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On Hilbert dynamical systems

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To the memory of our dear friend, Dan Rudolph, from whom we learned so much

Abstract. Returning to a classical question in harmonic analysis, we strengthen an old result of Walter Rudin. We show that there exists a weakly almost periodic function on the group of integers $\mathbb{Z}$ which is not in the norm-closure of the algebra $B(\mathbb{Z})$ of Fourier–Stieltjes transforms of measures on the dual group $\hat{\mathbb{Z}} = \mathbb{T}$, and which is *recurrent*. We also show that there is a Polish monothetic group which is reflexively but not Hilbert representable.

0. *Introduction*

Walter Rudin [**23**] was the first to show that the algebra $WAP(\mathbb{Z})$ of weakly almost periodic (WAP) functions on $\mathbb{Z}$ is strictly larger than the algebra $H(\mathbb{Z}) = \overline{B(\mathbb{Z})}$, the norm closure in the Banach space $\ell_\infty(\mathbb{Z})$ of the algebra $B(\mathbb{Z})$ of Fourier–Stieltjes transforms of complex measures on the dual group $\hat{\mathbb{Z}} = \mathbb{T}$. Many other examples of functions in $WAP(\mathbb{Z})\setminus H(\mathbb{Z})$ followed (see e.g. [**5**]). However, in all of these examples the function in question is trivial from the perspective of dynamical systems in that it is not *recurrent*.

Recall that a function $f \in \ell_\infty(\mathbb{Z})$ is *recurrent* if there is a sequence $n_k \nearrow \infty$ with $\lim_{k\to\infty} \sup_{j\in\mathbb{Z}} |f(j) - f(j + n_k)| = 0$. The more interesting functions in $WAP(\mathbb{Z})$ are the recurrent ones, and, moreover, they are persistent in the sense that every shift-invariant subalgebra of $WAP(\mathbb{Z})$ which is $\mathbb{Z}$-generated by a recurrent function consists entirely of recurrent functions (see Lemma 6.1 below). From the perspective of harmonic analysis, a Fourier–Stieltjes transform $\hat{\mu}$ of a continuous measure μ on $\mathbb{T}$ is recurrent if and only if μ is Dirichlet [**16**]. These functions are in $H(\mathbb{Z})$ and it was natural to ask if there can be recurrent functions in $WAP(\mathbb{Z})$ which do not belong to $H(\mathbb{Z})$. This was the initial question that motivated our work.

It turns out that another open problem concerning Polish monothetic groups is closely related to this question. The problem is the following: is there a Polish monothetic group P which can be represented as a group of isometries of a reflexive Banach space but cannot be represented as a group of unitary operators on a Hilbert space? In this work, we will

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see how these questions are related and show that the answer to both of them is affirmative (Theorem 4.2 and Corollary 5.2). In particular, there do exist many recurrent functions that are in $WAP(\mathbb{Z})$ but not in $H(\mathbb{Z})$.

We also have a result concerning Hilbert-representable groups. It states that if a Polish monothetic group P is Hilbert-representable and $K \subset P$ is a compact subgroup, then the quotient group P/K is also Hilbert-representable (Corollary 5.3).

Our proofs are based on the theory of topological dynamics and rely on a well-known construction of Banaszczyk [**2**]. We also use an idea of Megrelishvili [**18**], who showed that the topological groups $L_{2n}([0, 1])$, for n a positive integer, are reflexively but not Hilbert representable. For the sake of simplicity, we state our results in the most basic setup, where the acting group is $\mathbb{Z}$ and the dynamical systems are usually assumed to be point-transitive.

We refer the reader to the following related recent works: Gao and Pestov [**10**], Megrelishvili [**20**], Ferri and Galindo [**8**], and Galindo [**9**].

1. *Some preliminaries from topological dynamics*

A *dynamical system* (or sometimes just a *system*) is for us a pair (X, T), where X is a compact Hausdorff space and $T : X \to X$ is a self-homeomorphism. With (X, T) we associate an action of the group of integers $\mathbb{Z}$ via the map $n \mapsto T^n$. The *orbit* of a point $x \in X$ is the set $\mathcal{O}_T(x) = \{T^n x : n \in \mathbb{Z}\}$. The *orbit closure* of x is the set $\bar{\mathcal{O}}_T(x) = \overline{\mathcal{O}_T(x)}$. The system (X, T) is *point-transitive* if there is a point $x \in X$ with $\bar{\mathcal{O}}_T(x) = X$. Such a point is called a *transitive point* and the collection of transitive points (when it is not empty) is denoted by X_{tr}. For a point-transitive metric system, X_{tr} is a dense G_δ subset of X. We will mostly work in the category of *pointed* dynamical systems (X, x_0, T), where the latter is a point-transitive dynamical system with a distinguished point $x_0 \in X_{\mathrm{tr}}$. The restriction of T to a closed invariant subset $Y \subset X$ in a dynamical system (X, T) defines a dynamical system (Y, T). Such a system is called a *subsystem* of (X, T).

A continuous surjective map $\pi : (X, T) \to (Y, S)$ between two dynamical systems (X, T) and (Y, S) which intertwines the $\mathbb{Z}$-actions (i.e. $\pi(Tx) = S\pi(x)$ for every $x \in X$) is called a *homomorphism* of dynamical systems and we sometimes say that Y is a *factor* of X or that X is an *extension* of Y. When dealing with pointed systems, a homomorphism $\pi : (X, x_0, T) \to (Y, y_0, S)$ is further assumed to satisfy $\pi(x_0) = y_0$. An extension $\pi : (X, T) \to (Y, S)$ is called *almost one-to-one* if there is a dense G_δ subset $X_0 \subset X$ with $\pi^{-1}(\pi(x)) = \{x\}$ for every $x \in X_0$. The extension $\pi : (X, x_0, T) \to (Y, y_0, S)$ is called a *group extension* if there is a compact subgroup $K \subset \mathrm{Homeo}(X)$ such that each $k \in K$ is an automorphism of (X, T) (i.e. $Tk = kT$ for every $k \in K$) and such that the quotient dynamical system $(X/K, T)$ is isomorphic to (Y, S).

A point x in a metric dynamical system (X, T) is called *recurrent* if there is a sequence $\{n_k\} \subset \mathbb{Z}$ with $|n_k| \to \infty$ such that $\lim T^{n_k} x = x$. Note that in a point-transitive system, if the set of isolated points is not empty, then it coincides with the orbit of a transitive point. On the other hand, when there are no isolated points in a point transitive system, then every point of X_{tr} is recurrent. We call a point-transitive system with no isolated points a *recurrent-transitive system*.

Let $\ell_\infty(\mathbb{Z})$ be the Banach space (and C^*-algebra) of bounded complex-valued functions on $\mathbb{Z}$ with the sup norm $\|f\|_\infty = \sup_{n\in\mathbb{Z}} |f(n)|$. We write $S : \ell_\infty(\mathbb{Z}) \to \ell_\infty(\mathbb{Z})$ for the

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shift operator, where $Sf(n) = f(n+1)$, $(n \in \mathbb{Z})$. A norm-closed, S-invariant, conjugacy-invariant subalgebra of $\ell_\infty(\mathbb{Z})$, containing the constant function $\mathbf{1}$, will be called a $\mathbb{Z}$-*algebra*.

