialynicki–Birula map yields an isomorphism of v-sheaves

$$\mathrm{BB} : \mathrm{Gr}_{G,\mu} \xrightarrow{\sim} \mathcal{F}(G,\mu)^\diamond. \tag{6.1}$$

In the level of geometric points, the above BB map is given by

$$\mathrm{Gr}_{G,\mu}(C,C^+) \ni g[\mu^{-1}] \longmapsto [\bar{g}] \in \mathcal{F}(G,\mu)^\diamond(C^\flat, C^{\flat,+}) = G(C)/P_\mu(C),$$

with $[\mu^{-1}] := \mu(\xi)^{-1} G(B_{\mathrm{dR}}^+)/G(B_{\mathrm{dR}}^+) \in \mathrm{Gr}_G(C,C^+)$. We will relate $\mathrm{Gr}_{G,\mu} \cap \mathrm{Gr}_{P,\lambda}$ to some subspace of the flag variety $\mathcal{F}(G,\mu)$. Assume $\lambda = w\mu$ for some $w \in G(\bar{F})$ normalizing T and write $\mathcal{F}(G,\mu)^w$ as the P-orbit of the element

$$[w] \in \mathcal{F}(G,\mu) = G/P_\mu.$$

In particular, we have

$$\mathcal{F}(G,\mu)^w = P/P \cap P_{w\mu} \hookrightarrow \mathcal{F}(G,\mu) = G/P_\mu.$$

We claim that, under the BB map (6.1) above, $\mathrm{Gr}_{G,\mu} \cap \mathrm{Gr}_{P,\lambda} \hookrightarrow \mathrm{Gr}_{G,\mu}$ corresponds to the subdiamond

$$\mathcal{F}(G,\mu)^{w,\diamond} \hookrightarrow \mathcal{F}(G,\mu)^\diamond.$$

Indeed, in the level of geometric points

$$\begin{aligned}
&\mathrm{Gr}_{G,\mu}(C,C^+) \cap \mathrm{Gr}_{P,\lambda}(C,C^+)\\
&\quad = G(B_{\mathrm{dR}}^+) \cdot [\mu^{-1}] \cap (N(B_{\mathrm{dR}}) M(B_{\mathrm{dR}}^+) \cdot [\lambda^{-1}])\\
&\quad = P(B_{\mathrm{dR}}^+) w \cdot [\mu^{-1}],
\end{aligned}$$

where the last equality holds by [HKM12, Proposition 13.1]. Therefore, under the identification (6.1), $\mathrm{Gr}_{G,\mu} \cap \mathrm{Gr}_{P,w\mu}$ and $\mathcal{F}(G,\mu)^{w,\diamond}$ have the same geometric points. On the other hand, for a general affinoid perfectoid space S over $\mathrm{Spd}(\bar{F})$, a morphism $S \to \mathrm{Gr}_{G,\mu}$ factors

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through $\mathrm{Gr}_{G,\mu} \cap \mathrm{Gr}_{P,\lambda}$ if and only if this is the case after restricting to every geometric point of S. Therefore, to conclude the proof of our claim, it suffices to check that a morphism

$$\alpha : S^\sharp \longrightarrow \mathcal{F}(G,\mu)$$

factors through the subspace

$$\mathcal{F}(G,\mu)^w \subset \mathcal{F}(G,\mu)$$

if $\alpha(\bar{x}) \in \mathcal{F}(G,\mu)^w$ for every geometric points $\bar{x}$ of S. Let $U \subset \mathcal{F}(G,\mu)$ be an open subspace, which contains $\mathcal{F}(G,\mu)^w$ as a Zariski closed subspace. Because, for every geometric point $\bar{x}$ of S, $\alpha(\bar{x}) \in \mathcal{F}(G,\mu)^w$, it follows that $\mathrm{Im}(\alpha) \subset U$. By [Sch15, Lemma II.2.2], the pullback of $\mathcal{F}(G,\mu)^w \hookrightarrow U$ through the morphism $\alpha : S^\sharp \to U$ defines a Zariski closed immersion $T \hookrightarrow S^\sharp$ of affinoid perfectoid spaces, together with the commutative diagram.

