106.20 When do we have 1 + 1 = 11 and 2 + 2 = 5?

The phrase "2 + 2 = 5" is a cliché or a slogan used in political speeches, propaganda or literature, most notably in the novel *1984* by George Orwell. More recently, we came across a YouTube short film comedy, *Alternative Math*, produced by IdeaMan Studios (see [1]). It is a hilarious exaggeration of a teacher who is dragged through the mud for teaching that $2 + 2 = 4$ and not 22, as Danny, a young student, kept on insisting. In the movie, Danny and the whole community sincerely believe $1 + 1 = 11$ and $2 + 2 = 22$. Jokes aside, we ask the question whether a polynomially defined group law "$\oplus$" defined over the field of rationals such that $1 \oplus 1 = u$ and $2 \oplus 2 = v$ can simultaneously be satisfied for arbitrary integers u and v.

Introduction

In the standard mathematical world of arithmetic, the truth of 'two plus two equals four' is self-evident. However, a protagonist in 'Notes from the Underground', by Fyodor Dostoevsky says:

> "I admit that twice two makes four is an excellent thing, but if we are to give everything its due, twice two makes five is sometimes a very charming thing too."

Taking inspiration from the works of fiction by Orwell* and Dostoevsky, we ask rather seriously: can we really have $2 + 2 = 5$? If 'yes', then what about $1 + 1$ in such a mathematical system? And, what if we desire both $1 + 1 = 11$ and $2 + 2 = 5$ to hold together?†

Here we ask a more general question: when does the group law "$\oplus$" satisfy both $1 \oplus 1 = u$ and $2 \oplus 2 = v$? To investigate this, let us fix a field K. The case of interest for us will be when K is a finite extension of $\mathbb{Q}$. We will mainly consider the case of trivial extension, i.e. when K is $\mathbb{Q}$. We further impose the condition that $\oplus$ is defined by a polynomial $P(x, y) \in K[x, y]$, i.e. $x \oplus y = P(x, y)$.

Characterisation of the polynomial group operation $x \oplus y = P(x, y)$

The polynomial $x + y$ provides a natural additive group structure on K and the polynomial xy gives a natural multiplicative group structure on $K \setminus \{0\}$. One can use bijections from K to K to define other group laws on K and its subsets. More precisely, it can be checked that a bijection $f : K \to K$ gives rise to an alternative group structure

(i) on K by the definition $x \oplus y = f^{-1}\big(f(x) + f(y)\big)$, or

(ii) on $K \setminus \{f^{-1}(0)\}$ by $x \oplus y = f^{-1}\big(f(x) f(y)\big)$.

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* "Freedom is slavery. Two and two make five." - 1984.

† In 2017, while election campaigning in the state of Gujarat, India, the Indian Prime Minister Narendra Modi declared, "1+1 is not 2 but 11 and together we will take Gujarat to new heights". Another political leader, the general secretary of the Communist Party of India (Marxist), Mr. Sitaram Yechury said in 2017: "Politics is not just arithmetic. Two plus two could become twenty two if we strengthened people's struggle."

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Let $a, b \in K$ and $a \neq 0$. Assuming $f(x) = ax + b$, a simple calculation shows that the group law in case (i) is given by

$$x \oplus y = P(x, y) = x + y + \frac{b}{a}, \tag{1}$$

and in case (ii) the group law is

$$x \oplus y = P(x, y) = axy + bx + by + \frac{b^2 - b}{a}. \tag{2}$$

It is a consequence of the commutative and associative properties of the group law that the group laws given in (1) and (2) exhaust all the possible groups that can be obtained by polynomials. In other words, one cannot use a totally arbitrary polynomial to define the group law $\oplus$, because the commutative and associative properties of the group law $\oplus$ force the polynomial $P(x, y)$ to be in one of the two forms given in (1) and (2). The interested readers may refer to the proof of Lemma 5 in [2].

In case (i), from (1), it is clear that one can achieve either $1 \oplus 1 = 11$ or $2 \oplus 2 = 5$ but not both, whereas it will be shown in the next section that in case (ii) one can achieve both by using the group law given in (2).

Remark 1

(i) Clearly, by setting $b = 0$ in (1) one can recover the 'ordinary' addition on K, $x \oplus y = P(x, y) = x + y$. Similarly, by setting $a = 1$ and $b = 0$ in (2) the natural multiplicative group operation on $K \backslash \{0\}$, i.e. $x \oplus y = P(x, y) = xy$ is recovered.

