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# ON CLASSES OF LIFETIME DISTRIBUTIONS WITH UNKNOWN AGE

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Some class properties of the used better (worse) than aged [UBA (UWA)] and the used better (worse) than aged in expectation [UBAE (UWAE)] classes of lifetime distributions are considered. Relationships with the decreasing (increasing) mean residual lifetime [DMRL (IMRL)] class and the decreasing (increasing) variance residual lifetime [DVRL (IVRL)] class are established. Discrete UBA and UWA distributions are introduced and studied. Characterizations of UBA and UWA distributions are derived by using discrete aging properties of mixed Poisson distributions. Applications of these results to queueing theory and ruin are then considered. In particular, preservation of UBA (UWA) and UBAE (UWAE) under a transform of life distributions is given.

## 1. INTRODUCTION

Let $F(x) = 1 - \bar{F}(x)$ be the distribution function (d.f.) of a nonnegative random variable $X$ with finite mean $0 < \int_0^\infty \bar{F}(x)\,dx < \infty$. Define

$$r(x) = E(X - x | X > x) = \frac{\int_x^\infty \bar{F}(t)\,dt}{\bar{F}(x)} = \frac{\int_0^\infty \bar{F}(x+t)\,dt}{\bar{F}(x)}, \qquad x \geq 0, \qquad \textbf{(1.1)}$$

to be the mean residual lifetime of $X$ or $F$. Denote the equilibrium distribution of $F$ by

$$F_1(x) = 1 - \bar{F}_1(x) = \int_0^x \bar{F}(t)\,dt \left( \int_0^\infty \bar{F}(t)\,dt \right)^{-1}, \qquad x \geq 0. \qquad \textbf{(1.2)}$$

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If $F$ is absolutely continuous with failure rate

$$\mu(x) = -\frac{d}{dx}\ln\bar{F}(x) = \frac{F'(x)}{\bar{F}(x)},$$

then

$$\bar{F}(x) = e^{-\int_0^x \mu(y)\,dy}. \tag{1.3}$$

Furthermore, if $\mu(x)$ satisfies $0 < \mu(\infty) = \lim_{x\to\infty}\mu(x) < \infty$, then it is well known [and easily established using (1.1) and L'Hôpital's rule] that

$$r(\infty) = \lim_{x\to\infty} r(x) = \frac{1}{\mu(\infty)}. \tag{1.4}$$

In order to describe aging properties of devices with unknown age, Alzaid [1] introduced the two aging concepts of the used better than aged (UBA) and used better than aged in expectation (UBAE) in terms of $\mu(\infty)$, and absolutely continuous distributions are considered there. The two aging classes have a close connection with age-smooth failure distributions discussed by Bhattacharjee [2]. Di Crescenzo [4] went further into the study of the two aging classes of distributions by using stochastic orderings. We first give the definitions of UBA and UBAE and their dual classes in terms of $r(\infty)$.

Definition 1.1: *The d.f. $F$ is said to be used better* (*worse*) *than aged or* UBA (UWA) *if* $0 < r(\infty) < \infty$ *and, for all* $t \geq 0$ *and* $x \geq 0$, *it is the case that*

$$\bar{F}(x+t) \geq (\leq)\ \bar{F}(t)e^{-x/r(\infty)}. \tag{1.5}$$

Definition 1.2: *The d.f. $F$ is said to be used better* (*worse*) *than aged in expectation or* UBAE (UWAE) *if* $0 < r(\infty) < \infty$ *and, for all* $x \geq 0$,

$$r(x) \geq (\leq)\ r(\infty). \tag{1.6}$$

The above definitions of UBA and UBAE are essentially those of Alzaid [1] and Di Crescenzo [4], except that the requirement of absolute continuity may be removed as a result of (1.4). These are convenient modifications in that connections with other lifetime classes of distributions are more easily established, as will become evident in the next section. We remark that the definitions of UBA (UWA) and UBAE (UWAE) may be generalized by replacing $1/r(\infty)$ by $s_0 > 0$, where $-s_0$ is the left abscissa of convergence of the Laplace transform $\tilde{f}(s) = \int_0^\infty e^{-sx}\,dF(x)$ by virtue of (1.3) and Widder [17, Sect. II.2]. For the present purpose, however, we find it expedient to use the present definitions.

