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Hyperbolic polygons and NEC groups

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1. *Introduction*

Beardon [1] gave a procedure for constructing a polygon with prescribed angles. For each ordered set of angles Beardon's polygon is unique up to congruence. The polygon obtained this way has an inscribed circle. It is possible to obtain by means of these polygons a fundamental region for a non-Euclidean crystallographic (NEC) group with a given signature having equal angles in each of the cycles: see [5]. In [5] the minimal number of sides of a convex fundamental region of an NEC group is calculated, and regions are explicitly obtained achieving the bound.

In this paper we study the general construction of hyperbolic polygons with prescribed angles, and calculate in §3 the number of parameters upon which the construction depends. Beardon's method appears as a particular case of our construction. In §4 we obtain convex fundamental regions with minimal number of sides for each NEC signature (as a generalization of the results of [5]); we prove that the number of parameters of the construction equals the dimension of the Teichmüller space, and that these regions characterize the classes of Teichmüller space. Earlier definitions and results on NEC groups are given in §2.

2. *Preliminaries*

Let D denote the hyperbolic plane. We use the model of the disc. Let G be the group of the isometries of D. An NEC group Γ is a discrete subgroup of G with compact quotient space. NEC groups are classified according to their signature, which has the form

$$(g, \pm, [m_1, \ldots, m_r], \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}) \tag{2·1}$$

where all the numbers m_i, n_{ij} are integers greater than or equal to 2, and g, r and k are non-negative integers (cf. [3]). This signature determines a presentation of the group Γ, which as shown in [8] is given by

$$\left.\begin{array}{lll} \textit{generators} & x_i & (i = 1, \ldots, r) \\ & e_i & (i = 1, \ldots, k) \\ & c_{ij} & (i = 1, \ldots, k, j = 0, \ldots, s_i) \\ & a_i, b_i & (i = 1, \ldots, g) \text{ (if the sign is '+')} \\ & \textit{or } d_i & (i = 1, \ldots, g) \text{ (if the sign is '−')} \end{array}\right\} \tag{2·2}$$

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and

relations

$$\left.\begin{aligned} & x_i^{m_i} = 1 \\ & e_i^{-1} c_{i0}\, e_i = c_{is_i} \\ & c_{i,j-1}^2 = c_{ij}^2 = (c_{i,j-1}\, c_{ij})^{n_{ij}} = 1 \\ & x_1 \dots x_r\, e_1 \dots e_k\, a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1 \quad \text{(if the sign is '+')} \\ \text{or } & x_1 \dots x_r\, e_1 \dots e_k\, d_1^2 \dots d_g^2 = 1 \qquad\qquad \text{(if the sign is '−').} \end{aligned}\right\} \tag{2·3}$$

A fundamental region of Γ is a subset R of D such that

(1) $\bigcup_{g\in\Gamma} g(R) = D$;

(2) $g(\mathring{R}) \cap \mathring{R} = \varnothing$ for $g \in \Gamma\backslash\{\mathrm{id}\}$;

(3) the hyperbolic area of $R\backslash\mathring{R}$ is zero.

Given an NEC-group Γ, all fundamental regions of Γ have the same area, denoted by $|\Gamma|$, given by

$$|\Gamma| = 2\pi\left(\alpha g + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right)\right), \tag{2·4}$$

where $\alpha = 2$ for sign '+' and $\alpha = 1$ for sign '−'. Γ actually exists if and only if $|\Gamma| > 0$: see [**9**].

Wilkie [**8**] obtained a canonical region. We describe it in detail in §4. From it, he deduced the presentation (2·2)–(2·3).

If Γ is an NEC-group with signature (2·1), its orientable elements form a Fuchsian group, called the canonical Fuchsian subgroup Γ^+ of Γ, whose signature as shown in [**7**] is

$$(\alpha g + k - 1,\ +,\ [m_1, m_1, \dots, m_r, m_r, n_{11}, \dots, n_{ks_k}],\ \{\text{——}\}). \tag{2·5}$$

The Teichmüller space of an NEC group Γ (see [**4**]) is homeomorphic to a Euclidean ball of dimension $d(\Gamma) = \frac{1}{2}d(\Gamma^+)$ (see [**6**]).

3. *Hyperbolic polygons*

This section is devoted to the general construction of hyperbolic polygons with prescribed angles. First of all we need an earlier result on hyperbolic trigonometry. Consider the triangle of Figure 1 where O is the centre of D.

LEMMA 3·1.

(i) $$\cos\gamma = \frac{(\sinh^2 b + 1)^{\frac{1}{2}}(\sinh^2 b \sin^2\alpha + \sin^2\beta)^{\frac{1}{2}} \sin\alpha - \cos\alpha\cos\beta}{1 + \sinh^2 b \sin^2\alpha}$$

(ii)

$$\tanh c = \frac{\sinh b(\sin^2\beta - \sin^2\alpha)}{\cos\alpha \sin^2\beta(\sinh^2 b + 1)^{\frac{1}{2}} - \sin\alpha\cos\beta(\sinh^2 b \sin^2\alpha + \sin^2\beta)^{\frac{1}{2}}} \quad (\text{if } \alpha \neq \beta)$$

(iii) $$\sin\beta = \frac{\sinh b \sin\alpha}{((\cosh b \cosh c - \sinh b \sinh c)^2 - 1)^{\frac{1}{2}}}.$$

Proof. Let $A = \sinh a$, $A^* = \cosh a$, and similarly for b and c.