$$\begin{array}{ccc} T & \longrightarrow & \mathcal{F}(G,\mu)^w \\ \downarrow & & \downarrow \\ S^\sharp & \xrightarrow{\alpha} & \mathcal{F}(G,\mu) \end{array}$$

Moreover, for a morphism $f : T' \to S^\sharp$ of perfectoid spaces, the composed map $\alpha \circ f$ factors through $\mathcal{F}(G,\mu)^w$ if and only if f factors through T (cf. [Sch15, Remark II.2.3]). Our assumption on α implies that the induced map $T(C,C^+) \to S^\sharp(C,C^+)$ is bijective for all algebraically closed perfectoid fields C together with an open bounded valuation subring $C^+ \subset C$. Hence, by [Sch17, Lemma 5.4], $T \tilde{\to} S^\sharp$ and the morphism α factors through $\mathcal{F}(G,\mu)^w$, as wanted. This finishes the proof of our claim.

On the other hand, the map $\mathrm{Gr}_{G,\mu} \cap \mathrm{Gr}^{ss}_{P,\lambda,b_M} \to \mathrm{Gr}^{wa}_{M,\lambda,b_M}$ is the pullback of the open subdiamond $\mathrm{Gr}^{wa}_{M,\lambda,b_M} \subset \mathrm{Gr}_{M,\lambda}$ through the morphism $\mathrm{Gr}_{G,\mu} \cap \mathrm{Gr}_{P,\lambda} \to \mathrm{Gr}_{M,\lambda}$, which by the claim above can be identified with the natural map

$$\mathcal{F}(G,\mu)^{w,\diamond} = (P/P \cap P_\lambda)^\diamond \longrightarrow \mathcal{F}(M,\lambda)^\diamond.$$

So our proposition follows from the fact that the map $P/P \cap P_\lambda \to M/M \cap P_\lambda$ is an affine fibration with fiber isomorphic to $N/N \cap P_\lambda$. □

6.1 Non-emptiness

Theorem 6.2. *Suppose G is quasi-split. A Harder–Narasimhan stratum $\mathrm{Gr}^{\mathrm{HN}=[b']}_{G,\mu,b}$ is non-empty if and only if there exists an M-dominant cocharacter $\lambda \in S_M(\mu)$ such that the generalized Kottwitz set (cf. § 2.1.2 for the definition)*

$$B(M, \lambda + \kappa_M(b'_M), \lambda^\diamond \nu_{b'_M})$$

contains a reduction $[b_M]$ of b to M.

Remark 6.3.

(i) In the previous proposition, we may replace $S_M(\mu)$ by $\Sigma(\mu)_{M\text{-max}}$. Indeed, for any $\lambda \in S_M(\mu)$ we can always find $\lambda_{\max} \in \Sigma(\mu)_{M\text{-max}}$ such that $\lambda \leq_M \lambda_{\max}$. The result follows from the fact that

$$B(M, \lambda + \kappa_M(b'_M), \lambda^\diamond \nu_{b'_M}) \subseteq B(M, \lambda_{\max} + \kappa_M(b'_M), \lambda^\diamond_{\max} \nu_{b'_M}).$$

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(ii) When $G=\mathrm{GL}_n$, an analogous result for the non-emptiness of Harder–Narasimhan strata of the flag varieties is obtained by Orlik [Orl06, Theorem 1]. See Proposition 6.5 for the precise version.

(iii) When b is basic, the previous conditions are equivalent to the following: there exists a reduction b_M of b to M, and there exists an M-dominant cocharacter $\lambda\in S_M(\mu)$ such that

$$\kappa_M(b_M)-\lambda=\kappa_M(b'_M)\quad\text{in }\pi_1(M)_\Gamma.$$

In particular, $[b']\in B(G,\kappa_G(b)-\mu,\nu_b\mu^{-1})$. This is compatible with the description of the non-emptiness of Harder–Narasimhan strata for $b=1$ in [NV23, Proposition 3.13].