An important example of a UBA distribution arising often in reliability, queueing theory, and ruin probability is the mixture of Erlang d.f.'s with density

$$F'(x) = \sum_{j=1}^{r} q_j \frac{\beta(\beta x)^{j-1}e^{-\beta x}}{(j-1)!}, \qquad x \geq 0, \tag{1.7}$$

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where $\beta > 0$, $q_j \geq 0$, $j = 1,2,\ldots$, and $\sum_{j=1}^{r} q_j = 1$, see Esary, Marshall, and Proschan [7] and Tijms [16] for details.

Alzaid [1] demonstrated that the UBA class is a subclass of the UBAE class [a result which follows easily from (1.1) and (1.6)] and that if $F$ is absolutely continuous and has increasing failure rate (IFR) with $0 < \mu(\infty) < \infty$, then $F$ is UBA. It is not hard to see that similar implications also hold among their dual classes, namely UWA, UWAE, and DFR classes. Di Crescenzo [4] discussed further class implications among UBAE, NBUE, and HNBUE, and their dual classes using stochastic orderings. In this paper, we first consider further the relationships among UBA, UBAE, DMRL, and DVRL and their dual classes. Then, we introduce and discuss discrete UBA and UWA distributions in Section 3. Characterizations of the UBA and UWA distributions are given by using discrete UBA and UWA properties of mixed Poisson distributions. Finally, in Section 4, we consider applications of these results in queueing theory and ruin. In particular, the preservation of UBA (UWA) and UBAE (UWAE) under a transform of a life distribution is given. The transform arises often in queueing theory and ruin.

## 2. CLASS PROPERTIES

The following theorem demonstrates that the UBAE (UWAE) class essentially coincides with the class for which $F_1(x)$ is UBA (UWA).

THEOREM 2.1: *The d.f. $F$ is* UBAE (UWAE) *if and only if for all $x \geq 0$ and $t \geq 0$,*

$$\bar{F}_1(x+t) \geq (\leq)\ \bar{F}_1(t)e^{-x/r(\infty)}. \tag{2.1}$$

PROOF: Note that the equilibrium distribution function $F_1(x)$ has failure rate

$$\mu_1(x) = -\frac{d}{dx}\ln \bar{F}_1(x) = \frac{1}{r(x)} \tag{2.2}$$

with $\mu_1(\infty) = 1/r(\infty)$. Then, (2.1) is simply a restatement of the failure rate ordering criterion (1.B.5) of Shaked and Shanthikumar [15, p. 13] between the d.f. $1 - \bar{F}_1(x+t)/\bar{F}_1(t)$ and the exponential d.f. $1 - e^{-x/r(\infty)}$, with failure rates $1/r(x+t)$ and $1/r(\infty)$, respectively. ■

Clearly, (2.2) implies that the mean residual lifetime $r_1(x) = \int_x^\infty \bar{F}_1(t)\,dt/\bar{F}_1(x)$ of $F_1$ satisfies $r_1(\infty) = 1/\mu_1(\infty) = r(\infty)$ by (1.4). Thus, Theorem 2.1 implies that if $0 < r(\infty) < \infty$, then the d.f. $F$ is UBAE (UWAE) if and only if its equilibrium distribution function $F_1$ is UBA (UWA).

A larger class than the IFR (DFR) class is the decreasing (increasing) mean residual lifetime or DMRL (IMRL) class for which $r(x)$ is nonincreasing (nondecreasing) in $x$ for $x \geq 0$. The above relationship between UBA (UWA) and UBAE (UWAE) classes is the same as that between IFR (DFR) and DMRL (IMRL) classes. The following result gives the relationship between UBA (UWA) and DMRL (IMRL), which extends Theorem 1 of Alzaid [1] to a larger class.

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THEOREM 2.2: *If $F$ is* DMRL (IMRL) *with* $0 < r(\infty) < \infty$, *then $F$ is* UBA (UWA).