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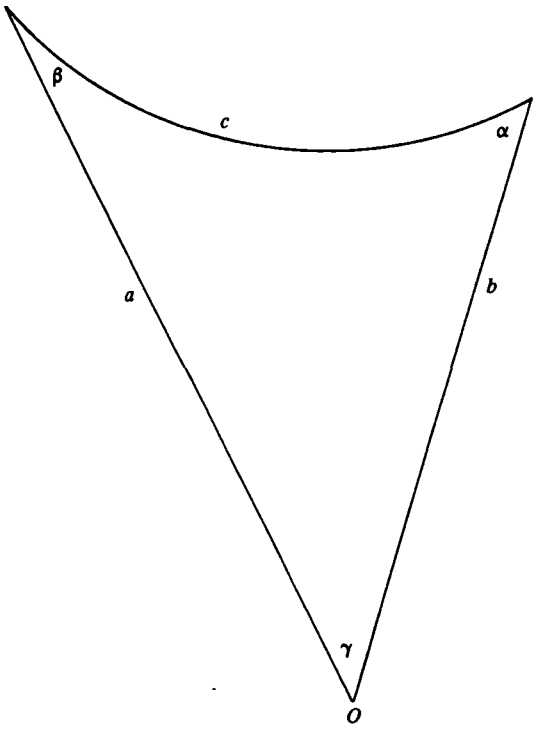


Fig. 1.

(i) From the Cosine Rules (see [2], p. 148) we have

$$A^*B^* - AB\cos\gamma = \frac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta}$$

and so

$$\cos\gamma = \frac{A^*B^*\sin\alpha\sin\beta - \cos\alpha\cos\beta}{1 + AB\sin\alpha\sin\beta}.$$

From the Sine Rule

$$A = \frac{B\sin\alpha}{\sin\beta},$$

and from $A^{*2} - A^2 = 1$ we obtain

$$\cos\gamma = \frac{(B^2+1)^{\frac{1}{2}}(B^2\sin^2\alpha + \sin^2\beta)^{\frac{1}{2}}\sin\alpha - \cos\alpha\cos\beta}{1 + B^2\sin^2\alpha}.$$

(ii) If α were equal to β, we would have

$$\tanh\tfrac{1}{2}c = \tanh b\cos\beta.$$

Let $\alpha \neq \beta$. From Cosine Rule I

$$A^* = B^*C^* - BC\cos\alpha \quad \text{and} \quad B^* = A^*C^* - AC\cos\beta.$$

Multiplying the first equation by B^* and the second by A^*, we have

$$B^{*2}C^* - BB^*C\cos\alpha = A^{*2}C^* - AA^*C\cos\beta$$

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and $$C^*(B^{*2}-A^{*2}) = C(BB^*\cos\alpha - AA^*\cos\beta),$$

and hence $$\tanh c = \frac{B^{*2}-A^{*2}}{BB^*\cos\alpha - AA^*\cos\beta}.$$

From the Sine Rule

$$A = \frac{B\sin\alpha}{\sin\beta} \quad \text{and} \quad A^* = \frac{(B^2\sin^2\alpha+\sin^2\beta)^{\frac{1}{2}}}{\sin\beta}$$

and thus,

$$\tanh c = \frac{(B^2+1)-\dfrac{B^2\sin^2\alpha+\sin^2\beta}{\sin^2\beta}}{B(B^2+1)^{\frac{1}{2}}\cos\alpha - B\dfrac{\sin\alpha}{\sin\beta}\dfrac{(B^2\sin^2\alpha+\sin^2\beta)^{\frac{1}{2}}\cos\beta}{\sin\beta}}$$

$$= \frac{B(\sin^2\beta-\sin^2\alpha)}{\cos\alpha\sin^2\beta(B^2+1)^{\frac{1}{2}} - \sin\alpha\cos\beta(B^2\sin^2\alpha+\sin^2\beta)^{\frac{1}{2}}}.$$

(iii) From $A^* = B^*C^* - BC\cos\alpha$, $A = B\sin\alpha/\sin\beta$ and $A^{*2}-1 = A^2$, we obtain

$$\sin\beta = \frac{B\sin\alpha}{((B^*C^*-BC\cos\alpha)^2-1)^{\frac{1}{2}}}.$$

We now attack the general construction of polygons with prescribed angles. A particular case of this situation was studied by Beardon in [1], where polygons that have an inscribed circle were constructed.

THEOREM 3·2. *Let $\{\theta_i; i = 1,\dots,n\}$ be an ordered set of angles satisfying $0 < \theta_i < \pi$ and $\sum_{i=1}^{n}\theta_i < (n-2)\pi$. Then for each choice of $(n-3)$-tuple $(\zeta_1,\dots,\zeta_{n-3})$ with $0 < \zeta_i < \theta_i$ there is a convex hyperbolic polygon P, depending on the ζ_i, whose angles are $\theta_1,\dots,\theta_n$. In particular, when $\zeta_i = \frac{1}{2}\theta_i$, we obtain the polygon found by Beardon in* [1].

Proof. First of all let us observe that any convex hyperbolic polygon has at least two consecutive angles whose bisectors meet at an interior point. Without loss of generality, we may suppose that this point is O, the centre of D, and that the bisectors of θ_{n-1} and θ_n meet at O. We label the subscripts of the vertices modulo n, and they will be numbered anti-clockwise. Let us suppose that the polygon exists. We are going to prove that it depends upon the parameters ζ_i with $i = 1,\dots,n-3$. We join O to each vertex P_i, and let ρ_i be the hyperbolic length of OP_i. Each angle θ_i is divided into ζ_i and $\hat{\zeta}_i = \theta_i - \zeta_i$ (see Figure 2).

