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A storage model with continuous infinitely divisible inputs

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Abstract. This paper presents a general theory of storage for dams subject to a steady release, and with continuous inputs having infinitely divisible distributions. The paper extends some earlier work by Downton, Lindley, Smith and Kendall ((10), see discussion), and gives a detailed account of results previously sketched by the authors (6).

1. *Introduction.* Possibly the most interesting problem in the theory of storage, with no direct analogue in queueing theory, is that formulated by Moran (13) for the dam fed by a continuous input. At time $t \geqslant 0$, let a dam of infinite capacity have the continuous random content $Z(t) \geqslant 0$; the input $X(t) \geqslant 0$ which has entered it during time t is continuous, and follows the infinitely divisible gamma distribution, while the dam is subject to a steady release at constant unit rate except if it is empty, when the release ceases.

For $Z(t) > 0$, exactly as for similar processes with discrete inputs, the dam content may be represented by the equation

$$Z(t+\delta t) = Z(t) + \delta X(t) - (1-r)\,\delta t, \tag{1·1}$$

where $r\,\delta t$ $(0 \leqslant r \leqslant 1)$ is that part of the interval $(t, t+\delta t)$ during which the dam is empty. However, for continuous inputs $X(t)$, having infinitely divisible distributions such as Moran's gamma type

$$dK(x,t) = k(x,t)\,dx = \{\rho\Gamma(t)\}^{-1}(x/\rho)^{t-1}e^{-x/\rho}\,dx \tag{1·2}$$

the constructive definition (1·1) fails when $Z(t) = 0$, since the time intervals when the dam is empty are no longer well defined. In this case, Gani and Pyke (7) have shown that the definition

$$Z(t) = \max\{\sup_{0\leqslant\tau\leqslant t}(X(t) - X(\tau-) - t + \tau),\ X(t) - t + z_0\} \quad (t \geqslant 0), \tag{1·3}$$

where $Z(0) = z_0 > 0$ is the initial dam content, overcomes the difficulty and provides the limit of a corresponding dam process in discrete time with probability one at each t.

It is possible to obtain the stationary solution for the distribution of the continuous dam content by choosing an appropriate discrete analogue, which leads to a Markov

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chain equation previously considered by Kendall (9). From this, Moran, using an ingenious limiting method, derives the Laplace–Stieltjes transform

$$\Psi(\theta) = \frac{(1-\rho)\theta}{\theta - \log(1+\rho\theta)} \quad (\theta > 0, \rho < 1) \tag{1·4}$$

for the stationary distribution of the dam content $Z(t)$. The transform (1·4) has since been inverted by Daniels (see discussion in (10)) to obtain the stationary distribution function (d.f.) of $Z(t)$ in the form

$$\Phi(z) = \lim_{t\to\infty} \Pr\{Z(t) \leqslant z\} = 1 - (1-\rho)\int_0^\infty \frac{(z+u)^{u-1} e^{-(z+u)/\rho}}{\rho^u \Gamma(u)} du. \tag{1·5}$$

More recently Downton (5), by a somewhat different limiting method, has generalized Moran's results.

The use of limiting methods to derive certain continuous stationary dam solutions has been discussed by the authors (6), while Meisling (12) has considered similar limiting techniques in the theory of queues. A comparison of these with Takács's (15) approach to queueing theory leaves little doubt as to the advantages of the latter method; a direct time-dependent integro-differential equation for the transition d.f.

$$F(z_0, \tau; z, \tau+t) = \Pr\{Z(t+\tau) \leqslant z | Z(\tau) = z_0\} \tag{1·6}$$

of the dam content, if obtainable, would clearly yield results well beyond the scope of the stationary distribution theory.

It is the purpose of this paper to present such an equation, and to outline a general theory of storage for dams subject to a steady release, and fed by continuous inputs having infinitely divisible distributions. Using Laplace–Stieltjes transforms, this equation is solved and a general formula is obtained for $F(z_0, \tau; z, \tau+t)$ in terms of the input distribution, and the probability of emptiness $F(z_0, \tau; 0, \tau+t)$ of the dam. The probability distribution for the 'wet period' T of the dam has been obtained by Kendall (10); the relation of $F(z_0, \tau; 0, \tau+t)$ to it is now established. The first passage problem for $Z(t)$ is also examined, and a study finally made of the stationary behaviour of $Z(t)$.

We should remark here that since the completion of both our 1959 paper (6) and the present work, two papers by Beneš (2), and Reich (14), have appeared in 1960 and 1961 respectively, presenting results not unlike ours. The approach and methods of these papers are, however, quite different from those we shall use.

2. *Continuous inputs with infinitely divisible distributions.* Lévy (11), Doob (4), and Gnedenko and Kolmogorov (8) have described a class of stochastic processes $X(t)$ in continuous time t, with infinitely divisible d.f.'s $K(x, t)$ such that for some suitable θ their Laplace–Stieltjes transform is given by

$$\phi(\theta, t) = \int_{-\infty}^{\infty} e^{-\theta x} d_x K(x, t) = e^{-t\xi(\theta)}. \tag{2·1}$$

Among these, continuous non-negative processes with infinitely divisible distributions (sometimes referred to as of 'Poisson' type), and having finite means and variances per unit time are of special relevance to storage theory, where they are identifiable with continuous inputs into a dam.

