Solving multivariate optimisation problems using inequalities

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Introduction

Optimisation problems are among the most practical applications of calculus to everyday life, and a survey of exercises in various calculus textbooks will provide a teacher with many interesting scenarios for framing intriguing questions on this topic. Whether it is finding a container's dimensions that yield the least surface area for a given volume, or finding that ideal movie ticket price which will maximise a theatre's revenue, students can usually relate to these problems. Pólya in his book *Plausible reasoning* makes the following remarks about the attraction of extrema problems:

> Problems concerned with greatest and least values, or maximum and minimum problems, are more attractive, perhaps, than other mathematical problems of comparable difficulty, and this may be due to a quite primitive reason. Every one of us has his personal problems. We may observe that these problems are very often maximum or minimum problems of a sort. We wish to obtain a certain object at the lowest possible price, or the greatest possible effect with a certain effort, or the maximum work done within a given time and, of course, we wish to run the minimum risk. Mathematical problems on maxima and minima appeal to us, I think, because they idealize our everyday problems. We are even inclined to imagine that Nature acts as we would like to act, obtaining the greatest effect with the least effort [1, p. 121].

Although Pólya seems to have adults in mind when he wrote the above, even younger students with little life experience can relate to themes such as greatest size, minimum time, maximum profit, and lowest cost.

Since these types of problems are ubiquitous, some readers may be interested in becoming more aware of alternative methods of solution. An excellent reference on this subject is Niven's *Maxima and minima without calculus* [2]. Many techniques relying on algebra, arithmetic inequalities, and/or geometric arguments are clearly outlined and well explained in this work. Regarding these techniques, Niven points out that, 'these elementary methods are often easier to implement than calculus methods, although they may lack the generality of more analytic methods'. On the other hand, Niven notes that while calculus is good 'for solving some problems in maxima and minima, the method is not universal. There are many problems that are awkward, if not impossible, to solve with elementary calculus' [1, p. ix]. Thus some benefits of learning additional methods of solving optimisation problems include more efficient and elegant solution methods as well as the ability to solve a wider range of problems.

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Many other authors have highlighted the fact that arithmetic inequalities can be used to solve optimisation problems. For example, Bagnato [3], explained how the weighted Arithmetic-Geometric mean inequality (AM-GM) is foundational to the theory in geometric programming, which enables optimisation in problems in the field of operations research. Boas [4] discusses a method that utilises the (AM-GM) inequality to find the extreme values of a polynomial over specific intervals. Using Holder's inequality, Klamkin [5] found the shortest line segment through a given point in the first quadrant. To avoid an arduous derivative computation, Azarian in [6] determined the maximum value of a complicated function with Jensen's Inequality. To find the maximum value of the Cobb-Douglass production function, Kojić [7] relied on the AM-GM inequality to solve elegantly this well-known problem from economics.

Although much of the literature on this subject is focused on problems having, in the end, a single independent variable, there are many types of intriguing problems from multivariate calculus that can be solved with inequalities. In this paper some select classes of optimisation problems are discussed which can be solved using the AM-GM inequality. The limited selection of problems that are considered here were chosen because of their inclusion in the exercises in popular university calculus textbooks such as Stewart [8] and Larson [9]. Since the AM-GM inequality will be foundational, its statement and proof will precede the forthcoming examples and applications.

The Arithmetic-Geometric Mean Inequality (AM-GM)