We write σ_i, κ_i, $\hat{\sigma}_i$, $\hat{\kappa}_i$ respectively for $\sin\zeta_i$, $\cos\zeta_i$, $\sin\hat{\zeta}_i$ and $\cos\hat{\zeta}_i$. From the Sine Rule we have

$$\sinh\rho_i = \frac{\sinh\rho_1\hat{\sigma}_1\dots\hat{\sigma}_{n-1}}{\sigma_2\dots\sigma_i}, \quad (i = 2,\dots,n) \tag{3·1}$$

and moreover $$\sinh\rho_n = \frac{\sinh\rho_1\sigma_1}{\hat{\sigma}_n}.$$

Hence $$\prod_{i=1}^{n}\sigma_i = \prod_{i=1}^{n}\hat{\sigma}_i.$$

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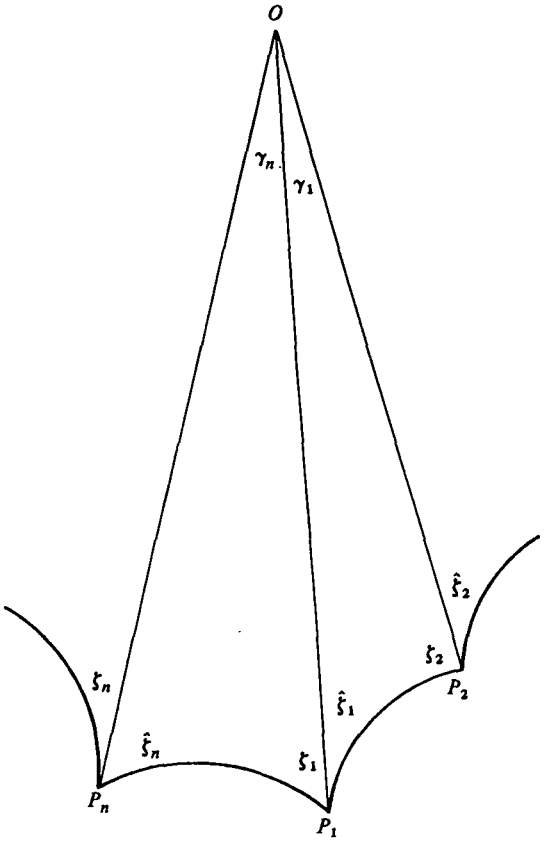


Fig. 2.

By choice of O, we have $\zeta_{n-1} = \hat{\zeta}_{n-1} = \frac{1}{2}\theta_{n-1}$, and $\zeta_n = \hat{\zeta}_n = \frac{1}{2}\theta_n$, so that

$$\prod_{i=1}^{n-2} \sigma_i = \prod_{i=1}^{n-2} \hat{\sigma}_i.$$

We thus obtain ζ_{n-2} as a function of the ζ_i with $i = 1, \ldots, n-3$, as we have

$$\tan \zeta_{n-2} = \frac{\sin \hat{\zeta}_1 \ldots \sin \hat{\zeta}_{n-3} \sin \theta_{n-2}}{\sin \zeta_1 \ldots \sin \zeta_{n-3} + \sin \hat{\zeta}_1 \ldots \sin \hat{\zeta}_{n-3} \cos \theta_{n-2}}. \tag{3·2}$$

Conversely, taking ζ_{n-2} defined by (3·2), $\zeta_{n-1} = \frac{1}{2}\theta_{n-1}$ and $\zeta_n = \frac{1}{2}\theta_n$, we may construct the polygon with arbitrary ζ_i with $0 < \zeta_i < \theta_i$ for $i = 1, \ldots, n-3$, as follows.

We must construct n triangles OP_iP_{i+1} (see Figure 3) satisfying $\sum_{i=1}^{n} \gamma_i = 2\pi$. From Lemma 3·1 (i) we have

$$\cos \gamma_i = \frac{(\sinh^2 \rho_i + 1)^{\frac{1}{2}} (\sinh^2 \rho_i \hat{\sigma}_i + \hat{\sigma}_{i+1}^2)^{\frac{1}{2}} \hat{\sigma}_i - \hat{\kappa}_i \kappa_{i+1}}{1 + \sinh^2 \rho_i \hat{\sigma}_i^2}. \tag{3·3}$$

Substituting in (3·3) the expressions (3·1) we obtain

$$\cos \gamma_i = \frac{\left(\frac{\sinh^2 \rho_1 \hat{\sigma}_1^2 \ldots \hat{\sigma}_{i-1}^2}{\sigma_2^2 \ldots \sigma_i^2} + 1\right)^{\frac{1}{2}} \left(\frac{\sinh^2 \rho_1 \hat{\sigma}_1^2 \ldots \hat{\sigma}_i^2}{\sigma_2^2 \ldots \sigma_i^2} + \sigma_{i+1}^2\right)^{\frac{1}{2}} \hat{\sigma}_i - \hat{\kappa}_i \kappa_{i+1}}{1 + \left(\frac{\sinh^2 \rho_1 \hat{\sigma}_1^2 \ldots \hat{\sigma}_i^2}{\sigma_2^2 \ldots \sigma_i^2}\right)}.$$