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Let $X(t)$ $(0 \leqslant X(t) < \infty)$ be such a non-negative input process, for which, as in Lévy (11), the function $\xi(\theta)$ of (2·1) may be written as

$$\xi(\theta) = \int_0^\infty (1 - e^{-\theta u})\, \lambda(u)\, du \quad (\theta > 0), \tag{2·2}$$

where $\lambda(u) \geqslant 0$ is finite for all $u > 0$, but $\lambda(u) \to \infty$ as $u \to 0$. The structure of the transform indicates that in a small interval of time δt, a jump $\delta X(t)$ of Poisson type occurs in the process, such that if its magnitude lies in

$$u < \delta X(t) \leqslant u + \delta u \quad (u > 0),$$

the parameter corresponding to it is $\lambda(u)\, \delta u$. This is equivalent to writing the probability of such a jump in δt (cf. Lévy (11), p. 176) as

$$dK(u, \delta t) = \lambda(u)\, du\, \delta t, \tag{2·3}$$

where terms which are $o(\delta t)$ are neglected. The mean and variance of the process per unit time are respectively

$$\rho = \int_0^\infty u\lambda(u)\, du < \infty; \quad \sigma^2 = \int_0^\infty u^2 \lambda(u)\, du < \infty. \tag{2·4}$$

Consider the d.f. $K(x, t)$; this is in effect a simple stationary Markov transition d.f. (cf. Doob (4), ch. VI, § 2). Since $X(t)$ has a continuous distribution, $K(x, t)$ is assumed differentiable in x for all $t > 0$ so that

$$\frac{d}{dx} K(x, t) = k(x, t) \quad (0 \leqslant x < \infty;\ t > 0). \tag{2·5}$$

For $t = 0$, it follows from equation (2·1) that

$$K(x, 0) = 1 \quad (x \geqslant 0);$$

and we have further, as is usual for Markov transition d.f.'s, the condition

$$\lim_{t \to 0} K(x, t) = 1 \quad (x > 0). \tag{2·6}$$

This is an immediate consequence of the continuity theorem for characteristic functions. Again, for $t > 0$, we may write (Doob (4); *loc. cit.*)

$$\Pr\{X(t) = 0\} = K(0, t) = \exp\left\{-t \int_0^\infty \lambda(u)\, du\right\},$$

and since this is zero, $\lambda(u) \to \infty$ as $u \to 0$.

Lastly, we prove that if $\rho < 1$ then

$$\lim_{t \to \infty} K(t + z, t) = 1. \tag{2·7}$$

For consider the input variance $v(X(t))$; from (2·1) it is readily seen that

$$v(X(t)) = \sigma^2 t = -t\xi''(0) = \int_0^\infty (x - \rho t)^2\, dK(x, t) > 0; \tag{2·8}$$

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taking $z > -(1-\rho)t$ without loss of generality, we obtain

$$\sigma^2 t \geqslant \int_{t+z}^{\infty} (x-\rho t)^2 \, dK(x,t) \geqslant [z+(1-\rho)t]^2 \int_{t+z}^{\infty} dK(x,t).$$

Thus

$$0 \leqslant 1-K(t+z,t) \leqslant \frac{t\sigma^2}{[z+(1-\rho)t]^2} \tag{2·9}$$

and (2·7) follows directly as $t \to \infty$.

We may now proceed with the derivation of the integro-differential equations for the dam process.

3. *The integro-differential equation for the d.f. of $Z(t)$.* The Markov process (1·1), in which the input $X(t)$ is infinitely divisible with the d.f. $K(x,t)$, is readily seen to be time-homogeneous. Thus for simplicity, with $\tau = 0$ in (1·6), the notation $F(z,t)$ may be used for the d.f. of $Z(t)$, the initial dam content still being $Z(0) = z_0 \geqslant 0$. For $0 \leqslant t < \infty$, let the transition d.f.

$$F(z,t) = \Pr\{Z(t) \leqslant z | Z(0) = z_0\} \tag{3·1}$$

be a point function, non-decreasing and everywhere continuous to the right in z such that $0 \leqslant F(z,t) \leqslant 1$. We may assume that the d.f. $F(z,t)$ has the following properties:

(i) $F(z,t) = 0$ for all $z < \max\{0, z_0 - t\}$;

(ii) $\lim_{z\to\infty} F(z,t) = 1$;

(iii) $F(z,t)$ is differentiable with respect to the time t for all values of z.

To derive the integro-differential equation for $F(z,t)$ it is helpful to consider a suitable discrete model for the dam in which both the time and the input are measured in units of magnitude Δ; as $\Delta \to 0$ the storage function $Z(n\Delta)$ $(n = 0, 1, 2, \ldots)$ in this discrete model will tend with probability 1 to $Z(t)$, the continuous time storage function defined in (1·3), as was shown by Gani and Pyke (7). The difference equation for the d.f. of $Z(n\Delta)$ then tends to the required integro-differential equation for $F(z,t)$.