Given a pointed dynamical system (X, x_0, T), we define a map $j_{x_0} : C(X) \to \ell_\infty(\mathbb{Z})$ by

$$j_{x_0} F(n) = F(T^n x_0), \qquad F \in C(X), n \in \mathbb{Z}.$$

It is easy to see that j_{x_0} is an isometry with $j_{x_0} \circ T = S \circ j_{x_0}$. We denote its image in $\ell_\infty(\mathbb{Z})$ by $\mathcal{A}(X, x_0)$. Clearly, $\mathcal{A}(X, x_0)$ is a $\mathbb{Z}$-algebra.

Conversely, given a $\mathbb{Z}$-algebra $\mathcal{A} \subset \ell_\infty(\mathbb{Z})$, we denote its Gelfand space (comprising the non-zero C^*-homomorphisms of $\mathcal{A}(X, x_0)$ into $\mathbb{C}$) by $X = |\mathcal{A}(X, x_0)|$. It is easy to see that the operator S induces a homeomorphism $T : X \to X$ and that the resulting dynamical system (X, T) is a point-transitive system. In fact, the point $x_0 = \mathrm{eva}_0 \in X$, which corresponds to the multiplicative complex-valued homomorphism defined on $\mathcal{A}$ by evaluation at 0, is a transitive point.

These operations are inverse to each other and we have

$$(|\mathcal{A}(X, x_0)|, \mathrm{eva}_0, S) \cong (X, x_0, T).$$

A $\mathbb{Z}$-algebra $\mathcal{A}$ is *cyclic* if there is a function $f \in \mathcal{A}$ such that $\mathcal{A} = \mathcal{A}_f$, where the latter is the smallest $\mathbb{Z}$-algebra that contains f.

Given $f \in \ell_\infty(\mathbb{Z})$, we can consider f as an element of the compact metrizable space $\Omega = [-\|f\|_\infty, \|f\|_\infty]^{\mathbb{Z}}$. Again denote by $S : \Omega \to \Omega$ the homeomorphism defined by $Sg(n) = g(n+1)$, $(g \in \ell_\infty, n \in \mathbb{Z})$. We let $X_f = \mathrm{cls}\{S^n f : n \in \mathbb{Z}\}$, where the closure is taken in Ω with respect to the product topology. It can be easily verified that $\mathcal{A}(X_f, f, S) = \mathcal{A}_f$. Note that as $\mathcal{A}_f$ is always separable, X_f is metrizable.

If $\pi : (X, x_0, T) \to (Y, y_0, S)$ is a homomorphism of pointed transitive systems then the diagram

$$\begin{array}{ccc} C(Y) & \xrightarrow{j_{y_0}} & \mathcal{A}(Y, y_0) \\ {\scriptstyle \pi^*}\downarrow & & \downarrow{\scriptstyle i} \\ C(X) & \xrightarrow[j_{x_0}]{} & \mathcal{A}(X, x_0) \end{array}$$

commutes. Here, $(\pi^* F)(x) = F(\pi x)$, for $F \in C(Y)$, $x \in X$, and i is the inclusion map. Conversely, if $B \subset C(X)$ is a closed conjugation-invariant, T-invariant subalgebra containing the constant functions, then the restriction map

$$\pi : (|\mathcal{A}(X, x_0)|, \mathrm{eva}_0, S) \to (|j_{x_0}(B)|, \mathrm{eva}_0, S)$$

is a pointed homomorphism of dynamical systems.

With every dynamical system (X, T), we associate its *enveloping semigroup* $E = E(X, T) \subset X^X$. This is the pointwise closure of the set $\{T^n : n \in \mathbb{Z}\}$, as a subset of X^X. $E(X, T)$ is a compact *right topological semigroup*, i.e. for each $p \in E(X, T)$, the right multiplication $q \mapsto qp$, $q \in E$ is continuous. The set $\{T^n : n \in \mathbb{Z}\}$ is contained in the topological center of E. (An element $p \in E$ is in the *topological center* of E if the corresponding left multiplication, $q \mapsto pq$, $q \in E$, is continuous.) In particular, for each

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$n \in \mathbb{Z}$, the map $p \mapsto T^n x$, $p \in E$ is a homeomorphism of E, so that, via multiplication by T, (E, T) is also a dynamical system.

A function $f \in \ell_\infty(\mathbb{Z})$ is an *almost periodic* (*weakly almost periodic*) function if its orbit$\{S^n f : n \in \mathbb{Z}\}$ is norm precompact (weakly precompact) in the Banach space $\ell_\infty(\mathbb{Z})$. We denote by $AP(\mathbb{Z})$ and $WAP(\mathbb{Z})$ the collections of almost periodic and weakly almost periodic functions in $\ell_\infty(\mathbb{Z})$, respectively. These are $\mathbb{Z}$-algebras with $AP(\mathbb{Z}) \subset WAP(\mathbb{Z})$. A point-transitive dynamical system (X, T) is called *almost periodic* if and only if $\mathcal{A}(X, x_0, T) \subset AP(\mathbb{Z})$ if and only if, for every F in the Banach space $C(X)$, the orbit $\{F \circ T^n : n \in \mathbb{Z}\}$ is norm precompact. The system (X, T) is *weakly almost periodic* if and only if $\mathcal{A}(X, x_0, T) \subset WAP(\mathbb{Z})$ if and only if, for every $F \in C(X)$, the orbit $\{F \circ T^n : n \in \mathbb{Z}\}$ is weakly precompact. It is well-known that a point-transitive system (X, T) is almost periodic if and only if it is equicontinuous and minimal. A theorem of Ellis and Nerurkar [**7**] based on a theorem of Grothendieck asserts that (X, T) is weakly almost periodic if and only if its *enveloping semigroup* $E(X, T)$ consists of continuous maps. Another characterization of WAP systems (which again goes back to Grothedieck) is that $E(X, T)$ is a commutative semigroup. Using this observation, it is also easy to deduce the following well-known *double-limit criterion* (stated here for a general topological group G, see, for example, [**24**]).

PROPOSITION 1.1. *For a topological group G, a bounded continuous function $f : G \to \mathbb{C}$ is WAP if and only if, whenever g_m, h_n are sequences in G such that the double limits*

$$a = \lim_{n\to\infty} \lim_{m\to\infty} f(g_m h_n) \quad \text{and} \quad b = \lim_{m\to\infty} \lim_{n\to\infty} f(g_m h_n)$$

exist, then $a = b$.

For WAP $\mathbb{Z}$-systems we have $E(X, T) \cong (X, T)$ (see, for example, [**4**] or [**11**]).

We summarize these results in the following theorem.

