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Lines, surfaces and duality

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In the paper [**12**] Shcherbak studied some duality properties of projective curves and applied them to obtain information concerning central projections of surfaces in projective three space. He also states some interesting results relating the contact of a generic surface with lines and the contact of its dual with lines in the dual space. In this paper we extend this duality to cover non-generic surfaces. Our proof is geometric, and uses deformation theory. The basic idea is the following. Given a surface X in projective 3-space we can consider the lines tangent to X, and measure their contact. The points on the surface with a line yielding at least 4-point contact are classically known as the flecnodal. (The reason is that the tangent plane meets the surface in a nodal curve, one branch of which has an inflexion at the point in question; see Proposition 7 below. The line in question is the inflexional tangent, which is clearly asymptotic.)

Now the dual surface X^* is a subset of the corresponding dual projective space. We wish to consider the contact of this surface with lines also. The main result we establish here is that the contact between the surface X and a tangent line is the same as the contact between the dual surface, at the dual point, and the dual line, namely the pencil of planes whose axis is the initial tangent line. (Contact here is in the usual sense of 'k-point' contact, i.e. infinitesimally k points in common.) In particular the flecnodal curves of X and X^* correspond (one of the results of Shcherbak). Of course for this result to be true we need to avoid parabolic points on the surface where the Gauss curvature vanishes, since the corresponding dual points are non-smooth.

1. *Introductory material*

Our basic reference for singularity theory will be [**13**], and for generic geometry [**14**] and [**2**]. As mentioned above we shall be concerned with the geometry of the dual of a smooth surface in projective 3-space. All of our results are local so we find it more convenient to work in affine 3-space which we denote by $\mathbb{R}^3$. (The results all hold over the complex numbers too.) The space of lines contained in $\mathbb{R}^3$ can be represented as an open subset of the Klein quadric K, in projective 5-space. One way to view this representation is to select from the line two points a and b and to associate to them the vector $(a-b, a\wedge b)$ in $\mathbb{R}^6$. Any other choice of points determines a non-zero multiple of the same vector; in other words we have a well defined point in the corresponding projective space. Moreover the components (x, y) of this vector satisfy the equation $(x-y).(x\wedge y) = 0$, and this quadric is the required space of lines. (Naturally this construction works for lines in projective 3-space; the quadric K is the space of all lines in projective space.)

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We shall be interested in the subset of lines which are tangent to a surface X in $\mathbb{R}^3$. Although the results we shall establish really are best formulated in projective terms it is useful to relate this material to the study of the family of orthogonal projections of X to planes as formulated in [2], p. 67. So suppose that $\mathbb{R}^3$ is now furnished with its Euclidean metric. Let S^2 denote the unit 2-sphere centred at the origin and TS^2 its tangent bundle. We consider the map $\pi: X \times S^2 \to TS^2$ given by

$$\pi(x, a) = (a, x - (x.a)\,a) = (a, \pi_a(x)) \in S^2 \times \mathbb{R}^3.$$

Note that the second vector is orthogonal to the first, so that the image of π does indeed lie in TS^2. There is a natural map $\alpha: TS^2 \to K$ defined by $\alpha(a, v) = (a, a \wedge v)$, and the image of α is the set of lines in $\mathbb{R}^3$. Note further that α is a 2–1 mapping, since $\alpha(a, v) = \alpha(-a, v)$; this simply reflects the fact that $\pi_a(x) = \pi_{-a}(x)$. Since our concerns are purely local this redundancy will not worry us. It should now be clear that the set of lines tangent to X is the set of critical values of π or rather the image of this by α. For π has a critical point (x, a) if and only if $\pi_a: X \to P_a$ has a critical point at x, where P_a is the tangent plane to S^2 at a.