The discrete model used here is specified by inputs $X(\Delta)$ over the time intervals $(n\Delta, n\Delta+\Delta)$; these are independent and have the probability distribution

$$\left.\begin{aligned} &\Pr\{X(\Delta) = 0\} = e^{-\lambda\Delta}, \\ &\Pr\{X(\Delta) = i\Delta\} = (1-e^{-\lambda\Delta})\frac{\lambda_i}{\lambda} \quad (i \geqslant 1), \end{aligned}\right\} \tag{3·2}$$

where

$$\lambda_i \equiv \lambda_i(\Delta), \quad \lambda \equiv \lambda(\Delta) = \sum_1^{\infty} \lambda_i;$$

we assume that

$$0 < \rho(\Delta) \equiv E[X(\Delta)] < \infty \quad \text{and} \quad 0 < \sigma^2(\Delta) \equiv E[X(\Delta)]^2 - \rho^2(\Delta) < \infty.$$

The release from the dam is Δ at the end of each time interval, except when the dam is empty. We have then the relations

$$Z(n\Delta+\Delta) = \left\{\begin{aligned} &Z(n\Delta)+X(\Delta)-\Delta \quad \text{if} \quad Z(n\Delta)+X(\Delta) > \Delta, \\ &0 \quad \text{otherwise}, \end{aligned}\right\} \tag{3·3}$$

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for the storage at times $n\Delta$, $(n+1)\Delta$; from (3·3) we obtain the identity

$$Z(n\Delta) = \max\{X(m\Delta) - m\Delta \ (0 \leqslant m \leqslant n),\ Z(0) + X(n\Delta) - n\Delta\}, \tag{3·4}$$

and it is this which leads to the definition of $Z(t)$ in the continuous time case given by Gani and Pyke.

To study the limiting behaviour of this discrete model, we assume that as $\Delta \to 0$ and $i\Delta \to u$,

$$\frac{\lambda_i}{\Delta} \to \lambda(u), \quad \lambda(\Delta) \to \infty \quad \text{but} \quad \lambda\Delta \to 0, \tag{3·5}$$

and

$$\frac{\rho(\Delta)}{\Delta} \to \rho \quad (0 < \rho < \infty), \quad \frac{\sigma^2(\Delta)}{\Delta} \to \sigma^2 \quad (0 < \sigma^2 < \infty). \tag{3·6}$$

From these assumptions it follows that

$$\left.\begin{aligned} \Pr\{X(\Delta) = i\Delta\} &= \frac{1-e^{-\lambda\Delta}}{\lambda\Delta}\lambda_i\Delta \sim \lambda(u)\,\delta u\,\delta t, \\ \Pr\{X(n\Delta) = 0\} &= e^{-\lambda n\Delta} \to 0 \quad \text{as} \quad \Delta \to 0,\ n\Delta \to t. \end{aligned}\right\} \tag{3·7}$$

Further, for the Laplace–Stieltjes transform $\phi(\theta, \Delta)$ of $X(\Delta)$ we obtain that

$$\begin{aligned} \log\phi(\theta,\Delta) &= \log\left\{e^{-\lambda\Delta} + \left(\frac{1-e^{-\lambda\Delta}}{\lambda}\right)\Sigma\lambda_i e^{-i\theta\Delta}\right\} \\ &= \log\left\{1 - \left(\frac{1-e^{-\lambda\Delta}}{\lambda}\right)\Sigma\dot{\lambda}_i(1-e^{-i\theta\Delta})\right\} \\ &= -\Delta\xi(\theta,\Delta) + o(\Delta), \end{aligned} \tag{3·8}$$

where

$$\xi(\theta,\Delta) = \sum_1^\infty \lambda_i(1-e^{-i\theta\Delta}) \to \int_0^\infty \lambda(u)\,(1-e^{-\theta u})\,du = \xi(\theta) \quad \text{(say)}. \tag{3·9}$$

The transform of $X(n\Delta)$ is thus given by

$$\phi(\theta, n\Delta) = e^{-n\Delta\xi(\theta,\Delta)+o(\Delta)} \to e^{-t\xi(\theta)}. \tag{3·10}$$

From (3·7) and (3·10), we see that the input $X(n\Delta)$ in the discrete model tends to the continuous process $X(t)$, having exactly the infinitely divisible d.f., described in § 2. We also remark that the assumptions (3·5) and (3·6) imply (2·4), so that the ρ and σ^2 of this section are in fact the mean and variance per unit time of the continuous process.