THEOREM 1.2. *Let (X, T) be a point-transitive dynamical system and let $E = E(X, T)$ be its enveloping semigroup. The following conditions are equivalent.*

(1) *The system (X, T) is WAP.*
(2) *E is a semitopological semigroup, i.e. both right and left multiplications are continuous.*
(3) *E is a commutative semigroup.*
(4) *E consists of continuous maps.*

A point x in a metric dynamical system (X, T) is an *equicontinuity point* if for every $\epsilon > 0$ there is a $\delta > 0$ such that $d(x, x') < \delta \Rightarrow d(T^n x, T^n x') < \epsilon$ for all $n \in \mathbb{Z}$. A dynamical system with a residual set of equicontinuity points is called an *almost equicontinuous* system. A recurrent-transitive metric almost equicontinuous system (X, T) is uniformly rigid and the the set of equicontinuity points in X coincides with X_{tr}. Moreover, the map $\Lambda_X \twoheadrightarrow X_{\text{tr}}$, $\lambda \mapsto \lambda x_0$ is a homeomorphism of the Polish group Λ_X onto the dense G_δ subset $X_{\text{tr}} \subset X$ [**15**].

By the results of Glasner and Weiss [**14**] and Akin *et al* [**1**], it follows that if (X, T) is a metric recurrent-transitive WAP dynamical system, then the system (X, T) is *uniformly rigid*, i.e. there is a sequence of positive integers $n_k \to \infty$ such that the sequence

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$\{T^{n_k} : k \in \mathbb{N}\}$ tends uniformly to the identity map on X (see [**12**]). In a uniformly rigid system, the uniform closure

$$\Lambda_X = \mathrm{cls}\{T^n : n \in \mathbb{Z}\} \subset \mathrm{Homeo}(X)$$

is a non-discrete Polish monothetic topological group. We then have $X_{\mathrm{tr}} = \Lambda_X x_0$.

When (X, T) is recurrent-transitive metric WAP, we know much more: the system (X, T) is *hereditarily almost equicontinuous*, i.e. every subsystem $Y \subset X$ is almost equicontinuous (see [**1, 13–15**]).

2. *Reflexive and Hilbert representability of groups and dynamical systems*

Definitions 2.1.

(1) We say that a Polish topological group P is *representable* on a reflexive Banach space V if there is a topological isomorphism of P into the group of isometries $\mathrm{Iso}(V)$ of V equipped with the strong operator topology. The group P is *reflexibly representable* if it is representable on some reflexive Banach space.

(2) The group P is *Hilbert-representable* if there is a topological isomorphism of P into the unitary group $\mathcal{U}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$ equipped with the strong operator topology (see Megrelishvili [**19, 20**]).

(3) As in Banaszczyk [**2**], we say that P is *exotic* if it does not admit any non-trivial continuous unitary representations. We say that P is *strongly exotic* if it does not admit any non-trivial weakly continuous representations in Hilbert spaces.

Definitions 2.2.

(1) A metric dynamical system (X, T) is called *reflexively representable* if there is a reflexive Banach space V, a linear isometry $U \in \mathrm{Iso}(V)$ and a weakly compact U-invariant subset Z of V such that the dynamical systems (X, T) and (Z, U) are isomorphic.

(2) A metric dynamical system (X, T) is called *Hilbert-representable* if there is a Hilbert space $\mathcal{H}$, a unitary operator $U \in \mathcal{U}(\mathcal{H})$ and a weakly compact U-invariant subset Z of $\mathcal{H}$ such that the dynamical systems (X, T) and (Z, U) are isomorphic.

(3) Let $B(\mathbb{Z})$ be the subalgebra of $\ell_\infty(\mathbb{Z})$ which consists of the Fourier–Steiltjes transforms of complex measures on the circle $\mathbb{T}$, i.e. $B(\mathbb{Z}) = \mathcal{F}(M(\mathbb{T}))$, where $\mathcal{F} : M(\mathbb{T}) \to \ell_\infty(\mathbb{Z})$ is the Fourier transform:

$$\mathcal{F}(\mu)(n) = \hat{\mu}(n) = \int_{\mathbb{T}} e^{\mathrm{int}}\, d\mu(t).$$

(4) Let $H(\mathbb{T}) = \overline{B(\mathbb{Z})}$ be the norm closure of $B(\mathbb{Z})$ in the Banach space $\ell_\infty(\mathbb{Z})$. Clearly, $H(\mathbb{Z})$ is a $\mathbb{Z}$-subalgebra of $\ell_\infty(\mathbb{Z})$.

(5) A point-transitive system (X, x_0, T) is called *Hilbert* if $\mathcal{A}(X, x_0) \subset H(\mathbb{Z})$. The elements of $H(\mathbb{Z})$ are called *Hilbert functions*.

Suppose (X, T) is reflexively representable. Thus we assume that there is a reflexive Banach space V, a linear isometry $U \in \mathrm{Iso}(V)$ and that $X \subset V$ is a weakly compact U-invariant subset, with $T = U \restriction X$. Let x_0 be a vector in $X \subset V$ and ϕ a vector in V^*. Set $F(x) = \langle x, \phi \rangle$ and $f(n) = F(T^n x_0) = \langle T^n x, \phi \rangle$ (such a function is called a *matrix*

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coefficient of the representation). Then it is easy to see how the weak compactness of the unit ball of V^* implies that the T-orbit of F in $C(X)$ is weakly precompact. Since $\mathcal{A}(X, x_0) \cong C(X)$, it follows that the S-orbit of f in $\ell_\infty(\mathbb{Z})$ is also weakly precompact, i.e. the function f is in $WAP(\mathbb{Z})$.

It turns out that the converse is also true. We have the following basic theorems of Shtern [**25**] and Megrelishvili [**19**] concerning reflexive representability. We formulate these results in the context of a general topological group G, where the C^*-algebra $LUC(G)$ of bounded, complex-valued, left uniformly continuos functions takes the place of $\ell_\infty(\mathbb{Z})$. Thus, for example, $WAP(G)$ is the G-subalgebra of $LUC(G)$ comprising functions whose G-orbit is weakly precompact.

THEOREM 2.3.

(1) *A topological group G can be faithfully represented on a reflexive Banach space V if and only if the $WAP(G)$ functions separate points and closed sets on G* [**19, 25**].
(2) Let G be a topological group; then, every $f \in WAP(G)$ is a matrix coefficient of a reflexive representation of G [**19**].
(3) A metrizable G-system (X, G) is WAP if and only if it is reflexively representable [**19**].

LEMMA 2.4. *If a point-transitive metric system (X, x_0, T) is Hilbert-representable, then there is a function $F \in C(X)$ such that the corresponding $f \in \mathcal{A}(X, x_0)$, i.e. the function $f \in \ell_\infty(\mathbb{Z})$ defined by $f(n) = F(T^n x_0)$, is positive definite and satisfies $\mathcal{A}(X, x_0) = \mathcal{A}_f$, where the latter is the $\mathbb{Z}$-subalgebra generated by f in $\ell_\infty(\mathbb{Z})$.*

Proof. By assumption, we can consider X as a weakly compact subset of a separable Hilbert space $\mathcal{H}$ and $T = U \restriction X$, where $U \in \mathcal{U}(\mathcal{H})$, the unitary group of $\mathcal{H}$. With no loss in generality, we also assume that $\|x_0\| = 1$ and that $\mathcal{H} = Z_U(x_0)$, where the latter is the U-cyclic space generated by x_0. Set $F(x) = \langle x, x_0 \rangle$. Then $F \in C(X)$ and

$$f(n) = F(U^n x_0) = \langle U^n x_0, x_0 \rangle$$

is indeed positive definite.