Lemma 6.4. *$\mathrm{Gr}^{wa}_{G,\mu,b}$ is non-empty if and only if $\mu^{ad,\diamond}\geq\nu_b^{ad}$ in G^{ad}. In particular, when $[b]\in B(G,\mu)$, $\mathrm{Gr}^{wa}_{G,\mu,b}$ is non-empty if and only if $\mu^\diamond\geq\nu_b$.*

Proof. By § 5.2, via the morphism $\pi:\mathrm{Gr}_G\to\mathrm{Gr}_{G^{ad}}$ induced from the quotient $G\to G^{ad}$, we have

$$\pi^{-1}(\mathrm{Gr}^{wa}_{G^{ad},\mu^{ad},b^{ad}})=\mathrm{Gr}^{wa}_{G,\mu,b}.$$

It follows that $\mathrm{Gr}^{wa}_{G,\mu,b}$ is non-empty if and only if so is $\mathrm{Gr}^{wa}_{G^{ad},\mu^{ad},b^{ad}}$. So we may assume that G is adjoint.

Necessity. Suppose $\mathrm{Gr}^{wa}_{G,\mu,b}\neq\emptyset$. Choose $x\in\mathrm{Gr}^{wa}_{G,\mu,b}(C)$ for some algebraically closed perfectoid field C containing $\breve{F}$. Let b_{M_1} be a reduction of b to a standard Levi subgroup M_1 such that b_{M_1} is basic in M and $\nu_{b_{M_1}}$ is G-anti-dominant. Let P_1 be the standard parabolic subgroup associated to M_1. By weak admissibility, for any $\chi\in X^*(P_1/Z_G)^+$,

$$\deg\chi_*((\mathcal{E}_{b,x})_P)=\deg\chi_*(\mathcal{E}_{b_{M_1},\mathrm{pr}_{M_1}(x)})=\langle\chi,\lambda-\nu_{b_{M_1}}\rangle\leq 0,\tag{6.2}$$

where $\mathrm{pr}_{M_1}(x)\in\mathrm{Gr}_{M_1,\lambda}(C)$ for some $\lambda\in S_{M_1}(\mu)$ by Lemma 2.10.

Claim. *We have $\lambda_{G\text{-dom}}\geq\nu_b$.*

The claim implies that $\mu\geq\lambda_{G\text{-dom}}\geq\nu_b$. Now it remains to prove the claim.

Proof. The inequality (6.2) implies that

$$\mathrm{Av}_{M_1}(\lambda)-\nu_{b_{M_1}}=\mathrm{Av}_{M_1}(\lambda-\nu_{b_{M_1}})\leq 0,$$

where $\mathrm{Av}_{M_1}:X_*(T)_{\mathbb{Q}}\to X_*(T)_{\mathbb{Q}}$ denotes the W_{M_1}-average, i.e.,

$$\mathrm{Av}_{M_1}(\lambda)=\frac{1}{|W_{M_1}|}\sum_{w\in W_{M_1}}w\lambda.$$

It follows that

$$\nu_b=w_0\nu_{b_{M_1}}\leq w_0\mathrm{Av}_{M_1}(\lambda)=\mathrm{Av}_{w_0M_1}((w_0\lambda)_{w_0M_1\text{-dom}})\leq(w_0\lambda)_{w_0M_1\text{-dom}}\leq\lambda_{G\text{-dom}},$$

as claimed. □

Sufficiency. Fontaine and Rapoport [FR05, Theorem 3] show that $\mathcal{F}(G,\mu,b)^{wa}=\mathcal{F}(G,\mu,b)^{ss}$ is non-empty if and only if $\nu_b\leq\mu^\diamond$. Then the result follows from Proposition 5.6. □