Now consider the d.f. $F(j\Delta, n\Delta)$ of $Z(n\Delta)$; from (3·3) we see that for $j > 0$,

$$F(j\Delta-\Delta, n\Delta+\Delta) = F(j\Delta, n\Delta)\,e^{-\lambda\Delta} + \sum_{i=1}^\infty F(j\Delta - i\Delta, n\Delta)\,(1-e^{-\lambda\Delta})\frac{\lambda_i}{\lambda}. \tag{3·11}$$

This can be written as

$$\begin{aligned} &\{F(j\Delta-\Delta, n\Delta+\Delta) - F(j\Delta-\Delta, n\Delta)\} - \{F(j\Delta, n\Delta) - F(j\Delta-\Delta, n\Delta)\} \\ &\qquad = \Delta\left(\frac{1-e^{-\lambda\Delta}}{\lambda\Delta}\right)\sum_1^\infty \lambda_i\{F(j\Delta - i\Delta, n\Delta) - F(j\Delta, n\Delta)\}. \end{aligned} \tag{3·12}$$

Suppose that

$$\Delta \to 0, \quad j\Delta \to z, \quad i\Delta \to u, \quad n\Delta \to t, \tag{3·13}$$

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then $F(j\Delta, n\Delta) \to F(z,t)$, the d.f. of $Z(t)$. If we divide (3·12) by Δ, and take limits as $\Delta \to 0$, a heuristic argument suggests that $\partial F/\partial z$ exists and is finite for all $z > 0$, and finally leads to the integro-differential equation

$$\frac{\partial}{\partial t}F(z,t) - \frac{\partial}{\partial z}F(z,t) = \int_0^\infty \{F(z-u,t) - F(z,t)\}\lambda(u)\,du \quad (z > 0). \tag{3·14}$$

Several points in this limiting process need further justification for the argument to be fully rigorous: it is hoped to provide these in a later note.

At $j = 0$, we have the relation similar to (3·11)

$$F(0, n\Delta + \Delta) = F(\Delta, n\Delta)\,e^{-\lambda\Delta}, \tag{3·15}$$

which can be written as

$$F(0, n\Delta + \Delta) - F(0, n\Delta) - \{F(\Delta, n\Delta) - F(0, n\Delta)\} = (e^{-\lambda\Delta} - 1)\,F(\Delta, n\Delta). \tag{3·16}$$

Dividing by Δ and letting $\Delta \to 0$, this leads to

$$\frac{\partial}{\partial z}F(z,t) = \infty \quad \text{at} \quad z = 0, \tag{3·17}$$

since $\lambda(\Delta) \to \infty$. Thus the equation (3·14) may be regarded as being also satisfied at $z = 0$. Since $F(0,t)$ is continuous to the right at $z = 0$, it may be obtained as

$$F(0,t) = \lim_{z \to 0+} F(z,t). \tag{3·18}$$

To the properties (i)–(iii) of $F(z,t)$ we might therefore add the following:

(iv) $F(z,t)$ is continuous for all points $z > 0$ in the range $0 \leqslant z < \infty$, but has a discontinuity $F(0,t)$ at $z = 0$, this being the probability of emptiness of the dam.

(v) $F(z,t)$ has a unique derivative at all points z in the range except for $z = 0$, where the right derivative $(\partial/\partial z)\,F(0,t) = \infty$. Subject to the suitable interpretation of $(\partial/\partial z)\,F(z,t)$ at $z = +0$, however, the integro-differential equation (3·14) for $F(z,t)$ may be regarded as valid for all values of $0 \leqslant z < \infty$.

4. *Transforms and general solutions for the dam process.* Let the Laplace–Stieltjes transform with respect to z of $F(z,t)$ be given by

$$\psi(\theta,t) = \int_{0-}^\infty e^{-\theta z}\,dF(z,t) \quad (\theta > 0), \tag{4·1}$$

while the transform with respect to time t of $\psi(\theta,t)$ is

$$\psi^*(\theta,s) = \int_0^\infty e^{-st}\,\psi(\theta,t)\,dt \quad (s > 0). \tag{4·2}$$

Taking transforms with respect to z of both sides of the equation (3·14), we obtain

$$\int_{0+}^\infty e^{-\theta z}\frac{\partial F}{\partial t}\,dz - \{\psi(\theta,t) - F(0,t)\} = \int_{0+}^\infty e^{-\theta z}\,dz \int_0^\infty \{F(z-u,t) - F(z,t)\}\lambda(u)\,du. \tag{4·3}$$

Now, integrating $\int_{0+}^\infty e^{-\theta z}\,F\,dz$

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by parts, it is easily seen to be equal to $\psi(\theta,t)/\theta$. Hence, assuming differentiation under the integral sign is justified, we have

$$\int_{0+}^{\infty} e^{-\theta z}\frac{\partial F}{\partial t}dz = \frac{1}{\theta}\frac{\partial \psi}{\partial t}. \tag{4·4}$$

After inverting the order of integration on the right-hand side of (4·3), this expression reduces to $-\xi(\theta)\psi(\theta,t)/\theta$, so that we finally obtain

$$\frac{\partial}{\partial t}\psi(\theta,t)-\theta[\psi(\theta,t)-F(0,t)] = -\xi(\theta)\psi(\theta,t), \tag{4·5}$$

where $\xi(\theta)$ is the function defined by (2·2). On integrating with respect to t, this leads to the result

$$\psi(\theta,t) = e^{-\{\xi(\theta)-\theta\}t-\theta z_0}-\theta\int_0^t F(0,t-\tau)\,e^{-\{\xi(\theta)-\theta\}\tau}\,d\tau. \tag{4·6}$$

This determines $F(z,t)$ uniquely, provided that $F(0,t)$ is known in the required range.