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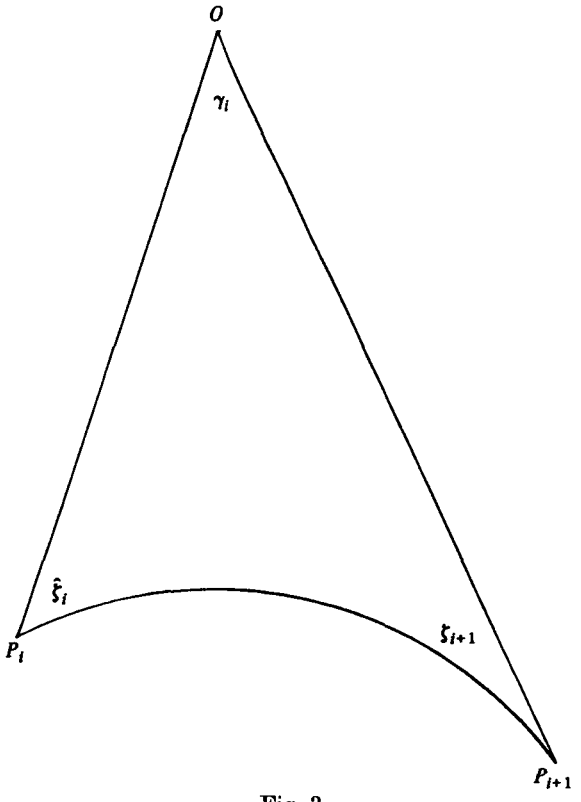


Fig. 3.

Let $f(\rho_1) = \Sigma_{i=1}^{n} \cos^{-1}(\gamma_i)$. This function is continuous. Moreover

$$\lim_{\rho_1 \to \infty} f(\rho_1) = \sum_{i=1}^{n} \cos^{-1}\left(\frac{\dfrac{\hat{\sigma}_1 \dots \hat{\sigma}_{i-1}}{\sigma_2 \dots \sigma_i} \dfrac{\hat{\sigma}_1 \dots \hat{\sigma}_i \hat{\sigma}_i}{\sigma_2 \dots \sigma_i}}{\dfrac{\hat{\sigma}_1^2 \dots \hat{\sigma}_i^2}{\sigma_2^2 \dots \sigma_i^2}} \right) = \sum_{i=1}^{n} \cos^{-1}(1) = 0$$

and

$$\lim_{\rho_1 \to 0} f(\rho_1) = \sum_{i=1}^{n} \cos^{-1}(\sigma_{i+1}\hat{\sigma}_i - \hat{\kappa}_i \kappa_{i+1}) = n\left(\pi - \sum_{i=1}^{n} (\hat{\zeta}_i + \zeta_{i+1})\right) = n\pi - \sum_{i=1}^{n} \theta_i > 2\pi.$$

Hence there exists a solution ρ_1^* such that $f(\rho_1^*) = 2\pi$, and the union of the n triangles OP_iP_{i+1} gives the desired polygon.

Now let $\zeta_i = \frac{1}{2}\theta_i$. In this case $\tan \zeta_{n-2} = (\sin \theta_{n-2})/(1 + \cos \theta_{n-2}) = \tanh \frac{1}{2}\theta_{n-2}$.

Observe that it is enough to take an angle $\zeta_i \neq \frac{1}{2}\theta_i$ in order to obtain a polygon not having an inscribed circle, and so this is different from the ones obtained by Beardon.

COROLLARY 3·3. *With the notation above, the perimeter of the hyperbolic polygon is*

$$\sum_{i=1}^{n} \tanh^{-1} \eta_i,$$

where

$$\eta_i = \left(\frac{\sinh \rho_1^* \hat{\sigma}_1 \dots \hat{\sigma}_{i-1}(\hat{\sigma}_{i+1}^2 - \sigma_i^2)}{(\sinh^2 \rho_1^* \hat{\sigma}_1^2 \dots \hat{\sigma}_{i-1}^2 + \sigma_2^2 \dots \sigma_i^2)^{\frac{1}{2}} \hat{\kappa}_i \hat{\sigma}_{i+1}^2 - \hat{\sigma}_i \kappa_{i+1} (\sinh^2 \rho_1^* \hat{\sigma}_1^2 \dots \hat{\sigma}_i^2 + \sigma_2^2 \dots \sigma_{i+1}^2)^{\frac{1}{2}}} \right).$$

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Proof. It suffices to apply Lemma 3·1 (ii).

COROLLARY 3·4. *In the above construction the lengths* λ_i *of the sides* P_iP_{i+1} *for* $i = 1, \ldots, n-3$ *may be taken as parameters.*

Proof. By Lemma 3·1 (iii), the lengths λ_i determine the angles ζ_i and so they may be used as parameters. Recall that the conditions $0 < \zeta_i < \theta_i$ imply restrictions on the possible values of λ_i.

Remark 3·5. In the next section we will be concerned with fundamental regions. There, the angles are classified in cycles such that the sum of the angles in each cycle is $2\pi/M$, with M a positive integer, and certain pairs of sides are identified. Thus, if the number of sides and the cycles are prescribed, the number of free parameters on which the construction depends is given by the sum of the parameters provided by the cycles and the parameters from Theorem 3·2, minus the number of pairs of identified sides. Explicitly, let n be the number of sides in the region, t the number of cycles, l_i the number of angles in the ith cycle and n^* the number of pairs of identified sides. We find that the number of parameters is

$$\sum_{i=1}^{t} (l_i - 1) + (n-3) - n^* = 2n - (t + n^* + 3). \tag{3·4}$$

4. *Fundamental regions*

The aim of this section is, for each NEC signature σ, the construction of hyperbolic polygons which are fundamental regions with the minimum possible number of sides for NEC groups with signature σ. As a consequence of the construction we deduce that the polygons characterize the classes in the Teichmüller space and the number of parameters equals the dimension of it. First we describe Wilkie's fundamental region by means of which he obtained the presentation (2·2)–(2·3).