The dual of X on the other hand arises from the family of projections to lines. Here we consider the mapping $X \times S^2 \to S^2 \times \mathbb{R}$ given by $(x, a) \mapsto (a, x.a)$. It was observed in [3] that the set of critical values of this mapping can be viewed as an affine dual of the surface X. This provides a useful way of studying the dual. However, it is more natural to use the standard representation of the set of planes in the projective space $P^3 = P$ as the dual projective space P^*, view X as a subset of P and X^*, the set of all tangent planes to X, as a subset of P^*. In this paper we will compare the geometry of X and X^*. For a generic X the dual X^* will have singularities. There will generally be a cuspidal edge corresponding to the parabolic or spinodal curve on X. This curve is also characterized as the locus of points of the surface where the tangent plane meets the surface in a cuspidal singularity or worse, in other words where the contact between the surface and its tangent plane is degenerate, i.e. worse than Morse (A_1). At exceptional points of this curve the contact, which is usually of type A_2, will be of type A_3; these are the swallowtail points, the projective analogues of the cusps of Gauss. (See [2] and [3].) For a generic surface these are the only local singular phenomena. The duals can also self-intersect in various ways, corresponding to bitangent and tritangent planes, and bitangent planes one of whose points of tangency is parabolic.

We also note that the dual of a line l in P is a line l^* in P^* and this correspondence yields a diffeomorphism $K \to K^*$ between the space of lines in P and the space of lines in P^*.

Finally we remark that since we are interested in the contact of lines with surfaces we can measure that contact in the time honoured manner by composing a parametrization of the line with a local defining equation for the surface and considering the multiplicity of the corresponding zero. We shall, at one point below, need to reverse this procedure, that is parametrize the surface and compose with equations of the line. The resulting mapping is then well defined up to contact equivalence (see [13] for a definition and discussion of this notion). Since the contact being measured is purely local we are considering the germ of a mapping $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ which will have rank 1. Any such germ which is finitely-$\mathscr{K}$-determined is contact equivalent to (x_1, x_2^k) for some k. This is then referred to as contact of type A_{k-1} and

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corresponds to k-point contact (that is k-fold roots) when measured in the usual way. For this assertion, and other related information see [**10**].

2. *The main results*

The following observation is the key to our results.

PROPOSITION 1. (i) *At a non-parabolic point p of X any tangent line is the limit of intersections of the tangent plane to X and tangent planes at nearby points. Conversely any line which is the limit of such intersections is tangent to X.*

(ii) *A line l is tangent to the surface X at the point p if and only if the corresponding pencil l^* is tangent to the dual X^* at the corresponding dual point p^*, the tangent plane to X at p. Moreover the tangency is of type A_1 for l at p if and only if that for l^* at p^* is of type A_1.*

Proof. (i) Here we take affine co-ordinates at $p \in X$ and write X locally as a graph $x_3 = f(x_1, x_2)$ where $f\colon \mathbb{R}^2, 0 \to \mathbb{R}, 0$ is a smooth function whose 1-jet at $(0,0)$ is zero. Indeed suppose that, after a rotation, the function f has a Taylor series which starts out $a_1 x_1^2 + a_2 x_2^2$. The condition that the point is not parabolic is equivalent to $a_1 . a_2 \neq 0$. Now one simply computes away; the tangent space to the surface at x is given by $x_3 = 0$. One considers the tangent spaces at the points $(b_1 t, b_2 t, f(b_1 t, b_2 t))$, computes their intersection with $x_3 = 0$ and lets t tend to 0. In the limit one obtains $x_3 = a_1 b_1 x_1 + a_2 b_2 x_2 = 0$. Varying the choices of b_1 and b_2 shows that any tangent line can be obtained in this way. The converse is straightforward too and is left to the reader. Note that in the case when say $a_1 = 0$ and $a_2 \neq 0$, so that p is parabolic, the limits of intersections only yield the line $x_3 = x_2 = 0$, except in the case when $b_2 = 0$, when the limiting process depends on the higher order terms of f (and would also depend on more than the limiting direction of the path along which we were considering the intersections).

(ii) Intuitively the proof goes as follows. Consider a line l^* tangent to X^*, and let us suppose that the point of tangency p^* is smooth. If we move l^* slightly it becomes a secant which cuts the dual in two points one of which we shall suppose is the point p^*. These intersections correspond to a pair of planes tangent to the surface, one at p and the other at a nearby point say q. The corresponding secant will be labelled l_q^*. These tangent planes will intersect in a line. We now let q tend to p and deform the secant l_q in P^* so that it approaches the line l^*. The line of intersection of the tangent planes to X at p and q, in the limit as q tends to p, converges to some line say l. By (a) above l is tangent to X at p. Moreover, as we moved q into p, the lines of intersection in P were axes of the corresponding pencils. So the same is true of the limit. In other words, the lines l and l^* are dual. For the converse, suppose given a line l through p tangent to X. By (a) it is the limit of intersections of the tangent plane to X at p with tangent planes to X at nearby points q. The corresponding secants in P^* represent pencils whose axes are the lines of intersection, and in the limit we obtain the result. We now establish this result by a direct calculation, and at the same time prove the final part of the claim.