The AM-GM inequality states that for any set of non-negative real numbers,

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geqslant \sqrt[n]{a_1 a_2 \ldots a_n},$$

where equality holds if, and only if, the a_k are all equal. Various proofs of this famous inequality can be found in virtually any book on inequalities. A basic proof of the case for $n = 2$ can be shown using the fact that

$$\left(\sqrt{a_1} - \sqrt{a_2}\right)^2 \geqslant 0.$$

G. Pólya in [10] outlined a fascinating proof of the general case, and this proof will be explained and elaborated on in this section. To start, let A denote the arithmetic mean, and G denote the geometric mean, of the numbers a_k, and let

$$x_k = -1 + \frac{a_k}{A}.$$

Since e^x is a convex function, it is greater than or equal to its tangent line at $x = 0$. That is, $e^x \geqslant x + 1$ for all x, with equality holding only at $x = 0$. Then by multiplication

$$\prod_{k=1}^{n} \exp(x_k) \geqslant \prod_{k=1}^{n} (x_k + 1).$$

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Equivalently,

$$\exp\left(\sum_{k=1}^{n} x_k\right) \geqslant \frac{1}{A^n}\prod_{k=1}^{n} a_k = \frac{G^n}{A^n}.$$

The left-hand side is equal to 1 since

$$\exp\left(\sum_{k=1}^{n} x_k\right) = \exp\left(-n + \frac{1}{A}\sum_{k=1}^{n} a_k\right) = \exp(0) = 1.$$

Therefore $A^n \geqslant G^n$ and $A \geqslant G$. A similar approach from [2] is to consider the tangent line of e^x at $x = 1$ to show that $e^x \geqslant ex$. Then after defining $x_k = a_k/A$, proceed as in the above.

It follows that if A is constrained to be a constant while the a_k are free to vary, then the product obtains its maximum value when all the a_k are equal to A. Similarly, if G is constant, then the sum of the a_k is minimum when these values are identical.

Functions of one variable

Before discussing the multivariate problems, a few types of illustrative single variable optimisation problems will be solved with the AM-GM inequality.

Let

$$f(x) = \frac{a}{x^m} + bx^n,$$

where the parameters a and b are positive real values and m and n are positive integers. (Whenever these parameters appear in subsequent problems, these descriptions will apply.) To determine when this function attains its minimum on the interval $(0, \infty)$, we rewrite it in this way:

$$f(x) = n \cdot \frac{a}{nx^m} + m \cdot \frac{bx^m}{m}.$$

Now view the function as consisting of n terms of a/nx^m and m terms of bx^n/m. Then by AM-GM we have

$$\frac{f(x)}{(n+m)} \geqslant \left[\left(\frac{a}{nx^m}\right)^n \cdot \left(\frac{bx^n}{m}\right)^m\right]^{1/(m+n)}.$$

Since the independent variable cancels on the right side of this inequality, the function $f(x)$ has the following minimum value on the interval $(0, \infty)$,

$$(n+m) \cdot \left[\left(\frac{a}{n}\right)^n \cdot \left(\frac{b}{m}\right)^m\right]^{1/(m+n)}.$$

The function assumes this minimum when $a/nx^m = bx^n/m$. That is, when

$$x = \left(\frac{am}{bn}\right)^{1/(n+m)}.$$

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Example: Find the radius r and the height h of a closed cylinder which would yield the smallest surface area S for a given fixed volume V_0.

Since $V_0 = \pi r^2 h$, and $S = 2\pi\left(r^2 + rh\right)$, solving for S in terms of r yields

$$S(r) = \frac{2V_0}{r} + 2\pi r^2.$$

Then $a = 2V_0$, $b = 2\pi$, $m = 1$ and $n = 2$. Using the above results, the minimum surface area is $3\left(2\pi V_0^2\right)^{1/3}$, and this occurs when the radius $r = (V_0/2\pi)^{1/3}$. The corresponding value of the height h is $2(V_0/2\pi)^{1/3}$.

Another single variable application is to find the *maximum* value of the function

$$f(x) = (x - a)^m (b - x)^n,$$

on the interval $[a, b]$. To proceed, we rewrite f as

$$f(x) = \frac{1}{n^m m^n}(nx - na)^m (mb - mx)^n.$$

For now ignore the constant $1/n^m m^n$, and view the remaining portion of f as a product of $m + n$ factors. Then by the AM-GM inequality

$$\sqrt[m+n]{(nx-na)^m(mb-mx)^n} \leqslant \frac{m(nx-na)+n(mb-mx)}{m+n} = \frac{mn(b-a)}{m+n}.$$

The right-hand side of the above is a constant. Thus

$$f(x) \leqslant \frac{1}{n^m m^n}\left(\frac{mn(b-a)}{m+n}\right)^{m+n},$$

where equality holds when $nx - na = mb - mx$, or specifically when $x = \dfrac{na + mb}{m + n}$.