Let Γ be an NEC group with signature (2·1). The Wilkie region is R, a hyperbolic polygon with sides labelled

$$\left.\begin{aligned} &\xi_1, \xi_1', \ldots, \xi_r, \xi_r', \epsilon_1, \gamma_{10}, \ldots, \gamma_{1s_1}, \epsilon_1', \ldots, \epsilon_k, \gamma_{k0}, \ldots, \gamma_{ks_k}, \epsilon_k', \alpha_1, \beta_1', \alpha_1', \beta_1, \ldots, \alpha_g, \beta_g', \alpha_g', \beta_g \\ \text{or} \quad &\xi_1, \xi_1', \ldots, \xi_r, \xi_r', \epsilon_1, \gamma_{10}, \ldots, \gamma_{1s_1}, \epsilon_1', \ldots, \epsilon_k, \gamma_{k0}, \ldots, \gamma_{ks_k}, \epsilon_k', \delta_1, \delta_1^*, \ldots, \delta_g, \delta_g^*, \end{aligned}\right\} \tag{4·1}$$

according to the sign '+' or '−' of (2·1).

The edges of R are identified (see (2·2)) by

$$\left.\begin{aligned} &x_i(\xi_i') = \xi_i \quad (i = 1, \ldots, r), \quad e_i(\epsilon_i') = \epsilon_i \quad (i = 1, \ldots, k), \\ &c_{ij}(\gamma_{ij}) = \gamma_{ij} \quad (i = 1, \ldots, k, j = 0, \ldots, s_i), \\ &a_i(\alpha_i') = \alpha_i, \quad b_i(\beta_i') = \beta_i, \quad d_i(\delta_i^*) = \delta_i \quad (i = 1, \ldots, g). \end{aligned}\right\} \tag{4·2}$$

Let us denote the angle between the sides λ and μ by $\langle \lambda, \mu \rangle$. The angles of R are

$$\left.\begin{aligned} &\langle \xi_i, \xi_i' \rangle = 2\pi/m_i \quad (i = 1, \ldots, r), \\ &\langle \epsilon_i, \gamma_{i0} \rangle + \langle \gamma_{is_i}, \epsilon_i' \rangle = \pi \quad (i = 1, \ldots, k), \\ &\langle \gamma_{i,j-1}, \gamma_{ij} \rangle = \pi/n_{ij} \quad (i = 1, \ldots, k, j = 1, \ldots, s_i). \end{aligned}\right\} \tag{4·3}$$

The sum of the remaining angles denoted by $\Sigma^{2ag+r+k-1} \langle\ \rangle$ is 2π.

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Let σ be an NEC signature. From now on let $s = \sum_{i=1}^{k} s_i$. We denote by $X(\sigma)$ the class of NEC groups with signature σ and by $T(\sigma)$ the quotient of $X(\sigma)$ by conjugation. With this notation we may state the next theorem.

THEOREM 4·1. *Let σ be an* NEC *signature. There exists a class P_σ of hyperbolic polygons associated with σ with the following properties*:

(*a*) *every polygon P of P_σ is a convex fundamental region of an* NEC *group $\Gamma \in X(\sigma)$ and the number of sides of P is minimal*;

(*b*) *the construction of the polygons of P_σ depends upon $d(\sigma)$ free parameters, where $d(\sigma)$ is defined as at the end of* §2;

(*c*) *there exists a bijection between $P_\sigma/\sim$ and $T(\sigma)$, where '$\sim$' denotes the equivalence relation of congruence of polygons.*

Proof. Let σ be as in (2·1). We shall consider separately the following four cases for the values of r and k:

(i) $r \neq 0 \neq k$, (ii) $r = 0, k \neq 0$, (iii) $r \neq 0, k = 0$, (iv) $r = k = 0$.

Throughout §4, all angles will belong to $(0, \pi)$. We group the sides of the polygons into the following blocks:

$$X_i = \xi_i \xi_i'$$

$$E_i = \epsilon_i \gamma_{i1} \dots \gamma_{is_i} \epsilon_i' \quad (\text{if } s_i \neq 0)$$

$$E_i^* = \epsilon_i \gamma_i \epsilon_i' \quad (\text{if } s_i = 0)$$

$$A_i = \alpha_i \beta_i' \alpha_i' \beta_i \quad (\text{if the sign of } \sigma \text{ is '+'})$$

$$D_i = \delta_i \delta_i^* \quad (\text{if the sign of } \sigma \text{ is '}-\text{'}).$$