Let us define the transform of $F(0,t)$ with respect to t by

$$F^*(s) = \int_0^{\infty} e^{-st}F(0,t)\,dt \quad (s>0). \tag{4·7}$$

(We shall return to this later in § 6.) Taking transforms with respect to t of (4·6), we readily obtain for $s>\theta-\xi(\theta)>0$, that

$$\psi^*(\theta,s) = \{s-\theta+\xi(\theta)\}^{-1}\{e^{-\theta z_0}-\theta F^*(s)\}, \tag{4·8}$$

an equation of the form derived by Beneš (1).

General equations for $F(z,t)$, as well as $F(0,t)$, may be established from (4·6), which we now write in the slightly modified form

$$\theta^{-1}\psi(\theta,t) = \theta^{-1}e^{-t\xi(\theta)+\theta(t-z_0)}-\int_0^t F(0,t-\tau)\,e^{-\tau\xi(\theta)+\theta\tau}\,d\tau \quad (\theta>0). \tag{4·9}$$

Here $F(z,t)$ is the inverse of $\theta^{-1}\psi(\theta,t)$. The inverse of the right-hand side may also be found; for since

$$\int_0^{\infty} e^{-\theta x}k(x,t)\,dx = e^{-t\xi(\theta)} \quad (\theta>0), \tag{4·10}$$

it follows that

$$\int_{z_0-t}^{\infty} e^{-\theta z}K(z+t-z_0,t)\,dz = \theta^{-1}e^{-t\xi(\theta)+\theta(t-z_0)}, \tag{4·11}$$

and

$$\int_{-\tau}^{\infty} e^{-\theta z}k(z+\tau,\tau)\,dz = e^{-\tau\xi(\theta)+\theta\tau}. \tag{4·12}$$

Thus, from (4·9) it is possible to derive the relation

$$F(z,t) = K(z+t-z_0,t)-\int_0^t F(0,t-\tau)\,k(z+\tau,\tau)\,d\tau \quad (z\geqslant 0), \tag{4·13}$$

where, from the properties of the transform for $F(z,t)$, recalling that $F(z,t)=0$ for $-t<z<0$, we may also obtain for $z'<0$ the following identity

$$K(z'+t-z_0,t) = \int_0^t F(0,t-\tau)\,k(z'+\tau,\tau)\,d\tau.$$

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Alternatively, putting $z = -z' > 0$, this can be written as

$$K(t-z-z_0,t) = \int_z^t F(0,t-\tau)\,k(\tau-z,\tau)\,d\tau \quad (z > 0). \tag{4·14}$$

Since
$$F(0,t) = \lim_{z\to+0} F(z,t),$$

it follows directly from (4·13) that the probability of emptiness of the dam would be given by

$$F(0,t) = K(t-z_0,t) - \int_0^t F(0,t-\tau)\,k(\tau,\tau)\,d\tau. \tag{4·15}$$

It is understood throughout that $K(x,t) = 0$ for $x < 0$, and $F(z,t) = 0$ for

$$z < \max\{0, z_0 - t\}.$$

We now consider in some detail the probability of first emptiness of the dam, or more simply the probability of its 'wet period'.

5. *The probability distribution of the wet period; first passage times.* The distribution of the wet period is a problem of some practical interest in storage theory. Suppose that the initial content of the dam is $Z(0) = z_0$, and let $T \equiv T(z_0)$ be the first subsequent time at which the dam runs dry, so that $Z(T) = 0$; then the duration T of this wet period is a random variable. Kendall (**10**) has obtained the probability distribution of T, as well as its Laplace–Stieltjes transform: the density function $g(z_0,T)$ of T is given by

$$g(z_0,T) = \begin{cases} 0, & \text{if} \quad T < z_0, \\ \dfrac{z_0}{T}\,k(T-z_0,T) & \text{if} \quad T \geqslant z_0 > 0. \end{cases} \tag{5·1}$$

The transform of this with respect to T is obtained as

$$g^*(z_0|s) = \int_{z_0}^{\infty} e^{-sT}\,g(z_0,T)\,dT = e^{-z_0\eta(s)} \quad (z_0 > 0), \tag{5·2}$$

where $\eta(s)$ satisfies the functional equation

$$\eta(s) = s + \xi\{\eta(s)\} \quad (s > 0). \tag{5·3}$$

From (5·2) and (5·3) it follows (cf. Takács (**15**); Kendall (**10**)) that the probability that the dam will eventually run dry is

$$\Pr\{T < \infty\} = 1 \quad \text{if} \quad \rho \leqslant 1, \tag{5·4 (i)}$$

$$\Pr\{T < \infty\} = e^{-z_0\zeta} \quad \text{if} \quad \rho > 1, \tag{5·4 (ii)}$$

where ζ is the largest positive root of the equation $\zeta = \xi(\zeta)$. Further, we note that (5·1) and (5·2) imply the relation

$$\int_{z_0}^{\infty} e^{-sT}\,\frac{z_0}{T}\,k(T-z_0,T)\,dT = e^{-z_0\eta(s)} \quad (z_0 > 0). \tag{5·5}$$