PROOF: Because $F_1(x)$ has failure rate $1/r(x)$ by (2.2), it is clear by (1.3) that

$$\bar{F}_1(x+t) = \bar{F}_1(t)e^{-\int_t^{x+t} 1/r(y)\,dy} \geq (\leq)\, \bar{F}_1(t)e^{-\int_t^{x+t} 1/r(\infty)\,dy} = \bar{F}_1(t)e^{-x/r(\infty)}.$$

Also, it follows from (1.1) and (1.2) that $\bar{F}(x) = \{\int_0^\infty \bar{F}(y)\,dy\}\bar{F}_1(x)/r(x)$. Thus,

$$\begin{aligned}\bar{F}(x+t) &= \left\{\int_0^\infty \bar{F}(y)\,dy\right\}\frac{\bar{F}_1(x+t)}{r(x+t)} \\ &\geq (\leq) \left\{\int_0^\infty \bar{F}(y)\,dy\right\}\frac{\bar{F}_1(t)e^{-x/r(\infty)}}{r(t)} = \bar{F}(t)e^{-x/r(\infty)},\end{aligned}$$

and so $F$ is UBA (UWA). ■

In addition, a larger class than the DMRL (IMRL) class is the decreasing (increasing) variance residual lifetime or DVRL (IVRL) class, which is equivalent to the class for which $F_1$ is DMRL (IMRL) (e.g., Fagiuoli and Pellerey [8]). The following proposition shows the relationship between DVRL (IVRL) and UBAE (UWAE).

*Proposition 2.1:* If $F$ is DVRL (IVRL) with $0 < r(\infty) < \infty$, then $F$ is UBAE (UWAE).

PROOF: $F$ is DVRL (IVRL) implies that $F_1$ is DMRL (IMRL) and, hence, UBA (UWA) by Theorem 2.2. Thus, $F$ is UBAE (UWAE) from the comments following Theorem 2.1, and the result follows. ■

## 3. DISCRETE UBA AND UWA DISTRIBUTIONS

Suppose that $N$ is a counting random variable with probabilities $\{p_n = \Pr(N = n); n = 0,1,2,\ldots\}$ and tail probabilities

$$a_n = \Pr(N > n) = \sum_{k=n+1}^{\infty} p_k, \qquad n = 0,1,2,\ldots.$$

Then, $\{p_n; n = 0,1,2,\ldots\}$ is said to be discrete increasing (decreasing) failure rate [D-IFR (D-DFR)] if the sequence $a_{n+1}/a_n$, $n = 0,1,2,\ldots$, is nonincreasing (nondecreasing) in $n$ for $n \geq 0$. A larger class than the D-IFR (D-DFR) class is the discrete decreasing (increasing) mean residual lifetime [D-DMRL (D-IMRL)] class for which

$$r_n = E(N - n | N > n) = \sum_{k=0}^{\infty} \frac{a_{n+k}}{a_n}, \qquad n = 0,1,2,\ldots \tag{3.1}$$

is nonincreasing (nondecreasing) in $n$ for $n \geq 0$. See, for example, Ebrahimi [5] and Klefsjö [11] for details.

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Let

$$P(z) = \sum_{n=0}^{\infty} p_n z^n, \qquad |z| < z_0,$$

be the probability generating function (p.g.f.) of $N$, with $z_0 \geq 1$ the radius of convergence. As shown by Willmot and Lin [19],

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{z_0} \tag{3.2}$$

if the indicated limit exists. Furthermore, if $\{p_n; n = 0,1,2,\ldots\}$ is D-IFR (D-DFR), then

$$a_{n+1} \geq (\leq) \frac{a_n}{z_0}, \qquad n = 0,1,2,\ldots. \tag{3.3}$$

The following theorem improves this result and shows that (3.3) holds for the larger class D-DMRL (D-IMRL).

THEOREM 3.1: *If* $\{p_n; n = 0,1,2,\ldots\}$ *is* D-DMRL (D-IMRL), *then* (3.3) *holds*.

PROOF: Let $N^*$ have the equilibrium probability function

$$p_n^* = \Pr(N^* = n) = \frac{a_n}{E(N)}, \qquad n = 0,1,2,\ldots,$$

where $E(N) = r_0 a_0$. Let

$$a_n^* = \sum_{k=n+1}^{\infty} p_k^* = \sum_{k=n+1}^{\infty} a_k/E(N), \qquad n = 0,1,2,\ldots,$$

be the associated tail. Then,

$$a_{n+1}^* = a_n^*(1 - h_{n+1}^*),$$

where

$$h_n^* = \Pr(N^* = n | N^* \geq n) = \frac{p_n^*}{p_n^* + a_n^*} = \frac{a_n}{a_n + \sum_{k=n+1}^{\infty} a_k} = \frac{1}{r_n}$$

is a discrete failure rate. Since $h_n^*$ is nondecreasing (nonincreasing), it is clear that $\{p_n^*, n = 0,1,2,\ldots\}$ is D-IFR (D-DFR). Now, from Feller [9, p. 260],

$$P^*(z) = \sum_{n=0}^{\infty} p_n^* z^n = \frac{P(z) - 1}{E(N)(z-1)}, \qquad |z| < z_0, \tag{3.4}$$