(ii) The identity element of the group defined by (1) is $f^{-1}(0) = -\frac{b}{a}$ and the group obtained is isomorphic to K. Similarly, $f^{-1}(1) = \frac{1-b}{a}$ is the identity element of the group defined by (2) and the group in this case is isomorphic to the multiplicative group of the non-zero elements of the field, i.e. $K^{\times}$.

Main results

The use of the following definition will be convenient in the rest of the paper.

Definition 1: Let K be a finite extension of $\mathbb{Q}$. We say that (u, v, P, K) is *true* if there exists a commutative group operation $x \oplus y$ given by a polynomial $P(x, y) \in K[x, y]$, such that

$$1 \oplus 1 = u \qquad \text{and} \qquad 2 \oplus 2 = v. \tag{3}$$

Remark 2

(i) In the following, we will continue to use the statement '(u, v, P, K) is true' as defined in Definition 1, even when the group might be defined on the set $K \backslash \{-b/a\}$, rather than on K.

(ii) We will assume $U = u - 1$ and $V = v - 2$ in the rest of the paper.

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Suppose $P(x, y)$ is given by (1), such that the group law "$\oplus$", defined by $x \oplus y = P(x, y)$, satisfies both $1 \oplus 1 = u$ and $2 \oplus 2 = v$. Then, we have $1 \oplus 1 = 1 + 1 + \frac{b}{a} = u$ and $2 \oplus 2 = 2 + 2 + \frac{b}{a} = v$. Put $c = \frac{b}{a}$. It follows that

$$(u, v, x + y + c, \mathbb{Q}) \text{ is true} \quad \Leftrightarrow \quad V - U = 1, \text{and } c = U - 1. \tag{4}$$

Next we consider the case when $P(x, y)$ is given by (2) and the group law "$\oplus$", defined by $x \oplus y = P(x, y)$, satisfies both $1 \oplus 1 = u$ and $2 \oplus 2 = v$. Since, we have $1 \oplus 1 = u$ and $2 \oplus 2 = v$, we need to solve for a, b in the following equations:

$$a^2 - b + 2ab + b^2 = au, \tag{5}$$

$$4a^2 - b + 4ab + b^2 = av, \tag{6}$$

Subtracting (5) from (6) gives

$$3a^2 + 2ab = a(v - u). \tag{7}$$

Since $a \neq 0$ by assumption, (7) implies $3a + 2b = v - u$, or $b = \frac{1}{2}(-3a - u + v)$. Substituting $b = \frac{1}{2}(-3a - u + v)$ in (5) leads to the quadratic equation

$$a^2 - 2(u + v - 3)a + (u - v)(u - v + 2) = 0,$$

$$\text{or,} \quad a^2 - 2(U + V)a + (U - V)^2 - 1 = 0, \tag{8}$$

where we recall that $U = u - 1$ and $V = v - 2$.

We note the final solutions

$$a = U + V \pm \sqrt{4UV + 1} \quad \text{and} \quad b = \tfrac{1}{2}(-3a - U + V + 1). \tag{9}$$

From now on, we assume that $P(x, y)$ is given by (2), and a and b are as in (9).

For $u = 11$ and $v = 5$, from the above equation, we see that the solution is possible in $\mathbb{Q}$. In this case $(11, 5, 2xy - 6x - 6y + 21, \mathbb{Q})$ and $(11, 5, 24xy - 39x - 39y + 65, \mathbb{Q})$ are both true. As remarked earlier, we note that $-\dfrac{b}{a} = \dfrac{6}{2} = 3$ is excluded from $\mathbb{Q}$ while defining the group by using the group law $x \oplus y = P(x, y) = 2xy - 6x - 6y + 21$, and similarly $-\dfrac{b}{a} = \dfrac{39}{24}$ is excluded from $\mathbb{Q}$, while defining the group using $x \oplus y = P(x, y) = 24xy - 39x - 39y + 65$. However, it is not always possible to have $1 \oplus 1 = u$ and $2 \oplus 2 = v$, defined by a polynomial group law $\oplus$ over rationals, i.e. for some choices of u and v, there is no P for which $(u, v, P, \mathbb{Q})$ is true. As an example, for $u = 11$ and $v = 22$ we have

$$a = 3\left(10 \pm \sqrt{89}\right), \; b = \tfrac{1}{2}(11 - 3a).$$

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In this case, $(11, 22, P, \mathbb{Q})$ is false for all P, but $\left(11, 22, P, \mathbb{Q}\left(\sqrt{89}\right)\right)$ is true for

$$P(x, y) = 3xy\left(10 \pm \sqrt{89}\right) - \tfrac{1}{2}(x + y)\left(79 \pm 9\sqrt{89}\right) + 6\left(10 \pm \sqrt{89}\right).$$