Example: A rectangular box has a length that is twice its width, and the sum of its three dimensions is a fixed distance D_0. Find the maximum volume.

Let the width be denoted by x. Then the length is $2x$, the height is $D_0 - 3x$, and the volume can be expressed as $V(x) = 6x^2\left(\frac{1}{3}D_0 - x\right)$. Let $a = 0$, $b = \frac{1}{3}D_0$, $m = 2$ and $n = 1$. Then using the above results, we obtain a maximum volume of $\frac{8}{243}D_0^3$, which occurs when $x = \frac{2}{9}D_0$.

In each of the above problems, the independent variable was eliminated from one side of the inequality. This occurred through addition and subtraction in the arithmetic mean or through multiplication and division in the geometric mean.

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Multivariate examples

The techniques used in the above examples can help solve some more difficult problems involving more than one variable. However, as should be expected, multivariate problems will often involve additional complications that do not arise in single variable problems. This section will focus on what are commonly referred to as 'non-constrained optimisation' problems in which the goal is to determine extreme values of an objective function that has no constraints on its independent variables. Below, a few classes of problems are discussed and, as in the previous section, this is a small sample of the wider variety of problems that are possible. In these problems the function domain will be restricted to positive real values of the independent variables x and y and, in this domain, a unique extremum can be found for each example discussed below.

A bivariate polynomial

Consider the general class of functions given by

$$f(x, y) = x^m y^n (a - mx - ny).$$

This function is a product of $(a - mx - ny)$ along with m factors of x and n factors of y. With this fact in mind, the maximum value of f can be found with a straightforward application of AM-GM. Specifically,

$$f(x, y) \leqslant \left(\frac{mx + ny + a - mx - ny}{m + n + 1}\right)^{(m+n+1)} = \left(\frac{a}{m + n + 1}\right)^{(m+n+1)}.$$

This maximum occurs when $x = y = a - mx - ny$. That is, when $x = y = \frac{a}{n+m+1}$.

Bivariate Rational Functions

One class of bivariate rational functions in which a local maximum can be found is given below. Let

$$P(x, y) = \frac{axy - nx - my}{x^{m+1} y^{n+1}}.$$

Rewrite the function as

$$P(x, y) = \frac{a}{x^m y^n} - \frac{n}{x^m y^{n+1}} - \frac{m}{x^{m+1} y^n} = \left(\frac{1}{x}\right)^m \left(\frac{1}{y}\right)^n \left(a - \frac{n}{y} - \frac{m}{x}\right).$$

Then view the final expression as consisting of $n + m + 1$ factors. Now apply AM-GM:

$$P(x, y) \leqslant \left(\frac{1}{n+m+1}\left(\frac{m}{x} + \frac{n}{y} + a - \frac{n}{x} - \frac{m}{y}\right)\right)^{n+m+1} = \left(\frac{a}{n+m+1}\right)^{n+m+1},$$

where equality holds when $\frac{n}{y} = \frac{m}{x} = \left(a - \frac{n}{x} - \frac{m}{y}\right)$, that is, when $x = \frac{3m}{a}$ and $y = \frac{3n}{a}$.

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A very similar approach can be applied to functions of the form

$$Q(x, y) = \frac{(ax - m)(by - n)}{x^{m+1}y^{n+1}} = \left(\frac{1}{x}\right)^m \left(\frac{1}{y}\right)^n \left(a - \frac{m}{x}\right)\left(b - \frac{n}{y}\right).$$

And as can be seen from the expression on the right-hand side, the function Q can be viewed as having $n + m + 2$ factors which, when added together in the AM-GM inequality, will yield a constant.