If $x \neq y$ are points in X, then, as $\mathcal{H} = Z_U(x_0)$, there is some $n \in \mathbb{Z}$ with

$$F(U^{-n}x) = \langle x, U^{-n}x_0 \rangle \neq \langle y, U^{-n}x_0 \rangle = F(U^{-n}y),$$

so that the sequence of functions $\{F \circ U^n\}_{n \in \mathbb{Z}}$ separates points on X, whence $\mathcal{A}(X, x_0) = \mathcal{A}_f$. □

LEMMA 2.5. *Suppose $f \in \ell_\infty(\mathbb{Z})$ is positive definite.*

(1) *The system (X_f, f, S) is Hilbert-representable, i.e. there exists a Hilbert space $\mathcal{H}$, a unit vector $x_0 \in \mathcal{H}$ and a unitary operator $U \in \mathcal{U}(\mathcal{H})$ such that $\mathcal{H} = Z_U(x_0)$, $X = w\text{-cls}\{U^n x_0 : n \in \mathbb{Z}\}$ and $(X_f, f, S) \cong (X, x_0, U)$.*
(2) *Every element g of $X_f = \bar{\mathcal{O}}(f) \subset [-\|f\|, \|f\|]^{\mathbb{Z}}$ has the form $g(j) = \langle U^j x_0, x \rangle$ for some $x \in X$. In particular, $g \in B(\mathbb{Z})$.*
(3) *For $g \in X_f = \bar{\mathcal{O}}(f)$, we have $g \in \mathcal{O}(f)$ if and only if there is $n \in \mathbb{Z}$ with $g(n) = \|g\| = \|f\|$, in which case $g = S^n f$. In particular, for $g \in X_f$, we have $g(0) = \|g\| = \|f\| \Rightarrow g = f$.*
(4) $(X_f)_{\mathrm{tr}} = \{g \in X_f : \|g\| = \|f\|\}$.

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Proof. (1) This is a consequence of Herglotz' theorem.

(2) Let $g \in X_f$. Then there exists a sequence $n_k \to \infty$ with

$$g(j) = \lim_{k\to\infty} f(j+n_k) = \lim_{k\to\infty} S^{n_k} f(j) = \lim_{k\to\infty} \langle U^j x_0, U^{n_k} x_0\rangle = \langle U^j x_0, x\rangle,$$

where we assume, with no loss in generality, that $x = w\text{-}\lim_{k\to\infty} U^{n_k} x_0$ exists. It follows that $g \in B(\mathbb{Z})$.

(3) Suppose $g \in X_f$ satisfies $g(0) = \|g\| = \|f\|$. As we have seen in part (1), there is an $x \in X$ with $g(j) = \langle U^j x_0, x\rangle$ for every $j \in \mathbb{Z}$, and our assumption reads

$$1 = g(0) = \langle x, x_0\rangle.$$

We conclude that $x = x_0$, hence

$$g(j) = \langle U^j x_0, x_0\rangle = f(j)$$

for all $j \in \mathbb{Z}$, i.e. $g = f$.

(4) Let $g \in (X_f)_{\mathrm{tr}}$. There is a sequence n_k with $\lim_{k\to\infty} S^{n_k} g(j) = \lim_{k\to\infty} g(j+n_k) = f(j)$ for every $j \in \mathbb{Z}$. In particular, $\lim_{k\to\infty} g(n_k) = f(0) = \|f\|$. Thus $\|f\| \le \|g\| \le \|f\|$, hence $\|f\| = \|g\|$. Conversely, assuming $\|f\| = \|g\|$, we have $\lim_{k\to\infty} g(n_k) = f(0) = \|f\|$. With no loss of generality, we can assume that $h = \lim_{k\to\infty} S^{n_k} g$ exists in X_f, so that $h(0) = \|f\|$. By part (3), $h = f$ and we conclude that $g \in (X_f)_{\mathrm{tr}}$. □

PROPOSITION 2.6. *A point-transitive system (X, x_0, T) is Hilbert-representable if and only if there is a positive definite function $f \in \ell_\infty(\mathbb{Z})$ such that $(X, x_0, T) \cong (X_f, f, S)$, where X_f is the orbit closure of f in $\ell_\infty(\mathbb{Z})$ under the shift S with respect to the pointwise-convergence topology. In particular, every Hilbert-representable system is Hilbert.*

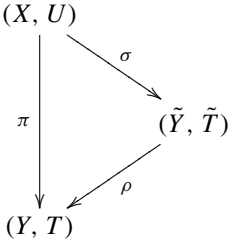
Proof. Combine Lemmas 2.4 and 2.5. □

PROPOSITION 2.7. *Every Hilbert system is WAP, i.e. $H(\mathbb{Z}) \subset WAP(\mathbb{Z})$.*

Proof. It is well-known (and easy to see) that $B(\mathbb{Z}) \subset WAP(\mathbb{Z})$. Since the algebra $WAP(\mathbb{Z}) \subset \ell_\infty(\mathbb{Z})$ is closed, we also have $H(\mathbb{Z}) \subset WAP(\mathbb{Z})$. □

3. *A structure theorem for Hilbert systems*

THEOREM 3.1. *Every metrizable recurrent-transitive Hilbert system (Y, T) admits an almost one-to-one extension $(\tilde{Y}, \tilde{T})$ which is a compact group-factor of a recurrent-transitive Hilbert-representable system (X, U). Thus there exists a commutative diagram*



(3.1)

with σ a group extension and ρ an almost one-to-one extension.

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Proof. (1) Let (Y, T) be a metrizable recurrent-transitive Hilbert system. Thus, picking a transitive point $y_0 \in Y$, we have $\mathcal{A}(Y, y_0) \subset H(\mathbb{Z})$. Since every function in $B(\mathbb{Z})$ is a linear combination of four positive definite functions, we can find a sequence f_n of positive definite functions (with $f_1 \equiv \mathbf{1}$ and $f_n(0) = \|f_n\|_\infty = 1$ $(n = 1, 2, \ldots)$) such that $\mathcal{A}(\{f_n : n \in \mathbb{N}\})$, the closed translation and conjugation-invariant algebra generated by the functions f_n, contains $\mathcal{A}(Y, y_0)$. For each n, there is a separable Hilbert space $\mathcal{H}_n$, a unit vector $x_n \in \mathcal{H}_n$ and a unitary operator $U_n \in \mathcal{U}(\mathcal{H}_n)$ such that $f_n(k) = \langle U_n^k x_n, x_n\rangle$(for all $k \in \mathbb{Z}$). Let $\mathcal{H} = \bigoplus_{n=1}^\infty \mathcal{H}_n$ (an ℓ_2-sum), $x_0 = \bigoplus_{n=1}^\infty x_n \in \mathcal{H}$ and $U = \bigoplus_{n=1}^\infty U_n \in \mathcal{U}(\mathcal{H})$. Set $X = \text{weak-cls}\{U^k x_0 : k \in \mathbb{Z}\}$. Then it is easy to verify that the Hilbert-representable system (X, U) admits (Y, T) as a factor, say $\pi : (X, U) \to (Y, T)$.