Differentiating with respect to s under the integral sign, we find that

$$\int_{z_0}^{\infty} e^{-sT}\,k(T-z_0,T)\,dT = e^{-z_0\eta(s)}\,\eta'(s) \quad (z_0 > 0). \tag{5·6}$$

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Clearly, the wet period $T(z_0)$ is a particular case of the first passage time of the process $Z(t)$, defined as follows:

$$T \equiv T(z, z_0) = \begin{cases} \inf\{t | Z(t) \leqslant z,\ Z(0) = z_0\} & \text{if} \quad z < z_0, \\ \inf\{t | Z(t) \geqslant z,\ Z(0) = z_0\} & \text{if} \quad z > z_0. \end{cases} \tag{5·7}$$

We assume that T has a density function $h(z_0; z, t)$; then if the initial dam content is $Z(0) = z_0$, the probability that its content will be z for the first time after an interval between $t, t+dt$ is given by $h(z_0, z, t)\,dt$. In particular, $T(z_0, 0) = T$, the wet period, and $h(z_0; 0, t) = g(z_0, t)$ gives its p.d.f.

As indicated by Darling and Siegert (3), the density function $h(z_0; x, t)$ satisfies the integral equation

$$\rho(z_0; z', t) = \int_0^t h(z_0; z, t-\tau)\,\rho(z; z', \tau)\,d\tau, \tag{5·8}$$

where either $z_0 < z < z'$ or $z_0 > z > z'$, and $\rho(z_0; z, t)$, the transition p.d.f. of the process, is defined as

$$\rho(z_0; z, t) = \begin{cases} F(z_0, 0; 0, t) & \text{for} \quad z = 0, \\ \dfrac{\partial}{\partial z} F(z_0, 0; z, t) & \text{for} \quad z > 0, \end{cases} \tag{5·9}$$

where $F(z_0, 0; z, t)$ has previously been written $F(z, t)$ for simplicity. This equation is obtained by direct enumeration of paths from z_0 to z'.

Let us denote the transforms of ρ and h by

$$\rho^*(z_1; z_2 | s) = \int_0^\infty e^{-st}\,\rho(z_1; z_2, t)\,dt \tag{5·10}$$

and

$$h^*(z_0; z | s) = \int_0^\infty e^{-st}\,h(z_0; z, t)\,dt; \tag{5·11}$$

the equation (5·8) then gives the relation

$$h^*(z_0; z | s) = \frac{\rho^*(z_0; z' | s)}{\rho^*(z; z' | s)} \tag{5·12}$$

for the transform of the first passage probability in terms of the transform of the transition probability functions. Since the latter can be obtained from the expression for $F(z, t)$ in (4·13), we may consider the problem as solved.

In particular, let us choose $z' = 0$ and let $z_0 > z$; then from (5·12) we obtain

$$\begin{aligned} h^*(z_0; z | s) &= \frac{\rho^*(z_0; 0 | s)}{\rho^*(z; 0 | s)} = \frac{e^{-z_0\eta(s)}}{\eta(s)} \Big/ \frac{e^{-z\eta(s)}}{\eta(s)} \\ &= e^{-(z_0 - z)\,\eta(s)} = g^*(z_0 - z | s) \quad (z_0 > z > 0), \end{aligned} \tag{5·13}$$

where $g^*(z_0 - z | s)$ is the transform (5·2) of the wet period when the initial dam content is $z_0 - z > 0$. Hence it follows that

$$h(z_0; z, t) = g(z_0 - z, t) \quad (t \geqslant z_0 - z > 0). \tag{5·14}$$

Thus the probability of first passage from z_0 to z is the same as the probability of first passage from $z_0 - z$ to 0; this is obvious from other considerations.

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6. *The probability of emptiness* $F(0,t)$. It is now possible to solve the integral equation (4·15) to obtain the probability of emptiness $F(0,t)$; this probability can most easily be derived from the limiting form of the known sum of wet period probabilities for the analogous discrete inputs in intervals Δ. We find that

$$F(0,t) = \begin{cases} 0 & \text{if} \quad t < z_0, \\ \int_{z_0}^{t} g(z,t)\,dz & \text{if} \quad t \geqslant z_0. \end{cases} \tag{6·1}$$

To verify this we consider $Y(t) = u + X(t) - t$, which is a time-homogeneous Markov process with the transition p.d.f. $k(y+t-u,\, t)$. Clearly, the wet period $T(u)$ of the dam can be written as

$$T(u) = \inf\{t \mid u + X(t) - t \leqslant 0\}, \tag{6·2}$$

so that $T(u)$ is the first passage time of $Y(t)$ from u to 0. By a direct enumeration of the paths from u to 0 for this process we obtain

$$g(u,t) + \int_{0+}^{t-u} g(u, t-\tau)\, k(\tau,\tau)\,d\tau = k(t-u,t) \quad (t \geqslant u > 0). \tag{6·3}$$

Integrating (6·3) over $z_0 \leqslant u \leqslant t$ we find that

$$\int_{z_0}^{t} g(u,t)\,du + \int_{z_0}^{t} du \int_{\tau=0+}^{t-u} g(u,t-\tau)\, k(\tau,\tau)\,d\tau = \int_{z_0}^{t} k(t-u,t)\,du,$$

or

$$\int_{z_0}^{t} g(u,t)\,du + \int_{\tau=+0}^{t-z_0} k(\tau,\tau)\,d\tau \int_{z_0}^{t-\tau} g(u,t-\tau)\,du = K(t-z_0,t) \quad (t \geqslant z_0).$$

On comparing this with (4·15), we see that (6·1) is the solution required for $F(0,t)$ (it is obviously the unique non-negative solution).