So we write our surface X locally as above. If we parametrize the space of all planes near $x_3 = 0$ by

$$(y_1, y_2, y_3) \equiv x_3 + y_1 x_1 + y_2 x_2 + y_3$$

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then the parametrization of the dual is given locally by

$$x \mapsto (-f_1, -f_2, x_1 f_1 + x_2 f_2 - f) = (y_1, y_2, y_3).$$

This is an immersion because $p = 0$ is not parabolic. We now choose any line in the tangent plane to X at 0 say $x_3 = x_1 - \alpha x_2 = 0$. The contact between the surface and the line at the origin is determined by composing the inclusion of X with the mapping to $\mathbb{R}^2$ whose components are the defining equations of the line. We suppose as above that the quadratic part of f at 0 is $a_1 x_1^2 + a_2 x_2^2$, so the 2-jet of this contact map is $(x_1 - \alpha x_2, a_1 x_1^2 + a_2 x_2^2)$. We next consider the pencil of planes which contain the above line. If the plane $x_3 + y_1 x_1 + y_2 x_2 + y_3 = 0$ contains this line then $\alpha y_1 + y_2 = y_3 = 0$, so to determine the contact between the dual and this line we compose the dual map above with these defining equations. The 2-jet of the resulting map is

$$(2\alpha a_1 x_1 + 2a_2 x_2 + O(2), a_1 x_1^2 + a_2 x_2^2).$$

The $O(2)$ refers to some quadratic terms, which do not effect the $\mathscr{K}$-type of the germ. Since this is also of rank 1, the dual line is tangent to the dual surface. It is now easy to see that, in the case when a_2 or $a_1 \alpha$ is non-zero, both maps are of type A_1 if $a_2 + a_1 \alpha^2 \neq 0$. (If $a_1 = a_2 = 0$ we have a flat umbilic point, and we can clearly say nothing. Note that such points are non-generic. If $a_2 = \alpha = 0$ then we have a parabolic point on X and we are projecting along an asymptotic direction.) For the converse, note that p is parabolic if and only if p^* is parabolic. This simply follows by comparing their contact with their tangent planes. So if l^* is tangent to X^* at p^* and the contact is of type A_1 then by the previous argument the same is true of the contact between $X^{**} = X$ and $l^{**} = l$ at $p^{**} = p$. ∎

We can deduce a number of interesting results from Proposition 1. First we need some auxiliary geometric constructions. Given the surface X, we can consider the space of all lines which are tangent to it. We shall denote this subset of K by $K(X)$. It is fairly easy to describe the local structure of this space for a generic surface.

PROPOSITION 2. *Let X be a generic surface in $\mathbb{R}^3$. The set $K(X)$ is locally diffeomorphic to the product of the discriminant of an A_k singularity $D(A_k)$ $(1 \leqslant k \leqslant 4)$ with $\mathbb{R}^{4-k}$ or the transverse union of such. In particular we have strata of type $A_k, 1 \leqslant k \leqslant 4$, $2A_1, 3A_1, 4A_1, A_1A_2, A_1A_3, 2A_2$, where, for example, the local model for A_1A_3 is $D(A_1) \times \mathbb{R}^3 \cup \mathbb{R} \times D(A_3)$.*