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and, thus, $a^*_{n+1} \geq (\leq)\ a^*_n/z_0$. Then,

$$a_{n+1} = E(N)p^*_{n+1} = \frac{E(N)a^*_{n+1}}{r_{n+1}-1} \geq (\leq)\ \frac{E(N)a^*_n}{z_0(r_n-1)} = \frac{E(N)p^*_n}{z_0} = \frac{a_n}{z_0},$$

and the result follows. ■

Discrete distributions satisfying (3.3) are of fundamental importance in the study of compound distributions. It has been shown in Grandell [10] and Willmot and Lin [19] that optimal exponential bounds for the tail probabilities of the compound distributions are obtained when compounding with a discrete distribution satisfying (3.3). By analogy with (1.5) in the continuous case, we have the following definition.

DEFINITION 3.1: *The discrete distribution* $\{p_n; n = 0,1,2,\ldots\}$ *is said to be discrete used better* (*worse*) *than aged* [D-UBA (D-UWA)] *if* $1 < z_0 < \infty$ *and*

$$a_{n+1} \geq (\leq)\ a_n/z_0, \qquad n = 0,1,2,\ldots. \tag{3.5}$$

Thus, we immediately obtain the following class implication.

*Proposition 3.1:* If $\{p_n; n = 0,1,2,\ldots\}$ is D-DMRL (D-IMRL) with $1 < z_0 < \infty$, then $\{p_n; n = 0,1,2,\ldots\}$ is D-UBA (D-UWA).

PROOF: The result follows from Theorem 3.1 and Definition 3.1. ■

Next, we consider an important class of discrete distributions, the mixed Poisson distributions, within the context of probabilistic modelling (e.g., Grandell [10]). The mixed Poisson distribution has probabilities

$$p_n = \int_0^\infty \frac{(\lambda x)^n e^{-\lambda x}}{n!}\, dF(x), \qquad n = 0,1,2,\ldots, \tag{3.6}$$

where $\lambda > 0$ and $F$ is a distribution on $[0,\infty)$.

Generally speaking, aging properties of $\{p_n; n = 0,1,2,\ldots\}$ and $F$ in the mixed Poisson distribution are "the same" (e.g., Block and Savits [3] and Grandell [10]). In fact, Block and Savits [3] and Klefsjö [11] gave characterizations of IFR, IFRA, NBU, NBUE, DMRL, and HNBUE and their dual classes by using discrete aging properties of the Laplace transform, which have a close connection with the mixed Poisson distribution. It will be seen that the same is also true for the UBA (UWA) class. That is to say, the aging property of UBA (UWA) of $F$ can be characterized by the similar aging property of the mixed Poisson distribution $\{p_n, n = 0,1,2,\ldots\}$.

Let $\tilde{f}(s) = \int_0^\infty e^{-sx}\, dF(x)$, $s \geq 0$, be the Laplace transform of $F$ and denote the discrete sequence of $\tilde{f}(s)$ by

$$\beta_n(s) = \frac{(-1)^n}{n!}\, \frac{d^n}{ds^n}\left[\frac{1-\tilde{f}(s)}{s}\right], \qquad n \geq 0,\ s \geq 0. \tag{3.7}$$

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Furthermore, set $\alpha_{n+1}(s) = s^{n+1}\beta_n(s)$ for $n \geq 0$, $s \geq 0$, and $\alpha_0(s) = 1$ for all $s \geq 0$. Then, it is well known that for $n \geq 0$ and $s \geq 0$,

$$\alpha_{n+1}(s) = s \int_0^\infty \frac{(su)^n}{n!} e^{-su} \bar{F}(u)\, du. \tag{3.8}$$

See, for example, (2.2) of Block and Savits [3].