Proof of Theorem 6.2. Necessity. Suppose $x\in\mathrm{Gr}^{\mathrm{HN}=[b']}_{G,\mu,b}(C)$. By Proposition 4.7, there exists a reduction (b_M,h) of b to M such that $\mathrm{pr}_M(h^{-1}x)\in\mathrm{Gr}^{wa}_{M,\lambda,b}(C)=\mathrm{Gr}^{\mathrm{HN}=[b'_M]}_{M,\lambda,b}(C)$ for some $\lambda\in S_M(\mu)$. Then $\kappa_M(b'_M)=\kappa_M(b_M)-\lambda^\sharp$ and $\nu^{ad}_{b_M}\leq_M\lambda^{ad,\diamond}$ by Lemma 6.4. It follows that $\nu_{b_M}\leq_M\lambda^\diamond\nu_{b'_M}$ and the result follows.

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Sufficiency. Suppose (b_M, h) and λ are as in the conditions. Then, again by Lemma 6.4, $\mathrm{Gr}^{wa}_{M,\lambda,b}$ is non-empty. Note that $h \cdot \mathrm{pr}_M^{-1}(\mathrm{Gr}^{wa}_{M,\lambda,b}) \subseteq \mathrm{Gr}^{\mathrm{HN}=[b']}_{G,\mu,b}$, and it follows that the HN stratum is non-empty. □

We have an analogous result for the non-emptiness of Harder–Narasimhan strata in the flag varieties. Let

$$S_M(\mu)_{cl} := \{\lambda \in W\mu \mid \lambda \text{ is } M\text{-dominant}\}.$$

Obviously, $S_M(\mu)_{cl} \subset \Sigma(\mu)_{M\text{-max}}$.

Proposition 6.5. *Suppose G is quasi-split. A Harder–Narasimhan stratum $\mathcal{F}(G,\mu,b)^{\mathrm{HN}=[b']}$ is non-empty if and only if there exists an M-dominant cocharacter $\lambda \in S_M(\mu)_{cl}$ such that the generalized Kottwitz set*

$$B(M, \lambda + \kappa_M(b'_M), \lambda\nu_{b'_M})$$

contains a reduction $[b_M]$ of b to M. Moreover, each non-empty Harder–Narasimhan stratum has classical points.

Proof. When $G = \mathrm{GL}_n$, this is proved by Orlik [Orl06]. For general G, the strategy is the same which is parallel to the proof of Theorem 6.2. In order to avoid the repetition, we only outline the strategy here. Fontaine and Rapoport determine in [FR05, Theorem 3] the condition when a semi-stable locus is non-empty (and contains classical points). The result then follows from the fact that the Harder–Narasimhan stratum is a union of some affine fibrations over the semi-stable locus of the Harder–Narasimhan stratification of flag variety for some Levi subgroup of G (cf. Proposition 6.1 and [Orl06, Proposition 7]). □

As an application of this result, we can determine which Harder–Narasimhan strata contain classical points.

Proposition 6.6. *Suppose G is quasi-split. A Harder–Narasimhan stratum $\mathrm{Gr}^{\mathrm{HN}=[b']}_{\mu,b}$ contains classical points if and only if there exists an M-dominant cocharacter $\lambda \in S_M(\mu)_{cl}$ such that the set*

$$B(M, \lambda + \kappa_M(b'_M), \lambda^{\diamond}\nu_{b'_M})$$

contains a reduction $[b_M]$ of b to M.

Proof. This follows from Proposition 6.5 combined with Theorem 5.10. □

Remark 6.7.

(i) When b is basic, this proposition is proved by Viehmann [Vie24, Theorem 5.5]. Indeed, Viehmann determines which Newton-strata contain classical points. It is the same as which Harder–Narasimhan strata contain classical points by Theorem 5.10.
(ii) When μ is minuscule, compared with Theorem 6.2, we see that every non-empty Harder–Narasimhan stratum contains classical points.