Proof. There are a number of ways to proceed. We shall think of the tangent lines as corresponding to the critical values of the map π discussed above. The key fact we use here is that the set of critical values of a smooth $\mathscr{A}_e$-versal unfolding of a map-germ is determined by the number of unfolding parameters and the contact type of the germ. To see this we note that if $F: \mathbb{R}^n \times \mathbb{R}^r, 0 \to \mathbb{R}^p \times \mathbb{R}^r, 0$ is an $\mathscr{A}_e$-versal unfolding of $f = F(-, 0)$ then $F_1: \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^p, 0 \to \mathbb{R}^p \times \mathbb{R}^r, 0$ defined by $F_1(x, u, y) = F(x, u) - y$ is a $\mathscr{K}_e$-versal unfolding of $f = F_1(-, 0, 0)$. Moreover, the set of critical values of F can be identified with the discriminant of F_1, which is determined by the $\mathscr{K}$-type of f and by r. Now the $\mathscr{K}$-type of the projection is, in turn, determined by the contact between X and the corresponding line. (See [**13**, **2**, **10**].) As mentioned above this contact can be measured at a point p by defining X locally as the zero set of a smooth submersion $g: \mathbb{R}^3, p \to R, 0$, parametrizing the line by $\gamma: \mathbb{R}, 0 \to \mathbb{R}^3, x$ and determining the multiplicity of 0 as a root of the composite $g \circ \gamma: \mathbb{R}, 0 \to \mathbb{R}, 0$. It follows, for example from [**1**] or [**10**], that one expects, generically, k-tuple roots for

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$1 \leqslant k \leqslant 6$. This is also a consequence of the classification of generic orthogonal projections $X \to \mathbb{R}^2$ given in [**1**, **9**] and discussed in [**2**], although it is much less deep. (Of course there are only six types of such mapping up to $\mathscr{K}$-equivalence, namely those represented by $(x_1, x_2) \to (x_1, x_2^k)$, for $1 \leqslant k \leqslant 6$.) We now note that the discriminant of such a mapping is simply that of an A_{k-1} function singularity if $k \geqslant 2$, and is empty if $k = 1$. Since the family of projections generically versally unfolds any individual germ ([**9**, **11**, **2**] and for a discussion of versality [**13**]) the result follows. The multi-germ case is similar and left to the reader. ∎

We next recall the geometric interpretation of these singularities. One such set, stated in Proposition 3(i) below, were given in [**2**], p. 74. We think that it may be useful to give an alternative collection (Proposition 3(ii)). The proof that they do characterize the contact is not difficult; indeed this second set of interpretations are intuitively quite clear, but they seem worth including. In the particular case of an A_3 singularity they justify the classical term 'flecnodal'. The A_4 contact points are known classically as biflecnodes, although there is good reason, in Proposition 3(ii) to prefer the term undunodal.

PROPOSITION 3. (i) *The $A_{\geqslant 2}$ singularities correspond to asymptotic directions of projection at non-elliptic points. The $A_{\geqslant 3}$ singularities correspond to asymptotic directions at points where the corresponding asymptotic curve has a simple zero of curvature, and the $A_{\geqslant 4}$ singularities to asymptotic lines where the corresponding asymptotic curve has a repeated zero of curvature.* (ii) *The A_{k-1} singularities, which correspond to tangent lines having k-point contact with the surface at the point in question, occur if and only if that line has $(k-1)$-point contact with one of the branches of the intersection of the surface and its tangent plane in that plane.*

Proof. For (i), see [**2**], p. 74. To prove (ii), as usual we write our surface in the form of a graph $x_3 = f(x_1, x_2)$, where the 1-jet of f is zero. Suppose that the line $x_3 = x_1 = 0$ has k-point contact with X at 0. It follows that $f(0, x_2)$ has leading term a non-zero multiple of x_2^k, so we can write

$$f = x_1(a_1 x_1 + a_2 x_2 + O(2)) + c x_2^k + O(k+1)$$

where $O(r)$ denotes terms of order at least r. Now the condition that $a_2 \neq 0$ is equivalent to the fact that p is not parabolic. Since f is, in the notation of [**13**], 2-$\mathscr{R}_1$-determined, it follows that the intersection of X with its tangent plane at p consists of two branches, with tangents $x_1 = 0, a_1 x_1 + a_2 x_2 = 0$. If we parametrize the former branch as $(\sum_{i=2}^{\infty} \alpha_i t^i, t)$ and substitute into the defining equation, then we find that the first non-zero α is α_{k-1} so that the line has $(k$-1)-point contact with the given branch in the tangent plane. (It is in this comparison of coefficients that we have used $a_2 \neq 0$.) ∎

Note that intuitively all that is happening above is that the contact between the line and the surface is getting higher as k increases, that is the line is more nearly contained in the surface. It is reassuring to have these interpretations in terms of classical differential geometric concepts, but the singularities actually yield the best description of the geometry.