(*a*)(i) The class P_σ is formed by polygons whose sides have the form

$$X_2 \dots X_r E_1 \dots E_k A_1 \dots A_g \quad \text{or} \quad X_2 \dots X_r E_1 \dots E_k D_1 \dots D_g \tag{4·4}$$

according to the sign of σ (note that if $s_i = 0$, then E_i must be replaced by E_i^*), and angles satisfying

$$\langle \xi_i, \xi_i' \rangle = 2\pi/m_i \quad (i = 2, \dots, r) \tag{4·5·i}$$

$$\left.\begin{aligned} \langle \epsilon_i, \gamma_{i1} \rangle + \langle \gamma_{is_i}, \epsilon_i' \rangle &= \pi/n_{i1} \quad (\text{if } s_i \neq 0) \\ \langle \epsilon_i, \gamma_i \rangle = \langle \gamma_i, \epsilon_i' \rangle &= \tfrac{1}{2}\pi \quad (\text{if } s_i = 0) \end{aligned}\right\} \tag{4·5·ii}$$

or

$$\langle \gamma_{i,j-1}, \gamma_{ij} \rangle = \pi/n_{ij} \quad (i = 1, \dots, k, j = 2, \dots, s_i) \tag{4·5·iii}$$

$$\sum^{2\alpha g + r + k - 1} \langle\ \rangle = 2\pi/m_1. \tag{4·5·iv}$$

For each choice in (4·5·ii, iv) the method studied in §3 allows us to obtain polygons with these angles.

Let x_i $(i = 2, \dots, r)$, c_{ij} $(i = 1, \dots, k, j = 1, \dots, s_i)$, e_i $(i = 1, \dots, k)$, a_i, b_i, d_i $(i = 1, \dots, g)$ be defined as in (4·2) and $c_{i0}(\gamma_i) = \gamma_i$ when $s_i = 0$. These transformations obviously exist, and x_i is elliptic with period m_i, c_{ij} and c_{i0} are reflections, e_i, a_i, b_i are hyperbolic and d_i is a glide-reflection.

We only need to search a neighbourhood around the vertices of the cycles of (4·5) which are different from those of (4·3).

Consider Figure 4, where $\phi_1 + \phi_2 = \pi/n_{i1}$.

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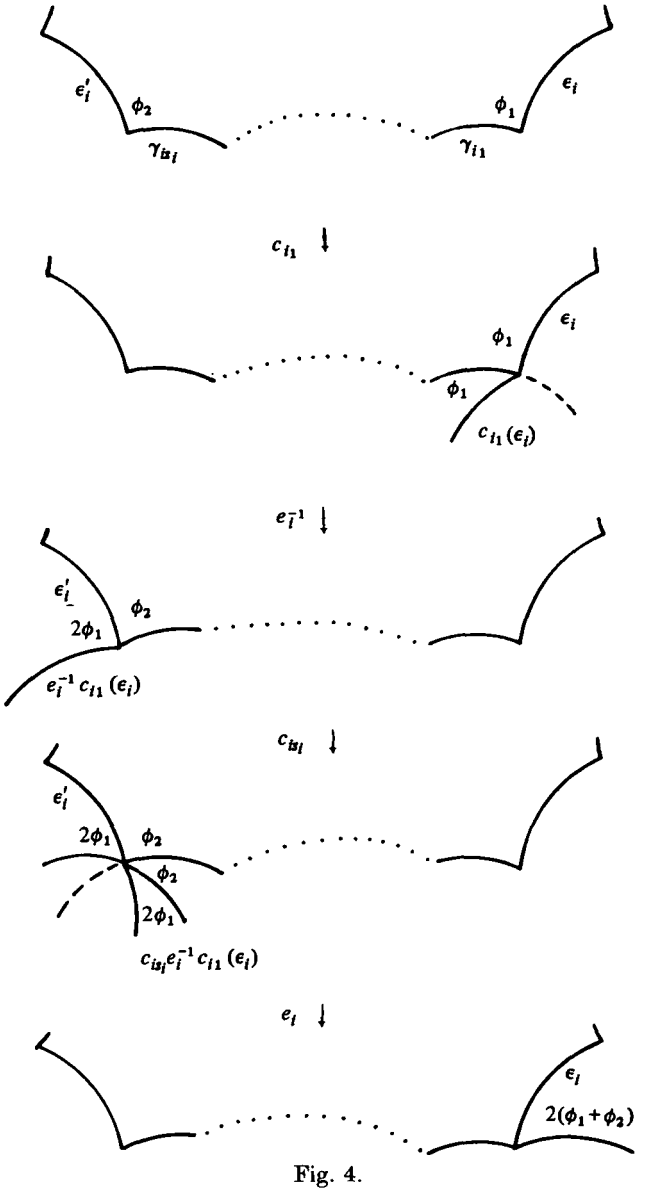


Fig. 4.

Then $e_i c_{is_i} e_i^{-1} c_{i1}$ is an elliptic transformation with period n_{i1}. We have obtained from the vertices of the cycles (4·5·ii) the following relations:

$$(e_i c_{is_i} e_i^{-1} c_{i1})^{n_{i1}} = 1, \quad (i = 1, \ldots, k).$$

Let us denote $e_i c_{is_i} e_i^{-1}$ by c_{i0}; then

$$(c_{i0})^2 = 1, \quad (c_{i0} c_{i1})^{n_{i1}} = 1, \quad e_i^{-1} c_{i0} e_i c_{is_i} = 1,$$

which are canonical Wilkie relations (see (2·3)).

When $s_i = 0$, the axes of e_i and c_{i0} coincide and then $e_i c_{i0} = c_{i0} e_i$. Consider Figure 5.