The transform of $F(0,t)$ with respect to t defined in (4·7) can now easily be obtained; we have

$$F^*(s) = \int_{z_0}^{\infty} e^{-st} F(0,t)\,dt = \int_{z_0}^{\infty} e^{-st}\,dt \int_{z_0}^{t} g(z,t)\,dz,$$

which after some reduction leads to

$$F^*(s) = \frac{e^{-z_0 \eta(s)}}{\eta(s)} \tag{6·4}$$

an equation of the form given by Beneš ((1), formula (2·6)).

Let us again consider the process $Y(t)$ and its paths from u to $-z$ (< 0) as an alternative argument for obtaining (6·1); direct enumeration now gives

$$k(t-z-u,t) = \int_{z}^{t-u} g(u,t-\tau)\, k(\tau-z,\tau)\,d\tau \quad (t-u \geqslant z > 0). \tag{6·5}$$

Integrating this over $z_0 \leqslant u \leqslant t-z$ we obtain

$$\begin{aligned} K(t-z-z_0,t) &= \int_{z_0}^{t-u} du \int_{\tau=z}^{t-u} g(u,t-\tau)\, k(\tau-z,\tau)\,d\tau \\ &= \int_{\tau=z}^{t-z_0} k(\tau-z,\tau)\,d\tau \int_{z_0}^{t-\tau} g(u,t-\tau)\,du. \end{aligned}$$

On comparison with (4·14), we see that (6·1) follows.

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7. *The equilibrium theory of Moran's process.* In this section we consider the behaviour of Moran's process when it is in equilibrium, and deduce some results obtained by other workers as a direct consequence of our previous sections. We first note that if $Z(0) = 0$, $Z(t)$ has the same distribution as

$$\sup_{0 \leqslant \tau \leqslant t} [X(\tau) - \tau], \tag{7·1}$$

so that if we define the first passage time as

$$U(z) = \inf\{t | X(t) - t > z\}, \tag{7·2}$$

then

$$\begin{aligned} F(0; z, t) &= \Pr\{X(\tau) - \tau \leqslant z (0 \leqslant \tau \leqslant t)\} \\ &= 1 - \Pr\{U(z) > t\}. \end{aligned} \tag{7·3}$$

Thus $F(0; z, t)$ is a monotone non-increasing function of t, so that $\lim_{t \to \infty} F(0; z, t)$ exists. Similarly, it may be shown that $\lim_{t \to \infty} F(z_0; z, t)$ exists for $z_0 > 0$, and is independent of z_0; we denote this limit by $\Phi(z)$ for $z \geqslant 0$. From (6·4) it can easily be shown that

$$\Phi(0) = \lim_{t \to \infty} F(z_0; 0, t) = \lim_{s \to 0+} sF^*(s) = \begin{Bmatrix} 0 & \text{if} & \rho \geqslant 1, \\ 1 - \rho & \text{if} & \rho < 1, \end{Bmatrix} \tag{7·4}$$

independently of z_0. Further, because of our assumption, the transform $\psi(\theta, t) \to \psi(\theta)$ as $t \to \infty$, where

$$\psi(\theta) = \lim_{s \to 0+} s\psi^*(\theta, s) = \begin{Bmatrix} 0 & \text{if} & \rho \geqslant 1, \\ \dfrac{\theta(1-\rho)}{\theta - \xi(\theta)} & \text{if} & \rho < 1. \end{Bmatrix} \tag{7·5}$$

This is the formula given by Downton, Lindley and Smith (see discussion in Kendall (**10**)); Daniels's inversion of this transform has already been referred to. The validity of the inversion will be seen to depend on the identities

$$\int_z^\infty k(t - z, t)\, dt = (1 - \rho)^{-1} \quad (z > 0, \rho < 1), \tag{7·6}$$

$$\int_{0+}^\infty k(t, t)\, dt = \rho(1 - \rho)^{-1} \quad (\rho < 1). \tag{7·7}$$

We may derive the limiting distribution $\Phi(z)$ directly from the equation (4·13). We first establish the following result:

$$\int_0^\infty k(t + z, t)\, dt \leqslant (1 - \rho)^{-1} \quad (-\infty < z < \infty, \rho < 1). \tag{7·8}$$