Now we wish to generalize Proposition 1 and establish a result which relates the type of contact between the surface and line, the dual surface and the dual line. Of course this is much easier to do away from the parabolic points; the problem here is

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that the dual of such a point is generally a singular point of X^*. We discuss briefly how to make sense of contact between a line and the dual X^* at non-smooth points of X^* below, but deal here with the smooth case.

THEOREM 4. *Let X be a smooth compact surface in $\mathbb{R}^3$ with the property that the complement of its parabolic set is dense in X. Let $K(X)$ denote the set of lines tangent to X and let $K^*(X)$ denote the set of lines in the dual space tangent to the dual of X. Then the following assertions hold*:

(i) *the dual map $K \to K^*$ is a diffeomorphism taking $K(X)$ to $K^*(X)$.*

(ii) *Moreover if p is not a parabolic point of X and the contact between X and l at p is finite, i.e. of type A_r for some finite r, then the contact between X^* and the dual l^* at p^* is of the same type.*

(iii) *If l is an asymptotic line to X at the non-parabolic point p then the dual line l is asymptotic to X^* at p^*. In particular p is elliptic (resp. hyperbolic) if and only if p^* is elliptic (resp. hyperbolic).*

(iv) *The non-parabolic part of the flecnodal curve of X corresponds, under the duality map, to the flecnodal curve of X^* (or rather the part of the flecnodal curve on the smooth part of X^*).*

(v) *The special hyperbolic points on the flecnodal curves corresponding to fourth-order contact of one of the branches of the intersection of the surface and tangent plane with its tangent line are also preserved.*

Remarks 5. Before proceeding with proofs a few comments are in order.

(i) Cases (iii)–(v) above, which are simply interpretations of (ii) for low values of r, were first established by Scherbak [**12**]. There were no proofs given.

(ii) Note that it is enough to establish the first part of this result on a dense set of X, since $K(X)$ is obviously compact. Proposition 1 does precisely that, so the first part of the result is already established.

(iii) The hypothesis that the parabolic set has dense complement is really extremely mild. It is relatively easy to see that for generic finite-dimensional families of surfaces each member of the family has this property. In other words, the complement of the set of embeddings of X in $\mathbb{R}^3$ with this property has infinite codimension. It is much milder than genericity, although the deductions (iii)–(v) above for the generic case are of natural interest.

Before giving the details of the proof of the second part of this theorem we first sketch out why this is a plausible result, and in so doing give the intuitive ideas behind the proof. Let us consider the generic case. Suppose that we have a line l having k-point contact, where $1 \leqslant k \leqslant 5$, with X at p. Then l lies in $K(X)$ and the part of this set corresponding to points near p is diffeomorphic to the discriminant of an A_{k-1} singularity. The first part of the above result assures us that l^* lies in $K^*(X)$, and that locally this too is diffeomorphic to the discriminant of an A_{k-1} singularity. Now if we could be sure that the dual also has generic contact with lines we would be done, for the discriminants of (versal unfoldings of) the A_k singularities determine those singularities. In the general case we need to refine this basic idea somewhat.

Proof of Theorem 4. We use the same basic idea as that in [**4**, **5**]. So suppose that we are considering the point $p = 0 \in X \subset \mathbb{R}^3$ which is non-parabolic. We write X locally as the graph of a function $x_3 = f(x_1, x_2)$ with f having zero 1-jet at the origin. We can similarly write the dual of X at the dual point p^* as a graph. Indeed with

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respect to the co-ordinates y_1, y_2, y_3 introduced above as a graph $y_3 = g(y_1, y_2)$, for some g, with the k-jet of f determining the $(k-1)$-jet of g. Suppose now that the contact between X and the line l at p is of type A_r. We consider the new germ of a surface given by $x_3 = f_{r+2}(x_1, x_2)$ where f_{r+2} is the $(r+2)$-jet of f. Clearly the line l still has contact of type A_r with this new surface, which we denote by X_{r+2}, at p. We then construct the corresponding dual. Since $r \geqslant 0$ the point $p = 0$ remains non-parabolic on X_{r+2}, so the dual is smooth at the corresponding dual point. We now work over the complex numbers; this is possible since our new surface and its dual are analytic. Since the line l has contact of type A_r with X_{r+2} at p, this is an isolated point of tangency. It follows from Proposition 1 that the dual line l^* will have isolated tangency with the dual X^*_{r+2} at p^*. (Proposition 1 remains valid over the complexes.) Consequently the contact in the dual space is also of type A_k for some k. Of course we have established, as yet, nothing about the original dual.