(2) Let $\mathcal{X}$ be the collection of all subsystems (X', U) of (X, U) with $\pi(X') = Y$. By Zorn's lemma, there is a minimal element (Z, U) in $\mathcal{X}$. Clearly, (Z, U) is a point-transitive Hilbert-representable system and, for convenience, we now rename it as (X, U). Also, let us rename $\mathcal{H} = Z_U(x_0)$, the cyclic space generated by $\{U^n x_0 : n \in \mathbb{Z}\}$, where x_0 is now some transitive point in X.

(3) Let $X_{\text{tr}} \subset X$ be the dense G_δ subset of the transitive points in X. Similarly, let $Y_{\text{tr}} \subset X$ be the dense G_δ subset of the transitive points in Y. Clearly, $\pi(X_{\text{tr}}) \subset Y_{\text{tr}}$, and we claim that these sets are equal. Suppose $y_1 \in Y_{\text{tr}}$. Let $x_1 \in X$ be some point with $\pi(x_1) = y_1$. By assumption, y_1 is a transitive point and there is a sequence $T^{n_i} y_1 \to y_0$. We can also assume that $U^{n_i} x_1 \to x_0'$ for some point $x_0' \in X$ with $\pi(x_0') = y_0$. It follows that $\pi(\bar{\mathcal{O}}_U(x_0')) = Y$ and, therefore, by minimality, $\bar{\mathcal{O}}_U(x_0') = X$. Thus x_0' is a transitive point and hence so is x_1, hence $y_1 = \pi(x_1) \notin X_{\text{tr}}$. We conclude that indeed $\pi(X_{\text{tr}}) = Y_{\text{tr}}$.

(4) Next recall that with every recurrent-transitive metrizable WAP system (W, R) there is an associated non-discrete Polish monothetic group $\Lambda_W = \text{unif-cls}\{R^n : n \in \mathbb{Z}\} \subset \text{Homeo}(W)$. Moreover, for every transitive point $w_0 \in W$, the map $g \mapsto gw_0$ is a homeomorphism from Λ_W onto the set W_{tr} of transitive points of W (see [**15**]). Now, using these observations, and with the notations as in the previous step, we note that the surjection $\pi : X_{\text{tr}} \to Y_{\text{tr}}$ defines a surjective Polish group homomorphism $\rho : \Lambda_X \to \Lambda_Y$. Moreover, we note that $\ker \rho = K$ is a compact subgroup of Λ_X.

(5) Set $\tilde{Y} = X/K$ and let $\sigma : (X, U) \to (\tilde{Y}, \tilde{T})$ denote the corresponding group homomorphism. We have now obtained diagram (3.1). □

4. *A Polish monothetic group P which is reflexive but not Hilbert-representable*

We will use Banaszczyk's construction of some families of Polish monothetic strongly exotic groups [**2**]. For our purposes, it suffices to consider one particular such example, which we now proceed to describe. Let $E = \ell_4(\mathbb{N})$ be the Banach space comprising the real-valued sequences $u = \{x_j\}_{j=1}^\infty$ with $|u| := (\sum_{j=1}^\infty x_j^4)^{1/4} < \infty$.

Let $\{e_n\}_{n=1}^\infty$ be the standard basis of unit vectors and choose a dense sequence $\{a_n\}_{n=1}^\infty \subset E$ with the property $a_n \in \text{span}\{e_j\}_{j<n}$ for every $n \geq 1$. Let Γ be the subgroup of E generated by the sequence $\{e_n + a_n\}_{n=1}^\infty$. It can be easily checked that $|\gamma| \geq 1$ for every $0 \neq \gamma \in \Gamma$, and that, on the other hand, $E = \Gamma + 2B_1(0)$, where $B_1(0)$ denotes the unit ball in E. Set $P = E/\Gamma$ and equip P with its quotient topology. It follows that P is a Polish topological group. By [**2**, Theorem 5.1, Remark 5.2], P is strongly exotic and monothetic.

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In Remark 5.2, Banaszczyk leaves the proof of the fact that P is monothetic as an exercise; for completeness we provide a proof of this fact at the end of this section, Proposition 4.3.

THEOREM 4.1. *The Polish monothetic group $P = \ell_4(\mathbb{N})/\Gamma$ is reflexively representable.*

Proof. We will follow the ideas of Megrelishvili's proof that the group $L_4([0, 1])$ is reflexively representable [**18**, Lemmas 3.3 and 3.4]. Note that Megrelishvili's proof works almost verbatim to show that $\ell_4(\mathbb{N})$ is reflexively representable.

We first define an invariant metric on P by

$$d(u + \Gamma, 0) = \|u + \Gamma\| = \inf_{\gamma\in\Gamma} |u + \gamma|, \quad u \in E.$$

Note that, for $u \in E$, we have

$$|u| \le \tfrac{1}{2} \Rightarrow \|u + \Gamma\| = |u|. \tag{4.1}$$

Next define the following continuous function on P:

$$F(u + \Gamma) = \min\{\|u + \Gamma\|, \tfrac{1}{10}\}.$$

We will show that F satisfies Grothendieck's *double-limit condition*, i.e. we have to show that, given two sequences $\{u_m + \Gamma\}_{m=1}^{\infty}$, $\{v_n + \Gamma\}_{n=1}^{\infty}$ in P, if the limits

$$a = \lim_m \lim_n F(u_m + v_n + \Gamma) \quad \text{and} \quad b = \lim_n \lim_m F(u_m + v_n + \Gamma) \tag{4.2}$$

exist, then necessarily $a = b$.

Of course, when $a = b = \frac{1}{10}$ we are done; so we now assume that $a < \frac{1}{10}$. Then, for some $\delta > 0$, eventually

$$a_m := \lim_{n\to\infty} F(u_m + v_n + \Gamma) = \lim_{n\to\infty} \|u_m + v_n + \Gamma\| < \tfrac{1}{10} - 2\delta.$$

Thus, for some m_0, we have $a_m \le \frac{1}{10} - 2\delta$ for every $m \ge m_0$, and, for a fixed $m \ge m_0$, there is an $n(m)$ such that, for $n \ge n(m)$, we have $\|u_m + v_n + \Gamma\| \le \frac{1}{10} - \delta$.