Now we consider, given $u, v \in \mathbb{Z}$, whether $(u, v, P, \mathbb{Q})$ is true (with P as defined in (2)). It is clear from (9) that

$$(u, v, P, \mathbb{Q}) \text{ is true } \Leftrightarrow 4UV + 1 \text{ is a perfect square.} \tag{10}$$

Consider the Diophantine equation

$$4UV + 1 = n^2. \tag{11}$$

for all possible integers U, V and n. First we assume that n is fixed. Clearly for a solution to exist $n^2 \equiv 1 \bmod 4$. This means n must be odd. Then, in fact, $n^2 \equiv 1 \bmod 8$. Let this be the case. Assume $n = 2m + 1$. Then, we have

$$UV = m(m + 1) = \tfrac{1}{4}\left(n^2 - 1\right). \tag{12}$$

Clearly, since n is odd, $n^2 \geqslant 1$, so $UV \geqslant 0$. We also note that, from (12), for a given n we can count the number of solutions (u, v), and it is given by $\sigma_0\left(\tfrac{1}{4}\left(n^2 - 1\right)\right)$ which is the number of divisors of $\tfrac{1}{4}\left(n^2 - 1\right)$.

From the above discussion, we obtain the following result.

Theorem 1

Assume $U = u - 1$ and $V = v - 2$ to be integers. Then

$$(u, v, P, \mathbb{Q}) \text{ is true } \Leftrightarrow UV = m(m + 1), \text{ for some } m \in \mathbb{Z}. \tag{13}$$

The following corollary immediately follows from Theorem 1.

Corollary 1: Let $U = u - 1$ and $V = v - 2$ be prime numbers. Then,

(i) $(u, v, P, \mathbb{Q})$ is true $\Rightarrow U = 2$ and $V = 3$ or vice versa;

(ii) $|U - V| = 1 \Rightarrow (u, v, P, \mathbb{Q})$ is true.

Proof:

(i) We have $UV = m(m + 1)$, coprime factors, whence the result.

(ii) Here, if $U = m$ then $V = m + 1$, or vice versa. In either case $UV = m(m + 1)$ and we are done.

Finally, we would like to add yet another "strange" operation $\oplus$ corresponding to the case $m = 1$ in Theorem 1. In this case $u = 2, v = 4$ and the group law is

$$x \oplus y = 6xy - 8x - 8y + 12.$$

Here we have $1 \oplus 1 = 2$ and $2 \oplus 2 = 4$, which looks like the ordinary addition, but it is a different group operation.

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106.21 A class of interesting integrals

In this Note we shall show that:

$$\int_0^{+\infty} \frac{\log x \arctan x}{x(1+x)}\,dx = \frac{\pi^3}{96}. \tag{1}$$

The calculation uses a trick, which reduces the integral to two easy ones. The same trick can be applied to show that: (k, m, n positive integers)

$$\int_0^{+\infty} \frac{\log^{2k-1} x \arctan x^m}{x(1+x^n)}\,dx = Q_{k,m,n}\,\pi^{2k+1},$$

$$\int_0^{+\infty} \frac{\log^{2k-2} x \log(1+x^m)}{x(1+x^n)}\,dx = R_{k,m,n}\,\pi^{2k}$$

with $Q_{k,m,n}$, $R_{k,m,n}$ rational numbers depending on k, m and n. These integrals cannot be found in tables of integrals such as Gradshteyn and Ryzhik's *Table of Integrals, Series, and Products* [1].

Proof: To prove (1), we denote the integral by I and split up the interval at $x = 1$. We then make the substitution $x = \frac{1}{t}$ in the resultant second integral. So we have

$$\begin{aligned} I &= \int_0^1 \frac{\log x \arctan x}{x(1+x)}dx + \int_1^0 \left(\log\frac{1}{t}\right)\left(\arctan\frac{1}{t}\right)\frac{t^2}{t+1}\left(-\frac{1}{t^2}\right)dt \\ &= \int_0^1 \frac{\log x \arctan x}{x(1+x)}dx - \int_0^1 \log t\left\{\frac{\pi}{2} - \arctan t\right\}\frac{1}{t+1}dt \\ &= \int_0^1 \log x \arctan x\left\{\frac{1}{x} - \frac{1}{1+x}\right\}dx - \int_0^1 \log x\left\{\frac{\pi}{2} - \arctan x\right\}\frac{1}{1+x}dx \\ &= \int_0^1 \frac{1}{x}\log x \arctan x\,dx - \frac{\pi}{2}\int_0^1 \frac{\log x}{1+x}dx. \end{aligned}$$

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