The following lemma is a restatement of Lemma 2.3 of Block and Savits [3].

LEMMA 3.1: *Let $x > 0$ be a continuity point of the d.f. $F$. If $s = s(n,x)$ and $\lim_{n\to\infty} n/s = x$, then*

$$\lim_{n\to\infty} \alpha_{n+1}(s) = \bar{F}(x). \tag{3.9}$$

THEOREM 3.2: *The d.f. $F$ is* UBA (UWA) *if and only if the mixed Poisson distribution $\{p_n; n = 0,1,2,\ldots\}$ given by* (3.6) *is* D-UBA (D-UWA) *for all $\lambda > 0$.*

PROOF: Let $z_0$ be the radius of convergence of the p.g.f. of $\{p_n, n \geq 0\}$ and $r_F(x) = (\int_x^\infty \bar{F}(t)\, dt)/\bar{F}(x)$ be the mean residual lifetime of $F$. First, note that it follows from Willmot and Lin [19, pp. 750–753] that $r_F(\infty) = 1/\{\lambda(z_0 - 1)\}$ and

$$a_n = \sum_{k=n+1}^{\infty} p_k = \lambda \int_0^\infty \frac{(\lambda x)^n e^{-\lambda x}}{n!} \bar{F}(x)\, dx, \qquad n = 0,1,2,\ldots, \tag{3.10}$$

from which integration by parts yields

$$a_{n+1} = \lambda \int_0^\infty \phi(x) \frac{(\lambda x)^n e^{-\lambda x}}{n!} \bar{F}(x)\, dx, \qquad n = 0,1,2,\ldots, \tag{3.11}$$

where

$$\phi(x) = \lambda \frac{\int_x^\infty e^{-\lambda y} \bar{F}(y)\, dy}{e^{-\lambda x}\bar{F}(x)}, \qquad x \geq 0. \tag{3.12}$$

Next, assume that $F$ is UBA (UWA). Then, from (3.12),

$$\begin{aligned}
\phi(x) &= \lambda \int_0^\infty e^{-\lambda y} \frac{\bar{F}(x+y)}{\bar{F}(x)}\, dy \\
&\geq (\leq)\ \lambda \int_0^\infty e^{-y\{\lambda + 1/r_F(\infty)\}}\, dy \\
&= \frac{\lambda}{\lambda + 1/r_F(\infty)} \\
&= \frac{1}{z_0}.
\end{aligned}$$

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Thus, from (3.11), $a_{n+1} \geq (\leq)\, a_n/z_0$ with $z_0 = 1 + 1/\{\lambda r_F(\infty)\}$ and $\{p_n;\ n = 0,1,2,\ldots\}$ is D-UBA (D-UWA).

Conversely, if $\{p_n;\ n = 0,1,2,\ldots\}$ is D-UBA (D-UWA), then

$$\frac{a_{n+1}}{a_n} \geq (\leq)\, \frac{1}{z_0} = \frac{\lambda}{\lambda + 1/r_F(\infty)},$$

from which it follows that

$$a_{n+m+1} \geq (\leq) \left\{ \frac{\lambda r_F(\infty)}{1 + \lambda r_F(\infty)} \right\}^{n+1} a_m \tag{3.13}$$

or

$$\alpha_{n+m+2}(\lambda) \geq (\leq) \left\{ \frac{\lambda r_F(\infty)}{1 + \lambda r_F(\infty)} \right\}^{n+1} \alpha_{m+1}(\lambda), \tag{3.14}$$

where $\alpha_{n+1}(\lambda) = a_n$. If $x$ and $y$ are continuity points of $F$, take $m = [ny/x]$ and $\lambda = n/x$. Then, $\lim_{n\to\infty} m/\lambda = y$ and $\lim_{n\to\infty}(n + m + 1)/\lambda = x + y$. Then, (3.14) may be expressed as

$$\alpha_{n+m+2}(\lambda) \geq (\leq)\, \frac{\alpha_{m+1}(\lambda)}{\{1 + x/nr_F(\infty)\}^{n+1}}. \tag{3.15}$$

Let $n \to \infty$ in (3.15), and from Lemma 3.1, it follows that $\bar{F}(x + y) \geq (\leq)\ \bar{F}(x)e^{-y/r_F(\infty)}$, which means that $F$ is UBA (UWA). ■

Theorem 3.2 gives both characterizations of UBA and UWA distributions and important examples of discrete UBA and UWA distributions.