Now, for $n \ge \max\{n(m), n(m_0)\}$, we have

$$\|u_m + v_n + \Gamma\| \le \tfrac{1}{10} - \delta \quad \text{and} \quad \|u_{m_0} + v_n + \Gamma\| \le \tfrac{1}{10} - \delta,$$

hence $\|u_m - u_{m_0} + \Gamma\| \le \frac{2}{10}$. Thus, by an appropriate choice of representatives, we can now assume that $\|u_m - u_{m_0} + \Gamma\| = |u_m - u_{m_0}| \le \frac{2}{10}$ for all $m \ge m_0$. By the same token, for $n \ge n(m_0)$, we can assume that also $\|u_{m_0} + v_n + \Gamma\| = |u_{m_0} + v_n| \le \frac{2}{10} - \delta$. This means that, for sufficiently large m and n, all the vectors u_m and $-v_n$ lie in a ball of radius $\frac{2}{10}$ around u_{m_0}. By the triangle inequality and (4.1),

$$\|u_m + v_n + \Gamma\| = |u_m + v_n| \le \tfrac{4}{10} - \delta$$

for all $m \ge m_0$ and $n \ge n(m_0)$.

Now, in E we have

$$|u_m + v_n|^4 = \sum_{i=1}^{\infty} u_m(i)^4 + 4u_m(i)^3 v_n(i) + 6u_m(i)^2 v_n^2(i) + 4u_m(i)v_n(i)^3 + v_n(i)^4.$$

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As all these vectors lie in a ball, we can assume, by passing to subsequences, that

$$\lim_{n\to\infty} v_n = v \quad \text{weakly in } \ell_4,$$
$$\lim_{n\to\infty} v_n^2 = v^2 \quad \text{weakly in } \ell_2,$$
$$\lim_{n\to\infty} v_n^3 = v^3 \quad \text{weakly in } \ell_{\frac{4}{3}},$$

whence

$$a_m = |u_m|^4 + 4\langle u_m^3, v\rangle + 6\langle u_m^2, v^2\rangle + 4\langle u_m, v^3\rangle + V^4,$$

where $V = \lim_{n\to\infty} |v_n|$.

Again, we can also assume that

$$\lim_{m\to\infty} u_m = u \quad \text{weakly in } \ell_4,$$
$$\lim_{m\to\infty} u_n^2 = u^2 \quad \text{weakly in } \ell_2,$$
$$\lim_{m\to\infty} u_n^3 = u^3 \quad \text{weakly in } \ell_{\frac{4}{3}}.$$

We now get

$$a = \lim_{m\to\infty} a_m = U^4 + 4\langle u^3, v\rangle + 6\langle u^2, v^2\rangle + 4\langle u, v^3\rangle + V^4,$$

where $U = \lim_{n\to\infty} |u_n|$.

Computing the double limit the other way, we similarly have

$$b = U^4 + 4\langle u^3, v\rangle + 6\langle u^2, v^2\rangle + 4\langle u, v^3\rangle + V^4,$$

whence $a = b$, as required.

We have thus shown that the function F satisfies the double-limit condition. Now observe that, by the same computations, for any fixed $v \in E$, the function $F_v(u) = F(u - v)$ also satisfies the double-limit property. By the double-limit criterion 1.1, we conclude that the family F_v, $v \in E$ is contained in $WAP(P)$. As, clearly, the collection $\{F_v\}_{v\in E}$ generates the topology on P, it follows that this family separates points and closed sets on P and we conclude from Theorem 2.3(1), as in [**18**], that the group P is indeed reflexively representable. □

THEOREM 4.2. *There is a Polish monothetic group P which admits a faithful representation on a reflexive Banach space but is not Hilbert-representable (in fact it is strongly exotic).*

Proof. Let $P = \ell_4(\mathbb{N})/\Gamma$ be the strongly exotic Polish monothetic group described by Banaszczyk [**2**]. By Theorem 4.1, P is reflexively representable and, being strongly exotic, it is not Hilbert-representable. □

PROPOSITION 4.3. *The group P is monothetic.*

Proof. Let $\pi_k : \ell_4(\mathbb{N}) \to \ell_4^k := E_k$ be the projection map onto the first k coordinates. Let $\Gamma_k = \pi_k(\Gamma)$ and $P_k = E_k/\Gamma_k$. Note that, for each $k \geq 1$, the group P_k is a k-dimensional torus, and, by Kronecker's theorem, the vectors $z = (z_1, \ldots, z_k) \in E_k$ such that $\{nz + \Gamma_k : n \in \mathbb{Z}\}$ is dense in P_k form a dense subset of E_k. We will construct, by induction, a

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Cauchy sequence $\{z^{(k)} : k \in \mathbb{N}\}$ in $\ell_4(\mathbb{N})$ whose limit $x = (x_1, x_2, \ldots) \in \ell_4(\mathbb{N})$ will have the property that $\{nx + \Gamma : n \in \mathbb{Z}\}$ is a dense subgroup of P.

Suppose we have already chosen $w^{(k)} = (z_1^{(k)}, z_2^{(k)}, \ldots, z_k^{(k)}) \in E_k$ with the property that, for some positive integer N_k,

$$\{nw^{(k)} + \Gamma_k : |n| \leq N_k\} \tag{4.3}$$

is $1/k$-dense in P_k. We set

$$z^{(k)} = (z_1^{(k)}, z_2^{(k)}, \ldots, z_k^{(k)}, 0, 0, \ldots) \in \ell_4(\mathbb{N}).$$

There exists an $\epsilon_k > 0$ such that the condition (4.3) (with $\pi_k(z)$ instead of $w^{(k)}$) is satisfied for every z in an ϵ_k-ball around $z^{(k)}$. As we have seen, there is then a vector

$$w^{(k+1)} = (z_1^{(k+1)}, z_2^{(k+1)}, \ldots, z_{k+1}^{(k+1)}) \in E_{k+1}$$

and a positive integer N_{k+1} such that the set $\{nw^{(k+1)} + \Gamma_{k+1} : |n| \leq N_{k+1}\}$ is $1/(k+1)$-dense in P_{k+1}, and such that, for

$$z^{(k+1)} = (z_1^{(k+1)}, z_2^{(k+1)}, \ldots, z_k^{(k+1)}, 0, 0, \ldots) \in \ell_4(\mathbb{N}),$$

we have $|z^{(k)} - z^{(k+1)}| < \min\{2^{-k}, \epsilon_k\}$. This completes the inductive construction and, clearly, the unique limit point $x = \lim_{k\to\infty} z^{(k)}$ is the required one. □

Remark 4.4. Consider an arbitrary jointly continuous action of the group P on a compact space X. Clearly, the system (X, T), where $T \in P$, is a topological generator, and admits a T-invariant probability measure, say μ. It follows that μ is P-invariant as well. Let $g \mapsto U_g$ be the corresponding Koopman representation of P on $L_2(X, \mu)$. As P is exotic, this unitary representation is a trivial representation. This of course means that μ is supported on the set of P-fixed points in X. What we have shown here is that the group P has the *fixed point on compacta* property, i.e. every jointly continuous action of P on a compact space admits a fixed point. This property is also called *extreme amenability* (see [**22**]).