If r and s denote positive integers, then it is possible to find the minimum of the function

$$f(x, y) = ax^m + by^n + \frac{c}{x^r y^s}.$$

First we express f as

$$f(x, y) = \alpha\frac{a}{\alpha}x^m + \beta\frac{b}{\beta}y^n + \gamma\frac{c}{\gamma x^r y^s},$$

where the positive integral constants α, β and γ will be determined later as functions of the parameters m, n, r and s. Applying AM-GM as in the previous examples yields

$$\left(\frac{f(x, y)}{\alpha + \beta + \gamma}\right)^{\alpha+\beta+\gamma} \geqslant \left(\frac{a}{\alpha}x^m\right)^{\alpha} \cdot \left(\frac{b}{\beta}y^n\right)^{\beta} \cdot \left(\frac{c}{\gamma x^r y^s}\right)^{\gamma}. \tag{1}$$

The expression on the right-hand side of (1) will be independent of x if $m\alpha = r\gamma$ and $n\beta = s\gamma$. We can find expressions for the constants α, β, γ by letting $\alpha = nr$, $\beta = ms$ and $\gamma = mn$. Substituting these values for the constants α, β, γ into the right-hand side of (1) yields

$$\left(\frac{a}{nr}\right)^{nr} \cdot \left(\frac{b}{ms}\right)^{ms} \cdot \left(\frac{c}{mn}\right)^{mn},$$

from which we can find the maximum value of this function. The point where the maximum is attained can be found by solving the system

$$\frac{a}{nr}x^m = \frac{b}{ms}y^n = \frac{c}{mnx^r y^s}.$$

In the above problems in this section cancellation of the independent variable occurred in a manner analogous to the single variable problems of the previous section. Often however, more work is required to rearrange the function algebraically so as to make this cancellation possible. Bagnato in [3] explains a similar method that utilises the weighted AM-GM inequality to solve problems of this type.

Lagrange Multiplier problems

Inequalities can sometimes save time in solving difficult problems that are presented in the Lagrange Multiplier sections of multivariable calculus. These are the constrained optimisation problems in contrast with the non-

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constrained problems of the previous section. The discussion below will be restricted to functions having three independent variables; however, the methods used here can be generalised to higher order dimensions, or simplified in the two variable case. As before, the independent variable values are restricted to positive real numbers. Additionally, parameters with a '0' subscript will be considered positive real constants.

The first problem is to minimise $f(x, y, z) = ax + by + cz$ subject to $x^m y^n z^p = T_0$. To start, rewrite the function f as

$$f(x, y, z) = m\left(\frac{a}{m}x\right) + n\left(\frac{b}{n}y\right) + p\left(\frac{c}{p}z\right),$$

and consider this as a sum of $(m + n + p)$ terms. Apply AM-GM to obtain

$$\frac{m\left(\frac{a}{m}x\right) + n\left(\frac{b}{n}y\right) + p\left(\frac{c}{p}z\right)}{m + n + p} \geqslant \left[\left(\frac{a}{m}x\right)^m \cdot \left(\frac{b}{n}y\right)^n \cdot \left(\frac{c}{p}z\right)^p\right]^{1/(m+n+p)}.$$

Then

$$f(x, y, z) \geqslant (m + n + p)\left[T_0\left(\frac{a}{m}\right)^m \cdot \left(\frac{b}{n}\right)^n \cdot \left(\frac{c}{p}\right)^p\right]^{1/(m+n+p)}.$$

The coordinates of the point for which the function assumes this minimum values can be found by solving $\frac{a}{m}x = \frac{b}{n}y = \frac{c}{p}z$ and $x^m y^n z^p = T_0$. Thus, for this problem, the minimum function value can be obtained much more easily with inequalities than by the Lagrange Multiplier method.