## 4. APPLICATIONS AND PRESERVATION OF UBA AND UBAE UNDER A TRANSFORM OF LIFE DISTRIBUTIONS

Many applications in single-server queueing and insurance ruin follow from the close connection with compound geometric distributions (e.g., Embrechts, Klüppelberg, and Mikosch [6, pp. 26–27], Ross [14, Sect. 7.4], and references therein). For example, in the $M/G/1$ queue or the classical ruin model, the equilibrium waiting time tail or the ruin probability is of the form

$$\psi(x) = \sum_{n=1}^{\infty} (1 - \rho)\rho^n \bar{F}^{*n}(x), \qquad x \geq 0, \tag{4.1}$$

where $\rho \in (0,1)$, $H(x) = 1 - \bar{H}(x)$ is the service time or the claim size d.f. and $F(x) = \int_0^x \bar{H}(t)\,dt / \int_0^\infty \bar{H}(t)\,dt$, the equilibrium d.f. of $H(x)$. Suppose that $\kappa > 0$ satisfies $\rho^{-1} = \int_0^\infty e^{\kappa x}\,dF(x)$. If $H(x)$ is DVRL with

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$$r_H(\infty) = \lim_{x\to\infty} \frac{\int_x^\infty \bar{H}(t)\,dt}{\bar{H}(x)} \in (0,\infty),$$

then $F$ is DMRL and, hence, UBA by Theorem 2.2. Thus, using the definition of UBA and similar arguments as those in Lin [12], it can be shown that

$$\rho e^{-\kappa x} \le \psi(x) \le \{1 - \kappa r_H(\infty)\}e^{-\kappa x}, \qquad x \ge 0, \tag{4.2}$$

whereas if $H$ is IVRL, then

$$\{1 - \kappa r_H(\infty)\}e^{-\kappa x} \le \psi(x) \le \rho e^{-\kappa x}, \qquad x \ge 0. \tag{4.3}$$

A transform of a life distribution $A$ arising often in both queueing and ruin has the form

$$\bar{F}(x) = \frac{\int_0^\infty e^{-\alpha t}\bar{A}(x+t)\,dt}{\int_0^\infty e^{-\alpha t}\bar{A}(t)\,dt}, \qquad x \ge 0, \tag{4.4}$$

where $-\infty < \alpha < \infty$, $\bar{A}(x) = 1 - A(x)$, and $F(x) = 1 - \bar{F}(x)$ is itself a d.f.

For example, if $\alpha > 0$, then the equilibrium waiting time tail in the $E_2/G/1$ queue is of the form (4.1) with $F(x)$ satisfying (4.4) (e.g., Willmot [18]). If $\alpha = -\kappa$ and $A(x)$ is the equilibrium d.f. of $H(x)$ in (4.1), then $F(x)$ is the limiting d.f. of the deficit at ruin given that ruin occurs (e.g., Willmot and Lin [20, p. 101]). If $A(x) = H(x)$ and $\alpha > 0$, the d.f. $F(x)$ satisfying (4.4) arises in connection with a discounted ruin model. See Lin and Willmot [13] and references therein.

Indeed, the transform $F$ can be viewed as a generalization of equilibrium distributions in that if $\alpha = 0$, $\bar{F}(x)$ is the survival function of the equilibrium distribution of $A$.

It follows from integration of both sides of (4.4) that

$$\int_x^\infty \bar{F}(y)\,dy = \frac{\int_0^\infty e^{-\alpha t}\int_x^\infty \bar{A}(y+t)\,dy\,dt}{\int_0^\infty e^{-\alpha t}\bar{A}(t)\,dt}$$

$$= \frac{\mu_A \int_0^\infty e^{-\alpha t}\bar{A}_1(x+t)\,dt}{\int_0^\infty e^{-\alpha t}\bar{A}(t)\,dt},$$