5. *A WAP recurrent-transitive system which is not Hilbert*

THEOREM 5.1. *The Banaszczyk group $P = \ell_4/\Gamma$ admits no non-trivial Hilbert system. Thus any non-trivial WAP action of P provides an example of a recurrent-transitive WAP system which is not Hilbert.*

Proof. (1) Let $T \in P$ be a generator of a dense cyclic subgroup in P. Suppose that P admits a non-trivial point-transitive Hilbert system (Y, y_0, P), and consider the $\mathbb{Z}$-system (Y, y_0, T). Since P is exotic, it is in particular MAP (= minimally almost periodic, i.e. it admits no non-trivial unitary finite-dimensional representations) and it follows that y_0 is a non-periodic, recurrent point of (Y, T). Let (X, x_0, U), with x_0 a unit vector in a Hilbert space $\mathcal{H}$, $U \in \mathcal{U}(\mathcal{H})$ a unitary operator on $\mathcal{H}$ and $X = w\text{-cls}\{U^n x_0 : n \in \mathbb{Z}\}$, be the extension provided by step (5) of Theorem 3.1 (diagram (3.1) above).

Let $G_X = \text{cls}\{U^n : n \in \mathbb{Z}\}$, where the closure is taken in the strong operator topology on $\mathcal{U}(\mathcal{H})$. It is not hard to see that there is a canonical topological isomorphism between

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the groups Λ_X and G_X. There are also canonical topological isomorphisms $\Lambda_X/K \cong \Lambda_Y \cong \Lambda_{\tilde{Y}}$. Note that our assumption that (Y, T) extends to a P-system means that the map $n \mapsto T^n$ extends to a continuous homomorphism $P \to \Lambda_Y$ with a dense image (see step (4) of the proof of Theorem 3.1).

By Proposition 2.6, we have an isomorphism $(X, U) \cong (X_f, S)$, where f is the positive definite function $f(n) = \langle U^n x_0, x_0 \rangle$. Now let $(\Omega, \mathcal{F}, \mu, R)$ be the Gauss dynamical system which corresponds to f; thus there is a $\mathbb{Z}$-sequence $\{\xi_n\}_{n\in\mathbb{Z}}$ of real-valued random variables defined on Ω, with joint Gauss distribution, such that $\xi_n = \xi_0 \circ R^n$ with $\mathbb{E}(\xi_n \xi_m) = f(n-m)$, and such that the σ-algebra generated by the sequence $\{\xi_n\}_{n\in\mathbb{Z}}$ is $\mathcal{F}$. (For more details on Gauss dynamical systems, see, for example, [**3, 17**].)

Let $\mathcal{H}_0 \subset L_2(\Omega, \mu)$ be the closed subspace spanned by $\{\xi_n\}_{n\in\mathbb{Z}}$ (the first Wiener chaos). Then $\mathcal{H}_0$ is isomorphic, as a Hilbert space, to $\mathcal{H}$ (which is the U-cyclic space spanned by x_0). In particular, we can think of Λ_X as a subgroup of $\mathcal{U}_0(\mathcal{H}_0)$ where it commutes with U_R, the Koopman operator on $L_2(\Omega, \mu)$ which corresponds to R. It follows that Λ_X can be realized as a group of measure-preserving transformations (i.e. a subgroup of $\mathrm{Aut}(\Omega, \mu)$; see, for example, [**17**]). In particular, we can now view K as a compact subgroup of $\mathrm{Aut}(\Omega, \mu)$ and then define the compact group-factor map $\pi_K : (\Omega, \mathcal{F}, \mu, R) \to (\Omega/K, \mathcal{F}/K, \mu_K, R_K)$.

Finally, considering $L_2(\Omega/K, \mu_K)$, we see that the Polish group Λ_X/K is faithfully represented on this Hilbert space. Since we have a continuous homomorphism $P \to \Lambda_Y \cong \Lambda_X/K$ with a dense image, this contradicts the fact that P is exotic and our proof is complete. □

COROLLARY 5.2. *There exists a recurrent-transitive WAP function $f \in WAP(\mathbb{Z})\backslash H(\mathbb{Z})$.*

Proof. Let P be a Polish monothetic exotic Banaszczyk group. Let $T \in P$ be a generator of a dense cyclic subgroup in P. As we have seen in Theorem 4.1, the functions $\{F_v\}_{v\in E}$ are in $WAP(P)$, and they separate points and closed sets on P. Therefore, their restrictions $f_v = F_v|\mathbb{Z}$ to the dense subgroup $\{T^n : n \in \mathbb{Z}\} \subset P$ are in $WAP(\mathbb{Z})$. As elements of $WAP(P)$, the functions F_v are recurrent and hence so are the functions f_v, $v \in E$. Finally, by Theorem 5.1, each f_v is an element of $WAP(\mathbb{Z})\backslash H(\mathbb{Z})$. □

COROLLARY 5.3. *If a Polish monothetic group P is Hilbert-representable, and if $K \subset P$ is a compact subgroup, then the quotient group P/K is Hilbert-representable.*

Proof. Follow the last steps of the proof of Theorem 5.1 with P taking the place of Λ_X. □

6. *Recurrent functions in $H(\mathbb{Z})\backslash B(\mathbb{Z})$*

LEMMA 6.1. *For every recurrent $f \in WAP(\mathbb{Z})$, the system (X_f, S) is uniformly rigid and the $\mathbb{Z}$-algebra $\mathcal{A}_f$ consists entirely of recurrent functions.*

Proof. The dynamical system $X_f = \mathrm{cls}\{S^n f : n \in \mathbb{Z}\}$ (in the pointwise-convergence topology) is WAP and therefore uniformly rigid. It then follows that the non-discrete Polish monothetic group $\Lambda_{X_f} = \mathrm{cls}\{S^n : n \in \mathbb{Z}\}$ (in the uniform-convergence topology on $\mathrm{Homeo}(X_f)$) acts on X_f as a group of automorphisms. In turn, this implies that every function $g \in \mathcal{A}(X_f, f) = \mathcal{A}_f$ is a recurrent point $g = \lim_{k\to\infty} S^{n_k} g$, for every sequence $n_k \nearrow \infty$ with S^{n_k} tending to the identity in Λ_{X_f}. □

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LEMMA 6.2. *If $\mathcal{A}$ is an infinite-dimensional $\mathbb{Z}$-subalgebra of $H(\mathbb{Z}) = \overline{B(\mathbb{Z})}$, then $\mathcal{A}\backslash B(\mathbb{Z})$ is a (norm) dense Borel subset of $\mathcal{A}$.*

Proof. The Fourier transform $\mathcal{F} : M(\mathbb{T}) \twoheadrightarrow B(\mathbb{Z})$ is a one-to-one onto bounded linear map ($\|\mathcal{F}(\mu)\|_\infty \leq \|\mu\|$). If $\mathcal{A} \subset \mathcal{F}(M(\mathbb{T})) = B(\mathbb{Z})$, then $\mathcal{F}^{-1}\mathcal{A} \subset M(\mathbb{T})$ is an infinite-dimensional C^*-algebra, which is impossible (see e.g. [**5**, Theorem 2.1]). Thus $\mathcal{A} \subsetneq B(\mathbb{Z})$. Moreover, if the algebra $\mathcal{A} \cap B(\mathbb{Z})$ contains a non-empty $\mathcal{A}$-norm-open subset, then it must coincide with $\mathcal{A}$, which again is impossible. Thus $\mathcal{A}\backslash B(\mathbb{Z})$ is norm-dense in $\mathcal{A}$.