When $z \geqslant 0$, we find from (4·13) that

$$1 \geqslant K(t + z, t) \geqslant \int_0^t F(0; 0, t - \tau)\, k(\tau + z, \tau)\, d\tau \geqslant F(0; 0, t) \int_0^t k(\tau + z, \tau)\, d\tau, \tag{7·9}$$

since $F(0; 0, t)$ is monotone non-increasing. Letting $t \to \infty$ in (7·9), and using (7·4) we obtain (7·8) for $z \geqslant 0$ and $\rho < 1$. A similar procedure with the identity (4·14)

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yields (7·8) for $z < 0$ and $\rho < 1$; in this case, however, we obtain the equality (7·6), since from (5·6) we have that

$$\int_z^\infty k(t-z,t)\,dt = \lim_{s\to 0+} e^{-z\eta(s)}\eta'(s) = (1-\rho)^{-1}. \tag{7·10}$$

Further, integrating the relation (6·3) over $u \leqslant t < \infty$, where u is put equal to z, we obtain

$$\int_z^\infty k(t-z,t)\,dt = \int_z^\infty g(z,t)\,dt\left\{1+\int_0^\infty k(t,t)\,dt\right\}, \tag{7·11}$$

which leads to (7·7) for $\rho < 1$.

Now let $T < t$, and write (4·13) as

$$F(z,t) = K(t+z-z_0,t) - \int_0^T F(0,t-\tau)\,k(\tau+z,\tau)\,d\tau - \int_T^t F(0,t-\tau)\,k(\tau+z,\tau)\,d\tau. \tag{7·12}$$

In this, assuming $\rho < 1$, let $t \to \infty$ first, and $T \to \infty$ afterwards. Now from (2·7) we know that

$$\lim_{t\to\infty} K(t+z-z_0,t) = 1 \quad (\rho < 1). \tag{7·13}$$

Further, using (7·4) we find that

$$\int_0^T F(0,t-\tau)\,k(\tau+z,\tau)\,d\tau \to (1-\rho)\int_0^T K(\tau+z,\tau)\,d\tau \quad (t\to\infty)$$

$$\to (1-\rho)\int_0^\infty k(\tau+z,\tau)\,d\tau \quad (T\to\infty), \tag{7·14}$$

while since by (7·8) $k(t+z,t)$ is integrable over $(0,\infty)$, it follows that

$$\int_T^t F(0,t-\tau)\,k(\tau+z,\tau)\,d\tau \leqslant \int_T^t k(\tau+z,\tau)\,d\tau \to 0 \quad (T,t\to\infty). \tag{7·15}$$

Thus from (7·12)–(7·15) we finally deduce that if $\rho < 1$,

$$1-\Phi(z) = (1-\rho)\int_0^\infty k(t+z,t)\,dt \quad (z \geqslant 0). \tag{7·16}$$

This is Daniels's formula; we note that the expression for $\Phi(z)$ does not involve the initial dam content z_0.

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REFERENCES

(1) BENEŠ, V. E. On queues with Poisson arrivals. *Ann. Math. Statist.* **28** (1957), 670–677.
(2) BENEŠ, V. E. Theory of queues with one server. *Trans. American Math. Soc.* **94** (1960), 282–294.
(3) DARLING, D. A. and SIEGERT, A. J. F. The first passage problem for a continuous Markov process. *Ann. Math. Statist.* **24** (1953), 624–639.
(4) DOOB, J. L. *Stochastic processes* (Wiley; New York, 1953).
(5) DOWNTON, F. A note on Moran's theory of dams. *Quart. J. Math. Oxford Ser.* (2), **8** (1957), 282–286.

https://doi.org/10.1017/S030500410003704X Published online by Cambridge University Press

(6) GANI, J. and PRABHU, N. U. Remarks on the dam with Poisson type inputs. *Aust. J. Appl. Sci.* **10** (1959), 113–122.
(7) GANI, J. and PYKE, R. The content of a dam as the supremum of an infinitely divisible process. *J. Math. Mech.* **9** (1960), 639–652.
(8) GNEDENKO, B. V. and KOLMOGOROV, A. N. *Limit distributions for sums of independent random variables* (Addison-Wesley; Cambridge, Mass., 1954).
(9) KENDALL, D. G. Some problems in the theory of queues. *J. Roy. Statist. Soc. Ser. B*, **13** (1951), 151–173.
(10) KENDALL, D. G. Some problems in the theory of dams. *J. Roy. Statist. Soc. Ser. B*, **19** (1957), 207–212.
(11) LÉVY, P. *Processus stochastiques et mouvement brownien* (Gauthier-Villars; Paris, 1948).
(12) MEISLING, T. Discrete-time queueing theory. *Operations Res.* **6** (1958), 96–105.
(13) MORAN, P. A. P. A probability theory of a dam with a continuous release. *Quart. J. Math. Oxford Ser.* (2), **7** (1956), 130–137.
(14) REICH, E. Some combinatorial theorems for continuous time parameter processes. *Math. Scand.* **9** (1961), 243–257.
(15) TAKÁCS, L. Investigation of waiting time problems by reduction to Markov processes. *Acta Math. Acad. Sci. Hungar.* **6** (1955), 101–129.

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