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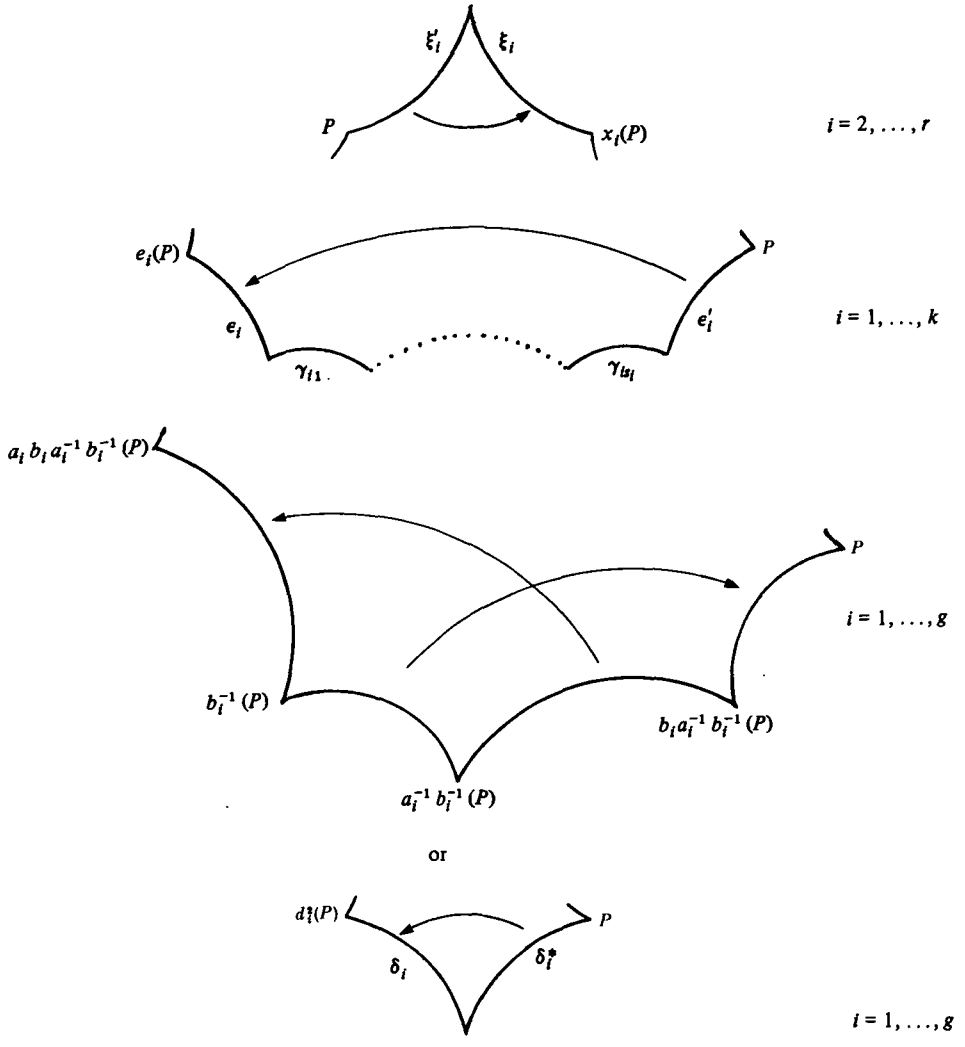


Fig. 5.

Let $\bar{P}$ be the point where β_g' (or δ_g^*) and ξ_2 meet; we have

$$v(\bar{P}) = (x_2 \ldots x_r e_1 \ldots e_k a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1})(\bar{P}) = \bar{P}$$

or

$$v(\bar{P}) = (x_2 \ldots x_r e_1 \ldots e_k d_1^2 \ldots d_g^2)(\bar{P}) = \bar{P},$$

and hence v is an elliptic transformation. From (4·5·iv) its period is m_1. If we denote it by x_1^{-1} we have $x_1^{m_1} = 1$,

and $x_1 \ldots x_r e_1 \ldots e_k a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1$ or $x_1 \ldots x_r e_1 \ldots e_k d_1^2 \ldots d_g^2 = 1$.

We have obtained all the relations of (2·3). Therefore the group Γ generated by x_i, e_i, c_{ij}, a_i, b_i (or d_i) is an NEC group with signature σ.

(*a*)(ii) Now let $r = 0$ and $k \neq 0$. Reordering the set of period cycles $(n_{i1}, \ldots, n_{is_i})$ with $i = 1, \ldots, k$, we may suppose that $s_1 \neq 0, \ldots, s_{k'} \neq 0, s_{k'+1} = 0, \ldots, s_k = 0$. We now construct a polygon with sides

$$\gamma_{11} \cdots \gamma_{1s_1} E_2 \ldots E_{k'} E_{k'+1}^* \ldots E_k^* A_1 \ldots A_g \quad \text{or} \quad \gamma_{11} \ldots \gamma_{1s_1} E_2 \ldots E_{k'} E_{k'+1}^* \ldots E_k^* D_1 \ldots D_g$$

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(according to the sign of the signature), and angles satisfying

$$\langle \gamma_{i,j-1}, \gamma_{ij} \rangle = \pi/n_{ij} \quad (i = 1, \dots, k', j = 2, \dots, s_i) \tag{4·6·i}$$

$$\langle \epsilon_i, \gamma_{i1} \rangle + \langle \gamma_{is_i}, \epsilon_i' \rangle = \pi/n_{i1} \quad (i = 2, \dots, k') \tag{4·6·ii}$$

$$\langle \epsilon_i, \gamma_i \rangle = \langle \gamma_i, \epsilon_i' \rangle = \tfrac{1}{2}\pi \quad (i = k'+1, \dots, k) \tag{4·6·iii}$$

$$\sum^{2\alpha g+k} \langle\ \rangle = \pi/n_{i1}. \tag{4·6·iv}$$

By an argument similar to that of (*a*)(i) we obtain the relations. We omit the remaining details.