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where $\mu_A = \int_0^\infty t\,dA(t)$ and $A_1(x) = \int_0^x \bar{A}(t)\,dt/\mu_A = 1 - \bar{A}_1(x)$. Thus, with $x = 0$, the mean results, namely

$$\int_0^\infty \bar{F}(y)\,dy = \frac{\mu_A \int_0^\infty e^{-\alpha t}\bar{A}_1(t)\,dt}{\int_0^\infty e^{-\alpha t}\bar{A}(t)\,dt}, \tag{4.5}$$

and the equilibrium d.f. $F_1(x) = 1 - \bar{F}_1(x)$ of $F$ thus satisfies

$$\bar{F}_1(x) = \frac{\int_x^\infty \bar{F}(y)\,dy}{\int_0^\infty \bar{F}(y)\,dy} = \frac{\int_0^\infty e^{-\alpha t}\bar{A}_1(x+t)\,dt}{\int_0^\infty e^{-\alpha t}\bar{A}_1(t)\,dt}, \tag{4.6}$$

which is of the same form as (4.4). It follows from (4.4)–(4.6) that the mean residual lifetime of $F$ satisfies

$$\begin{aligned} r_F(x) &= \frac{\bar{F}_1(x)}{\bar{F}(x)} \int_0^\infty \bar{F}(y)\,dy \\ &= \mu_A \frac{\int_0^\infty e^{-\alpha t}\bar{A}_1(t)\,dt}{\int_0^\infty e^{-\alpha t}\bar{A}(t)\,dt} \frac{\left\{e^{\alpha x}\int_x^\infty e^{-\alpha t}\bar{A}_1(t)\,dt\right\}\left(\int_0^\infty e^{-\alpha t}\bar{A}_1(t)\,dt\right)^{-1}}{\left\{e^{\alpha x}\int_x^\infty e^{-\alpha t}\bar{A}(t)\,dt\right\}\left(\int_0^\infty e^{-\alpha t}\bar{A}(t)\,dt\right)^{-1}} \\ &= \mu_A \frac{\int_x^\infty e^{-\alpha t}\bar{A}_1(t)\,dt}{\int_x^\infty e^{-\alpha t}\bar{A}(t)\,dt}. \end{aligned}$$

Define $r_A(x) = \mu_A \bar{A}_1(x)/\bar{A}(x)$ to be the mean residual lifetime of $A(x)$, and it follows that

$$r_F(x) = \frac{\int_x^\infty e^{-\alpha t}\bar{A}(t) r_A(t)\,dt}{\int_x^\infty e^{-\alpha t}\bar{A}(t)\,dt}, \qquad x \geq 0. \tag{4.7}$$

L'Hôpital's rule yields immediately that

$$\lim_{x\to\infty} r_F(x) = \lim_{x\to\infty} \frac{e^{-\alpha x}\bar{A}(x) r_A(x)}{e^{-\alpha x}\bar{A}(x)} = \lim_{x\to\infty} r_A(x);$$

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that is,

$$r_F(\infty) = r_A(\infty). \tag{4.8}$$

Thus, preservation of the UBA (UWA) and UBAE (UWAE) properties under transform (4.4) may now be established.

THEOREM 4.1: *Suppose that F satisfies* (4.4).

(a) *If A is* UBA (UWA), *then F is* UBA (UWA).
(b) *If A is* UBAE (UWAE), *then F is* UBAE (UWAE).

PROOF: To prove (a), note that (4.4) implies that

$$\bar{F}(x+y) = \frac{\int_0^\infty e^{-\alpha t}\bar{A}(x+y+t)\,dt}{\int_0^\infty e^{-\alpha t}\bar{A}(t)\,dt}$$

$$\geq (\leq)\ \frac{\int_0^\infty e^{-\alpha t}\bar{A}(x+t)e^{-y/r_A(\infty)}\,dt}{\int_0^\infty e^{-\alpha t}\bar{A}(t)\,dt} = \bar{F}(x)e^{-y/r_F(\infty)}$$

using (4.8) also. To prove (b), note that (4.7) and (4.8) imply that

$$r_F(x) \geq (\leq)\ \frac{\int_x^\infty e^{-\alpha t}\bar{A}(t)r_A(\infty)\,dt}{\int_x^\infty e^{-\alpha t}\bar{A}(t)\,dt} = r_A(\infty) = r_F(\infty);$$

that is, $F$ is UBAE (UWAE). ■

Preservation of UBA and UBAE under reliability operations have been considered; see, for example, Alzaid [1] for details.

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