Example 6.8. Let $G = \mathrm{GL}_3$, $\nu_b = (5/2^{(2)}, 0)$ and $\mu = (3,1,1)$. There are four non-empty Harder–Narasimhan strata with HN-vector $\nu_{b'}$ of the following form:

$$\left(\frac{1}{2}, \frac{1}{2}, -1\right), \quad (1,1,-2), \quad \left(\frac{3}{2}, \frac{3}{2}, -3\right), \quad (0,0,0).$$

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Except for the Harder–Narasimhan stratum corresponding to $(1,1,-2)$, the other three non-empty Harder–Narasimhan strata all contain classical points.

6.2 Dimension formula

Now we consider the dimension of a non-empty Harder–Narasimhan stratum $\mathrm{Gr}_{\mu,b}^{\mathrm{HN}=[b']}$. Let $S_{\mu,b,b'}$ be the set of $\lambda \in S_M(\mu)$ which satisfies

$$[b_M] \in B(M, \lambda^\sharp + \kappa_M(b'_M), \lambda^\diamond \nu_{b'_M})$$

for some reduction b_M of b to M. The set $S_{\mu,b,b'}$ is non-empty by Theorem 6.2. We give the dimension formula of this Harder–Narasimhan stratum. Here, for X a locally spatial diamond or an adic space, its *dimension*, written

$$\dim X,$$

is defined to be the dimension $\dim(|X|)$ of the associated locally spectral space $|X|$, that is, the supremum of all integers n for which there exist a chain $x_0, x_1, \ldots, x_n \in |X|$ of distinct points in $|X|$ such that x_i is a specialization of x_{i+1} for $i = 0, \ldots, n-1$ (cf. the paragraph after Remark 21.8 of [Sch17] for the case of locally spatial diamonds and [Hub96, § 1.8] for the case of adic spaces). We remark that, for T a locally spectral space, if $T = \bigcup_i T_i$ is a union by closed subsets of T, we have

$$\dim T = \max_i \dim T_i.$$

Furthermore, if $X = Z^\diamond$ is the diamond associated with an analytic adic space Z over $\mathbb{Z}_p$ we have $|X| = |Z|$ [Sch17, Lemma 15.6] and thus $\dim X = \dim Z$.

Theorem 6.9. *The notation as above. Assuming that μ is minuscule, then*

$$\dim \mathrm{Gr}_{G,\mu,b}^{\mathrm{HN}=[b']} = \langle \mu, \rho \rangle + \max_{\lambda \in S_{\mu,b,b'}} \langle \lambda, 2\rho_M - \rho \rangle.$$

Remark 6.10.

(i) We expect that the dimension formula still holds for a non-minuscule cocharacter μ although we cannot prove it for the moment.

(ii) Let $S^{op}_{\mu,b,b'} := \{-w_{M,0}\lambda \mid \lambda \in S_{\mu,b,b'}\}$. Replacing $S_{\mu,b,b'}$ by $S^{op}_{\mu,b,b'}$, we can write the dimension formula in Theorem 6.9 in a simpler form:

$$\dim \mathrm{Gr}_{G,\mu,b}^{\mathrm{HN}=[b']} = \max_{\lambda \in S^{op}_{\mu,b,b'}} \langle \mu + \lambda, \rho \rangle.$$

(iii) When b is basic, the set $S_{\mu,b,b'}$ consists of the elements $\lambda \in S_M(\mu)$ such that

$$\lambda^\sharp = \kappa_M(b_M) - \kappa_M(b'_M) \text{ in } \pi_1(M)_\Gamma.$$

In particular, $\mathrm{Av}_M(\lambda) = \nu_b - w_0 \nu_{b'}$. Hence, the dimension formula could also be written in the following form:

$$\dim \mathrm{Gr}_{G,\mu,b}^{\mathrm{HN}=[b']} = \langle \mu - \nu_{b'}, \rho \rangle + \max_{\lambda \in S_{\mu,b,b'}} \langle \lambda, \rho_M \rangle.$$