If $k' = 0$ (that is, if $s_i = 0$ for each i) the angles in (4·6·i) and (4·6·ii) do not appear. The two last types of cycles now become

$$\langle \epsilon_i, \gamma_i \rangle + \langle \gamma_i, \epsilon_i' \rangle = \tfrac{1}{2}\pi \quad (i = 2, \dots, k) \tag{4·7·i}$$

$$\sum^{2\alpha g+k} \langle\ \rangle = \pi. \tag{4·7·ii}$$

(Note that the polygon is now $\gamma_1 E_2^* \dots E_k^* A_1 \dots A_g$ or $\gamma_1 E_2^* \dots E_k^* D_1 \dots D_g$.)

(*a*)(iii) If $r \neq 0$ and $k = 0$ we construct a polygon with sides

$$X_2 \dots X_r A_1 \dots A_g \quad \text{or} \quad X_2 \dots X_r D_1 \dots D_g$$

with angles satisfying

$$\langle \xi_i, \xi_i' \rangle = 2\pi/m_i \quad (i = 2, \dots, r) \tag{4·8·i}$$

$$\sum^{2\alpha g+r-1} \langle\ \rangle = 2\pi/m_1. \tag{4·8·ii}$$

When the sign of the signature is '+' we obtain Fuchsian groups (see [1]).

(*a*)(iv) If $r = k = 0$ the polygon is

$$A_1 \dots A_g \quad \text{or} \quad D_1 \dots D_g,$$

with angles satisfying

$$\langle \alpha_1, \beta_1' \rangle = \langle \alpha_1', \beta_1 \rangle \tag{4·9·i}$$

$$\langle \beta_1', \alpha_1' \rangle + \langle \alpha_1, \beta_1' \rangle = \pi \tag{4·9·ii}$$

$$\sum^{4g-3} \langle\ \rangle = \pi - \langle \alpha_1, \beta_1' \rangle \tag{4·9·iii}$$

or

$$\langle \delta_1, \delta_1^* \rangle = \langle \delta_2, \delta_2^* \rangle \tag{4·10·i}$$

$$\langle \delta_1^*, \delta_2^* \rangle + \langle \delta_1, \delta_1^* \rangle = \pi \tag{4·10·ii}$$

$$\sum^{2g-3} \langle\ \rangle = \pi - \langle \delta_1, \delta_1^* \rangle. \tag{4·10·iii}$$

The first case corresponds to the well-known Fuchsian groups (see [9]).

We observe that by construction the number of sides of the above polygon is minimal. Indeed, if k' is the number of indices i such that $s_i \neq 0$, we have

(*a*)(i) $n = 2\alpha g + 2r + 3k - k' + s - 2$;

(*a*)(ii) $n = 2\alpha g + 3k - k' + s - 2$;

(*a*)(iii) $n = 2\alpha g + 2r - 2$;

(*a*)(iv) $n = 2\alpha g$.

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(Compare the result with [5].)

(*b*) Now let t and n^* be as in Remark 3·5. We have for the different cases of (*a*)

(i) $t = r+2(k-k')+s$, $n^* = \alpha g+r-1+k$;

(ii) $t = 2(k-k')+s$, $n^* = \alpha g+k-1$;

(iii) $t = r$, $n^* = \alpha g+r-1$.

(iv) In this case there exists a unique cycle subject to certain conditions, and under these circumstances $t = 3$ (see (4·9) and (4·10)), and $n^* = \alpha g$.

By (3·4) we see that the number of free parameters is $3\alpha g+3k+2r+s-6$, that is, it equals the dimension of the Teichmüller space of each NEC-group with signature σ.

(*c*) Let $P \in P_\sigma$ and let Γ_P be the group generated by the side-pairing transformations of P. We denote by $[P]$ the class of polygons which are congruent to P (that is, the sides with the corresponding label have the same length and the angles with the corresponding label are equal) and we denote by $[\Gamma_P]$ the class of NEC groups Γ' such that $\Gamma' = g\Gamma g^{-1}$ for some $g \in G$.

We define $f: P_\sigma/\sim \to T(\sigma)$ by $f([P]) = [\Gamma_P]$. We show that this correspondence is a bijection.

(i) If P_1 and P_2 are congruent there exists $g \in G$ such that $P_2 = g(P_1)$. Let v be the generator of Γ_{P_2} that identifies the sides λ and $\bar{\lambda}$ of P_2 (note that λ and $\bar{\lambda}$ may be the same side). Let $g^{-1}(\lambda)$ and $g^{-1}(\bar{\lambda})$ be the corresponding sides of P_1 and u the generator of Γ_{P_1} that identifies them. We have $u(g^{-1}(\lambda)) = g^{-1}(v(\lambda))$. Thus $v = gug^{-1}$, and therefore $\Gamma_{P_2} = g\Gamma_{P_1}g^{-1}$. Hence if $P_1, P_2 \in [P]$ then $\Gamma_{P_1}, \Gamma_{P_2} \in [\Gamma_P]$.

(ii) Conversely, let $\Gamma_{P_2} = g\Gamma_{P_1}g^{-1}$; then $g(P) = P^*$ is a fundamental region of an NEC group Γ_{P^*} and $\Gamma_{P_2} = \Gamma_{P^*}$. Thus f is one-to-one.

(iii) The surjectivity of f comes directly from part (*b*).

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