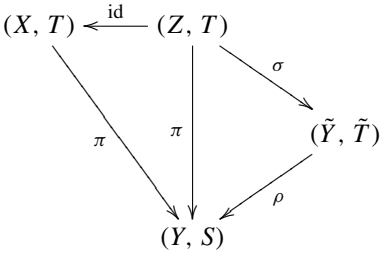
Finally, the algebra $B(\mathbb{Z})$, being a one-to-one continuous image of $M(\mathbb{T})$ (under $\mathcal{F}$), is a Borel subset of $H(\mathbb{Z}) = \overline{B(\mathbb{Z})}$, and hence so are the sets $\mathcal{A} \cap B(\mathbb{Z})$ and $\mathcal{A}\backslash B(\mathbb{Z})$. □

COROLLARY 6.3. *In every infinite-dimensional $\mathbb{Z}$-subalgebra $\mathcal{A} \subset H(\mathbb{Z})$, the set $\mathcal{A}_{\text{rec}}\backslash B(\mathbb{Z})$, where $\mathcal{A}_{\text{rec}}$ is the collection of recurrent functions in $\mathcal{A}$, is a (norm) dense subset of $\mathcal{A}_{\text{rec}}$. In particular, there are recurrent functions in $H(\mathbb{Z})$ which are not Fourier–Stieltjes transforms (i.e. are not in $B(\mathbb{Z})$).*

Proof. Combine Lemmas 6.1 and 6.2. □

7. *A structure theorem for weakly almost periodic systems*

THEOREM 7.1. *Let (X, T) be a metrizable recurrent-transitive WAP dynamical system and $\pi : (X, T) \to (Y, S)$ a factor. Then there is an almost one-to-one extension which is a compact group-factor of a recurrent-transitive subsystem $Z \subset X$. More explicitly, there is a commutative diagram*



with $Z \subset X$ a subsystem, σ a group extension, and ρ an almost one-to-one extension.

Proof. Just repeat steps (2) to (5) of the proof of Theorem 3.1 above. □

8. *Some open problems*

Problem 8.1. In Theorem 3.1, we have shown that every metrizable recurrent-transitive Hilbert system admits an almost one-to-one extension which is a group factor of a Hilbert-representable system. Can one get rid in this structure theorem of either one of these extensions, or maybe of both? Thus, our question is whether a metrizable recurrent-transitive Hilbert system is always (a) Hilbert-representable, or (b) a group factor of a Hilbert-representable system, or (c) an almost one-to-one factor of a Hilbert-representable system. Question (a) can be reformulated as follows: is a factor of a Hilbert-representable system also Hilbert-representable? This latter question is stated as [**21**, Problem 996].

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Problem 8.2. Is there a metrizable recurrent-transitive Hilbert system (X, T) with no non-trivial Hilbert-representable factors? In other words, is there a non-trivial $\mathbb{Z}$-algebra $\mathcal{A} \subset H(\mathbb{Z})$ with $\mathcal{A} \cap B(\mathbb{Z}) = \mathbb{C}$? Such a system will provide a counter-example to option (a) in Problem 8.1.

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References

[1] E. Akin, J. Auslander and K. Berg. When is a transitive map chaotic. *Convergence in Ergodic Theory and Probability*. Walter de Gruyter & Co., Berlin, New York, 1996, pp. 25–40.
[2] W. Banaszczyk. *Additive Subgroups of Topological Vector Spaces (Lecture Notes in Mathematics, 1466)*. Springer, Berlin, 1991.
[3] I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai. *Ergodic Theory*. Springer, New York, 1982.
[4] T. Downarowicz. Weakly almost periodic flows and hidden eigenvalues. *Topological Dynamics and Applications (Contemporary Mathematics, 215)*. American Mathematical Society, Providence, RI, 1998, pp. 101–120, a volume in honor of R. Ellis.
[5] C. F. Dunkl and D. E. Ramirez. *Topics in Harmonic Analysis*. Appleton-Century-Crofts, New York, 1971.
[6] R. Ellis. Distal transformation groups. *Pacific J. Math.* **8** (1957), 401–405.
[7] R. Ellis and M. Nerurkar. Weakly almost periodic flows. *Trans. Amer. Math. Soc.* **313** (1989), 103–119.
[8] S. Ferri and J. Galindo. Embedding a topological group into its WAP-compactification. *Studia Math.* **193** (2009), 99–108.
[9] J. Galindo. On unitary representability of topological groups. *Math. Z.* **263** (2009), 211–220.
[10] S. Gao and V. Pestov. On a universality property of some abelian Polish groups. *Fund. Math.* **179** (2003), 1–15.
[11] E. Glasner. *Ergodic Theory via Joinings (Mathematical Surveys and Monographs, 101)*. American Mathematical Society, Providence, RI, 2003.
[12] E. Glasner and D. Maon. Rigidity in topological dynamics. *Ergod. Th. & Dynam. Sys.* **9** (1989), 309–320.
[13] E. Glasner and M. Megrelishvili. Linear representations of hereditarily non-sensitive dynamical systems. *Colloq. Math.* **104**(2) (2006), 223–283.
[14] E. Glasner and B. Weiss. Sensitive dependence on initial conditions. *Nonlinearity* **6** (1993), 1067–1075.
[15] E. Glasner and B. Weiss. Locally equicontinuous dynamical systems. *Colloq. Math.* **84/85**(Part 2) (2000), 345–361.
[16] B. Host, J.-F. Méla and F. Parreau. Analyse harmonique des mesures. *Astérisque* **135–136** (1986), 261.
[17] M. Lemańczyk, F. Parreau and J.-P. Thouvenot. Gaussian automorphisms whose ergodic self-joinings are Gaussian. *Fund. Math.* **164** (2000), 253–293.
[18] M. Megrelishvili. Reflexively but not unitarily representable topological groups. *Topology Proc.* **25** (2000), 615–625, (2002).
[19] M. Megrelishvili. Fragmentability and representations of flows. *Topology Proc.* **27:2** (2003), 497–544.
[20] M. Megrelishvili. Reflexively but not Hilbert representable compact flows and semitopological semigroups. *Colloq. Math.* **110** (2008), 383–407.
[21] M. Megrelishvili. Topological transformation groups: selected topics. *Open Problems in Topology II*. Ed. E. Pearl. Elsevier, Amsterdam, 2007, pp. 421–435.
[22] V. Pestov. *Dynamics of Infinite Dimensional Groups. The Ramsey–Dvoretzky–Milman Phenomenon (University Lecture Series, 40)*. American Mathematical Society, Providence, RI, 2006.
[23] W. Rudin. Weak almost periodic functions and Fourier–Stieltjes transforms. *Duke Math. J.* **26** (1959), 215–220.
[24] W. Ruppert. *Compact Semitopological Semigroups (Lecture Notes in Mathematics, 1079)*. Springer, Berlin, 1984.
[25] A. Shtern. Compact semitopological semigroups and reflexive representability. *Russ. J. Math. Phys.* **2** (1994), 131–132.

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