In a similar manner the maximum value of $f(x, y, z) = x^m y^n z^p$ subject to $ax + by + cz = S_0$ can be found. Here, the roles of the function and constraint of the previous problem have been reversed compared. Now we first rewrite the constraint as,

$$m\left(\frac{a}{m}x\right) + n\left(\frac{b}{n}y\right) + p\left(\frac{c}{p}z\right) = S_0.$$

Then

$$\frac{S_0}{m + n + p} \geqslant \left[f(x, y, z)\left(\frac{a}{m}\right)^m \left(\frac{b}{n}\right)^n \left(\frac{c}{p}\right)^p\right]^{1/(m+n+p)}.$$

And upon isolating the function we can identify the maximum value of f through the expression

$$f(x, y, z) \leqslant \left(\frac{m}{a}\right)^m \left(\frac{n}{b}\right)^n \left(\frac{p}{c}\right)^p \left(\frac{S_0}{m + n + p}\right)^{(m+n+p)}.$$

As in the previous problem, the equality condition holds when

$$\frac{a}{m}x = \frac{b}{n}y = \frac{c}{p}z,$$

and specific values of the variables can be found with this set of inequalities and the constraint $ax + by + cz = S_0$.

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Some of the most interesting problems in constrained optimisation can be framed in the context of area/volume type application problems. For example, suppose it costs $\frac{1}{2}a$ dollars per square foot to make the top and bottom of a rectangular box; $\frac{1}{2}b$ dollars/ft^2 for the left and right sides; and $\frac{1}{2}c$ dollars/ft^2 for the front and back. Find the dimensions which would minimise the cost of making a closed box of fixed volume V_0. Mathematically, this can be expressed in the following way: Minimise $f(x, y, z) = axy + bxz + cyz$ subject to $xyz = V_0$.

A straightforward application of AM-GM to the three function terms yields

$$\frac{axy + bxz + cyz}{3} \geqslant \sqrt[3]{abc\,(xyz)^2}.$$

Equivalently,

$$f(x, y, z) \geqslant 3\,(abc)^{1/3} V_0^{2/3}.$$

The function f attains the minimum value when $axy = bxz = cyz$ and $xyz = V_0$. Then the ideal dimensions are

$$x = c\left(\frac{V_0}{abc}\right)^{\frac{1}{3}}, \qquad y = b\left(\frac{V_0}{abc}\right)^{\frac{1}{3}}, \qquad z = a\left(\frac{V_0}{abc}\right)^{\frac{1}{3}}.$$

In this section the constraint was crucial in enabling the elimination of the independent variables from the inequality. The constraint often determines the manner in which the objective function is to be rearranged before applying the AM-GM inequality.

Summary

Several classes of non-constrained optimisation problems from single variable and multivariable calculus were shown to be solved efficiently using techniques involving the AM-GM inequality. Certain constrained optimisation problems, which could also be attacked with the method of Lagrange Multipliers, were solved quickly using these elementary methods.

Appendices

Additional examples

1) Find the *maximum* value on $(0, \infty)$ of rational functions of the form

$$g(x) = \frac{x^n}{x^m + C},$$

where and $m > n$ and C is a positive real number.

Hint: Rewrite the function in the following form: $g(x) = \left(x^{m-n} + \dfrac{c}{x^n}\right)^{-1}$.

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2) Show that a relative maximum of

$$f(x, y) = x^m y^n (a - mx)(b - ny)$$

is

$$f(x, y) \leqslant \left(\frac{a}{m+1}\right)^{(m+1)} \left(\frac{b}{n+1}\right)^{(n+1)}$$

and find where it occurs.

3) Using the AM-GM inequality and properties of quadratics, show that

$$f(x, y) = a^4 x^{4n} + b^4 y^{4n} - 4a^2 b^2 x^n y^n + c$$

has a minimum value of $c - 2a^2 b^2$ at the point $(x, y) = (1, 1)$.

4) Determine the minimum value of

$$F(x, y, z) = ax^m + by^m + cz^p + \frac{d}{x^r y^s z^t}$$

by generalizing the result from the analogous two-variable problem discussed above.