Recall that Nguyen and Viehmann defined in [NV23, 3.17(iii)] the set $\Theta(\mu, b')$ of Harder–Narasimhan types consisting of $G(F)$-conjugacy classes of Harder–Narasimhan pairs of the form $(Q, \{\lambda\})$, where Q is a parabolic subgroup which is conjugate to the centralizer of $\nu_{b'}$ and $\{\lambda\}$ is a $Q(\bar{F})$-conjugacy class of cocharacters of $Q_{\bar{F}}$ such that $[b']$ has a reduction $[b'_{M_Q}]$ to some Levi factor M_Q of Q that is basic in M_Q, and such that

$$-\lambda^{\sharp_{M_Q}} = \kappa_{M_Q}(b') \in \pi_1(M_Q)_\Gamma \quad \text{and} \quad (-\lambda)_{G\text{-dom}} \leq \mu_{G\text{-dom}}.$$

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We may omit Q in the pairing $(Q, \{\lambda\})$ for $\Theta(\mu, b')$. When μ is minuscule, it is easy to check that there is an identification $S^{op}_{\mu,1,b'} = \Theta(\mu, b'^*)$, where $[b'^*]$ denotes the element in $B(G)$ such that $\nu_{b'^*} = -w_0\nu_{b'}$ and $\kappa(b'^*) = -\kappa(b')$ (cf. [CT22, Corollary 2.9]). Therefore, when $b = 1$, the dimension formula could also be written in the way that

$$\dim \mathrm{Gr}^{\mathrm{HN}=[b']}_{G,\mu,1} = \max_{\lambda\in\Theta(\mu,b')}\langle\mu+\lambda,\rho\rangle,$$

from which we could see that the upper bound of the dimension of a Harder–Narasimhan stratum given by Nguyen and Viehmann [NV23, Proposition 7.1] is actually reached. This also answers the question in [NV23, Remark 7.2(2)].

(iv) When G is split and b is basic, the set $S_{\mu,b,b'}$ consists of a unique element. It follows that, in this situation, we do not need to take the maximum in the dimension formula.

(v) When b is basic, the dimension of a Harder–Narasimhan stratum has already been studied by many people. For $G = \mathrm{GL}_n$, the dimension formula was proved by Fargues in [Far19, Proposition 23]. In [She23, Theorem 3.9], Shen has shown that any non-basic Harder–Narasimhan strata are parabolic induction by introducing a finer decomposition on a given Harder–Narasimhan stratum. As a byproduct, he also gave a description of the dimension as the maximum of the dimension of each stratum [She23, Remark 3.10]. However, we want to clarify that our dimension formula is not the same as Shen's. We have mentioned that, when G is split, the maximum disappears in our dimension formula while it always exists in Shen's even for $G = \mathrm{GL}_n$. In [NV23, Propositions 7.1 and 7.3], Nguyen and Viehmann give some upper bound for the dimension formula.

Proof of Theorem 6.9. By Proposition 6.1 combined with Lemma 6.4, we have that

$$|\mathrm{Gr}^{\mathrm{HN}=[b']}_{G,\mu,b}| = \bigcup_{\substack{\text{reduction } (b_M,g)\\ \text{of } b \text{ to } M}} \bigcup_{\lambda\in S_{\mu,b,b'}} g\cdot|(\mathrm{Gr}_{G,\mu}\cap \mathrm{Gr}^{ss}_{P,\lambda,b_M})|.$$