To ease the notation we now drop the suffix $r+2$, and write X for X_{r+2}, and f for f_{r+2} in what follows. We next embed the surface X in a family X_s in such a way that the A_r contact singularity between the line l and X at p is $\mathscr{K}$-versally unfolded by considering the contact of this family with the set of lines. For example, if we define the X_s's by $x_3 = f(x_1, x_2) + \sum_{i=0}^{r} s_i x_2^i$ then it is clear that the contact is versally unfolded. Moreover for all small s all points near p on X_s will be non-parabolic, since this is an open condition.

We now do the same with the dual surface, producing a family X_t^* which versally unfolds the contact with the dual line l^*. Again the points on the X_t^*'s for t small, which are near to p^* will be non-parabolic, so their duals, the X_t^{**}'s, will be locally smooth. It is now not difficult to produce a family of surfaces depending on both sets of parameters $(s, t) \in S \times T$ say, which contain the X_s's and the X_t^{**}'s (taking the duals of the latter yields the X_t^*'s). For locally these families can be written respectively as graphs $x_3 = F(x_1, x_2, s), x_3 = G(x_1, x_2, t)$, so we consider

$$x_3 = F(x_1, x_2, t) + G(x_1, x_2, t) - f(x_1, x_2),$$

with the surface corresponding to the parameters (s, t) being denoted by $X_{s,t}$. For $t = 0$ we obtain the X_s's, and for $s = 0$ the X_t^{**}'s. This family will versally unfold the contact with the line l, while the corresponding dual family $X^*_{s,t}$ will versally unfold the contact of X^* with the dual line l^*. The discriminant of the first unfolding lives in $S \times T \times K$ where K, as usual, denotes the set of all lines in our space. The discriminant of the dual unfolding lives in $S \times T \times K^*$ where K^* is the set of lines in the dual space. The key point is that Proposition 1 shows that these discriminant sets are diffeomorphic, for (s, t, l) is in the discriminant precisely when l is tangent to $X_{s,t}$, which occurs exactly when l^* is tangent to $X^*_{s,t}$, i.e. (s, t, l^*) lies in the dual discriminant. It is easy to see however that if the discriminants of two unfoldings of singularities of type A, of the same dimension, are diffeomorphic then they are right-equivalent singularities. This follows for example from the fact that if the set of vector fields tangent to the discriminant, when evaluated at the base-point, span a space which has codimension equal r, we have a singularity of type A_r.

To return to the original surface, we note that if the contact between the dual line and the dual surface determined by the $(r+2)$-jet of f is of type A_r then the same is true of the dual line and the original X^*. For this contact is determined by the $(r+1)$-jet of g. The second result now follows.

The remaining results are now all consequences of the earlier characterizations of

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the various degrees of contact of the surface with lines given in Proposition 3. (Note that we only have asymptotic directions at a point p in X if this point is non-elliptic. So the first assertion of (3) proves the second.) ∎

Remarks 6. (i) One should not be misled into thinking that the mapping taking the surface X to its dual X^* will preserve asymptotic curves. Clearly the assignment of dual lines is not via the differential of this map.

(ii) Having considered the contact between the surface and individual lines it is natural to bundle these lines together, to obtain a genuine mapping, and seek a duality here. One natural mapping to consider in the Euclidean case is orthogonal projection in a given direction. Thus we are bundling lines into two-dimensional parallel families. But note that the family of lines in K parallel to a given direction has as its dual in K^* the set of all lines in a given plane. In other words the dual family certainly do not arise as the fibres of any mapping.