5) Another important sub-class of problems among bivariate polynomials are the quadric surfaces

$$f(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + Fz,$$

where the capital letters denote real constants, and where A and C have the same sign. If the B term is zero, then extrema can be found by completing the square procedure in both x and y. The method for solution when B is non-zero is also elementary, but it is less familiar. To illustrate the method with an example, consider the function

$$f(x, y) = 4x^2 + 16xy + 25y^2 - 24x - 30y + 60.$$

This is an elliptic paraboloid in which level curves of the form $f(x, y) = c$ are ellipses whose major and minor axes are tilted about 18° clockwise from the coordinate axes. This function has a minimum value and, to find it, we complete the square using the x^2 and the xy terms. That is,

$$f(x, y) = 4\left(x^2 + 4xy + 4y^2\right) - 16y^2 + 25y^2 - 24x - 30y + 60.$$

Letting $w = x + 2y$ the function can be expressed as

$$g(w, y) = 4(w - 3)^2 + 9(y + 1)^2 + 15.$$

Then the minimum value is 15 and occurs at $(w, y) = (3, -1)$, that is, at $(x, y) = (5, -1)$. This solution could have also been found by completing the square using the y^2 and the xy terms.

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Creating constrained optimisation problems

A variety of multivariable optimisation problems can be created that are generalizations of the problems that we have considered. One method of doing so is to consider a non-singular matrix $A = [a_{i,j}]$ of real entries. Then choose a vector $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ having positive integral components, and let $\mathbf{b} = \langle b_1, b_2, b_3 \rangle = A\mathbf{w}$. If c_k for $k = 1, 2, 3$ are positive real values, then the function

$$f(x, y, z) = \sum_{k=1}^{3} c_k x^{a_{1,k}} y^{a_{2,k}} z^{a_{3,k}},$$

subject to the constraint $x^{b_1} y^{b_2} z^{b_3} = R_0$ has an absolute minimum which occurs at exactly one point in the first octant.

To see why this is true, first let

$$S_w = w_1 + w_2 + w_3.$$

Then expressing f in terms of $\mathbf{w}$ and using the AM-GM inequality yields

$$\sum_{k=1}^{3} \frac{w_k c_k x^{a_{1,k}} y^{a_{2,k}} z^{a_{3,k}}}{w_k} \geqslant S_w \left[\prod_{k=1}^{3} \left(\frac{c_k x^{a_{1,k}} y^{a_{2,k}} z^{a_{3,k}}}{w_k} \right)^{w_k} \right]^{1/S_w}.$$

Since $A\mathbf{w} = \mathbf{b}$ the product on the right can be simplified to

$$x^{b_1} y^{b_2} z^{b_3} \prod_{k=1}^{3} \left(\frac{c_k}{w_k} \right)^{w_k} = R_0 \prod_{k=1}^{3} \left(\frac{c_k}{w_k} \right)^{w_k}.$$

Let $C = \prod_{k=1}^{3} \left(\frac{c_k}{w_k} \right)^{w_k}$, then the minimum value of f is $S_w (R_0 C)^{1/S_w}$. The point for which equality in the AM-GM holds occurs is when

$$\frac{c_1 x^{a_{1,1}} y^{a_{2,1}} z^{a_{3,1}}}{w_1} = \frac{c_2 x^{a_{1,2}} y^{a_{2,2}} z^{a_{3,2}}}{w_2} = \frac{c_3 x^{a_{1,3}} y^{a_{2,3}} x^{a_{3,3}}}{w_3},$$

and $x^{b_1} y^{b_2} z^{b_3} = R_0$. This non-linear system has a unique solution because the matrix A is invertible. To prove this fact, let β be the common value of the expressions $c_3 x^{a_{1,k}} y^{a_{3,2}} z^{a_{3,k}} / w_k$ at the desired domain value. Then

$$\beta^{w_1} \beta^{w_2} \beta^{w_3} = \beta^{S_w} = R_0 C,$$

and, consequently,

$$x^{a_{1,k}} y^{a_{3,2}} z^{a_{3,k}} = \frac{w_k (R_0 C)^{1/S_w}}{c_k}.$$

Let $L_k = \ln\left(w_k (R_0 C)^{1/S_w} / c_k \right)$, then, after taking the logarithm of both sides of the above equation, we have

$$A \langle \ln(x), \ln(y), \ln(z) \rangle^{\mathrm{T}} = \langle L_1, L_2, L_3 \rangle^{\mathrm{T}}.$$