Note that since each $|\mathrm{Gr}_{G,\mu}\cap \mathrm{Gr}^{ss}_{P,\lambda,b_M}|$ in the union above is a dense open subset of $|\mathrm{Gr}_{G,\mu}\cap \mathrm{Gr}_{P,\lambda}|$, and the latter is itself a locally closed subset of $|\mathrm{Gr}_{G,\mu}| \simeq |\mathcal{F}(G,\mu)|$, it follows that the dimension of $|\mathrm{Gr}_{G,\mu}\cap \mathrm{Gr}^{ss}_{P,\lambda,b_M}|$ is the same as the dimension of its closure $\overline{|\mathrm{Gr}_{G,\mu}\cap \mathrm{Gr}^{ss}_{P,\lambda,b_M}|}$ in $|\mathrm{Gr}_{G,\mu}|$. Consequently,

$$|\mathrm{Gr}^{\mathrm{HN}=[b']}_{G,\mu,b}| = \bigcup_{\substack{\text{reduction } (b_M,g)\\ \text{of } b \text{ to } M}} \bigcup_{\lambda\in S_{\mu,b,b'}} g\cdot(\overline{|(\mathrm{Gr}_{G,\mu}\cap \mathrm{Gr}^{ss}_{P,\lambda,b_M})|}\cap|\mathrm{Gr}^{\mathrm{HN}=[b']}_{G,\mu,b}|)$$

and hence we have

$$\begin{aligned}
\dim \mathrm{Gr}^{\mathrm{HN}=[b']}_{G,\mu,b} &= \max_{\substack{(b_M,g)\\ \lambda\in S_{\mu,b,b'}}} \dim(\overline{|(\mathrm{Gr}_{G,\mu}\cap \mathrm{Gr}^{ss}_{P,\lambda,b_M})|}\cap|\mathrm{Gr}^{\mathrm{HN}=[b']}_{G,\mu,b}|)\\
&= \max_{\substack{(b_M,g)\\ \lambda\in S_{\mu,b,b'}}} \dim \mathrm{Gr}_{G,\mu}\cap \mathrm{Gr}^{ss}_{P,\lambda,b_M}\\
&= \max_{\substack{(b_M,g)\\ \lambda\in S_{\mu,b,b'}}} \dim N/N\cap P_\lambda + \dim(\mathrm{Gr}^{wa}_{M,\lambda,b_M})\\
&= \max_{\lambda\in S_{\mu,b,b'}} \dim N/N\cap P_\lambda + \langle\lambda,2\rho_M\rangle,
\end{aligned}$$

where the third equality follows from Proposition 6.1(ii) and the last equality follows from the fact that $\mathrm{Gr}^{wa}_{M,\lambda,b_M}$ is the semi-stable Harder–Narasimhan stratum and hence is open in $\mathrm{Gr}_{M,\lambda}$. So

$$\dim \mathrm{Gr}^{wa}_{M,\lambda,b_M} = \dim \mathrm{Gr}_{M,\lambda} = \langle\lambda,2\rho_M\rangle$$

is independent of the choice of (b_M, g). Therefore, it remains to show that

$$\dim N/N\cap P_\lambda = \langle\mu-\lambda,\rho\rangle.$$

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Now we compute this dimension:

$$\begin{aligned}\dim N/N\cap P_\lambda &= \sum_{\substack{\alpha\in\Phi_G^+-\Phi_M^+\\ s.t.\langle\lambda,\alpha\rangle<0}} -\langle\lambda,\alpha\rangle\\ &= \sum_{\substack{\alpha\in\Phi_G^+-\Phi_M^+\\ s.t.\langle\lambda,\alpha\rangle<0}} -\frac{1}{2}\langle\lambda,\alpha\rangle + \sum_{\substack{\alpha\in\Phi_G^+-\Phi_M^+\\ s.t.\langle\lambda,\alpha\rangle\geq 0}} \frac{1}{2}\langle\lambda,\alpha\rangle - \langle\lambda,\rho_N\rangle\\ &= \langle\mu,\rho\rangle - \langle\lambda,\rho_M\rangle - \langle\lambda,\rho_N\rangle\\ &= \langle\mu-\lambda,\rho\rangle.\end{aligned}$$

□

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Conflicts of interest

None.

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