(iii) It is, of course, of interest to discover what happens when we are at a parabolic point of X. Naturally the crucial problem is that the dual is no longer smooth, so it is rather difficult to make sense of what we mean by its contact with lines. We can circumvent this difficulty by using the fact that the dual comes with a natural desingularization. (Geometrically the dual has X, or if we are working in Euclidean space the normal bundle to X, as its normalization.) So the question of contact can always be pulled back to this smooth set. We consider the set $\{(x, H) \in X \times P^* : x \in H\}$; using the projection to the first factor we see that it is a bundle over X with fibre a projective plane. If we now project to the second factor, we obtain a smooth mapping whose critical set consists of pairs (x, H) where H is the tangent plane to X at x; in other words the critical set is smooth and indeed diffeomorphic to X. The set of critical values of this projection on the other hand is simply the dual X^*. We write Σ for the critical set, and $\pi : \Sigma \to X^*$ for the natural projection. Let $g : P^*, p^* \to \mathbb{R}^2, 0$ be the germ of a submersion whose zero fibre is a line through a point p^*. We can define the contact between this line and X^* at p^* by considering the composite $g \circ \pi : \Sigma, p \to \mathbb{R}^2, 0$. Of course this need not be a rank 1 germ, so the contact need no longer be encoded by a single positive integer, although we do get a well defined contact class. Clearly germs of corank 2 do emerge. To see this, consider projection of the dual along a tangent to the cuspidal edge. The critical set here is generically a pair of cusps, with transverse cuspidal tangents; one cusp corresponds to the outline of the cuspidal surface, while the other corresponds to the image of the cuspidal edge. We hope to describe the outlines of the dual in a later paper. Note that in the computational proof of Proposition 1 it actually followed that at a parabolic point the contact of the surface with a non-asymptotic direction (which is necessarily of type A_1) corresponds, on taking the dual surface, and dual line, to contact of corank 1.

(iv) If a line is bitangent to our surface X then the arguments above show that the dual line will be bitangent to the dual X^*. Indeed the analogues of the results above all work at the multi-local level. The reader can easily fill in the details.

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REFERENCES

[1] V. I. Arnold. Indices of singular points of 1-forms on a manifold with boundary, convolution of invariants of reflection groups, and singularities of projections of smooth surfaces. *Russian Math. Surveys* **34** (1979), 1–42.
[2] T. Banchoff, T. J. Gaffney and C. McCrory. *Cusps of Gauss Mappings* (Pitman, 1982).
[3] J. W. Bruce. The duals of generic hypersurfaces. *Math. Scand.* **49** (1981), 36–69.
[4] J. W. Bruce. On contact of hypersurfaces. *Bull. London Math. Soc.* **13** (1981), 51–54.
[5] J. W. Bruce. Envelopes duality and contact structures. *Proc. Sympos. Pure Math.* **40** (1983), 195–202.
[6] J. W. Bruce. Geometry of singular sets. *Math. Proc. Cambridge Philos. Soc.* **106** (1989), 495–509.
[7] J. W. Bruce and M. C. Romero-Fuster. Duality and projections of curves and surfaces in 3-space. *Quart. J. Math. Oxford Ser.* (2), **42** (1991), 433–441.
[8] J. W. Bruce. Generic geometry, transversality and projections. Preprint (1991).
[9] T. J. Gaffney. The structure of $T\mathscr{A}(f)$, classification and an application to differential geometry. *Proc. Sympos. Pure Math.* **40** (1983), 409–428.
[10] J. A. Montaldi. On contact between submanifolds. *Michigan Math. J.* **33** (1986), 195–199.
[11] J. A. Montaldi. On generic composites of mappings. *Bull. London Math. Soc.* **23** (1991), 81–85.
[12] O. P. Shcherbak. Projectively dual space curves and Legendre singularities. *Trudy Tbiliss. Univ.* **232–233** (1982), 280–336.
[13] C. T. C. Wall. finite determinacy of smooth map germs. *Bull. London Math. Soc.* **13** (1981), 481–539.
[14] C. T. C. Wall. Geometric properties of generic differentiable manifolds. In *Geometry and Topology, Rio de Janeiro 1976*, Lecture Notes in Math. vol. 597 (Springer-Verlag, 1977), pp. 707–774.

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