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Thus the unique point where the function is a minimum is

$$\langle x, y, z\rangle = \left\langle \exp\left(A^{-1}\mathbf{L}\right)_1, \exp\left(A^{-1}\mathbf{L}\right)_2, \exp\left(A^{-1}\mathbf{L}\right)_3 \right\rangle,$$

where $\left(A^{-1}\mathbf{L}\right)_k$ is the k th entry of the vector $A^{-1}\mathbf{L}$.

To generate a specific problem, we can start with a non-singular matrix such as

$$A = \begin{pmatrix} 2 & 4 & 1 \\ 3 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

Then, as discussed above, one possible corresponding function is

$$F(x, y, z) = x^2y^3z + x^4y^2z^3 + 4xy^3z^2.$$

By using $\mathbf{w} = \langle 1,1,1\rangle$, we obtain $A\mathbf{w} = \mathbf{b} = \langle 7,8,6\rangle$. Thus when combining F with any constraint of the form $x^7y^8z^6 = R_0$, it will possess a minimum value at a particular point. With some more experimentation, we can discover that with an $R_0 = 2$ the function F has a minimum value of 6 at $\left(2, 1, \frac{1}{2}\right)$.

Exercise: Prove that the function $g(x, y, z) = x^{b_1}y^{b_2}z^{b_3}$ subject to the constraint

$$\sum_{j=1}^{3} c_k x^{a_{1,j}} y^{a_{2,j}} z^{a_{3,j}} = S_0$$

has one absolute maximum which occurs at exactly one point in the first octant. (In this problem the parameters here have the same properties and conditions as in the previous problem.)

Questions for further study

Since the techniques discussed here were limited primarily to the AM-GM inequality, it may be instructive for some readers to see some problems that can be solved with other well-known inequalities. Here are several large *classes* of functions with extrema that can be found with inequalities. In each problem below the name of an inequality that can be used to solve the problem is included. Readers unfamiliar with these inequalities may consult the references below to find additional information that can shed light on these and other questions.

1) The Cauchy-Swartz inequality can be used to find the maximum value of the function $f(x, y, z) = ax + by + cz$ subject to $x^2 + y^2 + z^2 = S_0$.
2) With the aid of Jensen's inequality show that

$$\frac{x}{ay + bz} + \frac{y}{az + bx} + \frac{z}{ax + by} \geqslant \frac{3}{a + b},$$

and find the point (x, y, z) in terms of the parameters a and b such that equality holds.

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3) By appealing to Holder's inequality demonstrate that

$$\frac{\left(x^p + y^p + z^p\right)^{1/p}\left(x^q + y^q + z^q\right)^{1/q}}{xy + xz + yz} \geqslant 1,$$

where p and q are positive real numbers greater than 1, and $\frac{1}{p} + \frac{1}{q} = 1$.

4) Use the power mean inequality to find an extreme value of

$$f(x, y, z) = x^n + y^n + z^n,$$

subject to $g(x, y, z) = x^m + y^m + x^m = S_0$.

5) Show that the function

$$\frac{x^m}{y^n + z^p} + \frac{y^n}{z^p + x^m} + \frac{z^p}{x^m + y^n}$$

has a local minimum value of $\frac{3}{2}$ at the point (1, 1, 1). (Here p is a positive integer.) This problem can be solved with AM-GM, Cauchy or a number of other methods. However, it can be solved very efficiently with the less well-known Nesbitt's inequality.

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