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WARING LENGTH TWO FOR THE SUPERINVOLUTIVE L'VOV–KAPLANSKY COUNTEREXAMPLE ON UPPER TRIANGULAR MATRICES

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Abstract

Let F be an algebraically closed field of characteristic zero. Campedel, Fagundes and Ioppolo ['Upper triangular matrices with superinvolution: identities and images of multilinear polynomials', *Bull. Braz. Math. Soc. (N.S.)* **57** (2026), Article no. 27] recently established a qualitative break from the L'vov–Kaplansky conjecture by proving that the multilinear $*$-polynomial $f(y^+, z^+) = y^+z^+$ has a nonlinear image on the upper triangular matrix algebra $A_n = \mathrm{UT}_n(F)$ $(n \geq 4)$ under the alternating elementary $\mathbb{Z}_2$-grading and super-reflection superinvolution. We provide the strict quantitative counterpart to this phenomenon by determining the exact Waring length of the counterexample in the first nontrivial dimensions. For $n = 4$ and $n = 5$, we prove that the linear span of the image is the entire odd component $(A_n)_1$ and that the additive defect is strictly minimal, with Waring length 2. Our proof is constructive: we compute the products $(A_n)_0^+(A_n)_1^+$ via the orbit structure of matrix units and provide explicit two-value decompositions for arbitrary odd upper triangular matrices.

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1. Introduction

The L'vov–Kaplansky problem belongs to the broader study of polynomial images in associative algebras. If $f \in F\langle X\rangle$ is a noncommutative polynomial and A is an F-algebra, then the set $f(A)$ of values of f need not be closed under addition for formal reasons. Its additive structure depends on the interaction between the polynomial, the Jacobson radical and the invariant subspaces forced by trace, commutator and central-polynomial phenomena. In the full matrix case, the L'vov–Kaplansky conjecture predicts that the image of every multilinear polynomial on $M_n(F)$ is a vector space. The historical background includes Shoda's theorem that trace-zero matrices over characteristic-zero fields are commutators, and the extension of this commutator theorem by Albert and Muckenhoupt to arbitrary fields [1, 17]. The polynomial-image problem is substantially understood for 2×2 matrices: Kanel-Belov, Malev and Rowen proved the standard four-alternative description over quadratically closed fields, and

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Malev obtained extensions over arbitrary fields [13, 14]. The Waring viewpoint asks a finer additive question. When $f(A)$ is not already a vector space, how many polynomial values must be added to recover its linear span? This point of view was developed for noncommutative polynomial images by Brešar and, in the full matrix setting, by Brešar and Šemrl [3, 4].

The upper triangular matrix algebras $\mathrm{UT}_n(F)$ form a natural testing ground because their radical filtration makes polynomial images visible through superdiagonal constraints. The ordinary multilinear analogue of the L'vov–Kaplansky conjecture for $\mathrm{UT}_n(F)$ over infinite fields was proved by Gargate and de Mello, who showed that such images are powers of the Jacobson radical determined by a commutator-degree invariant [10]. In parallel, graded polynomial identities and graded images on upper triangular matrices have been studied beginning with the graded-identity theory of Di Vincenzo, Koshlukov and Valenti, and continuing with the image classifications of Fagundes and Koshlukov, and of Fagundes for graded $*$-polynomials in small dimensions [7–9]. Waring-type questions for ordinary upper triangular matrix algebras have also been investigated. Panja and Prasad introduced a general additive-width problem for polynomial images on triangular algebras, and Chen and Wang subsequently solved the corresponding Waring problem in that setting [6, 15]. These results show that linear-image theorems and Waring-width theorems are related, but not equivalent; the latter retain information lost after taking the linear span.

The superalgebraic setting adds a further layer of structure. Superalgebras, and their Lie and Jordan superalgebra shadows, go back to foundational work of Kac and to the Jordan-superalgebra structure theory of Racine and Zelmanov [12, 16]. In PI-theory, Aljadeff, Giambruno and Karasik showed that identities with involution are controlled by Grassmann envelopes of finite-dimensional superalgebras with superinvolution, thereby making superinvolutions a central object rather than an auxiliary decoration [2]. For upper triangular matrix superalgebras, Ioppolo and Martino classified superinvolutions and described the $*$-identities of $\mathrm{UT}_2(F)$ and $\mathrm{UT}_3(F)$ with superinvolution [11]. Campedel, Fagundes and Ioppolo [5] recently investigated the algebra of upper triangular matrices endowed with the alternating elementary grading and the super-reflection superinvolution. They determined the T_2^*-ideal of $\mathrm{UT}_4(F)$, computed the corresponding $*$-codimensions and proved that for $n \geq 4$, the multilinear $*$-polynomial $f(y^+, z^+) = y^+z^+$ possesses a nonlinear image. Their analysis further demonstrates that while e_{12} and e_{23} are individually attainable as values of f, their sum $e_{12} + e_{23}$ is not. This establishes a failure of the L'vov–Kaplansky property, but the precise additive defect of the image remains unresolved. We address that problem by determining the exact Waring length—the minimal number of summands required to recover the linear hull—in the foundational dimensions $n = 4$ and $n = 5$.

Let $A_n = \mathrm{UT}_n(F)$ have the alternating elementary grading and the super-reflection superinvolution, and let $(A_n)_1$ denote its odd homogeneous component. For $n = 4, 5$, we prove

$$\mathrm{span}_F f(A_n) = (A_n)_1, \quad (A_n)_1 = f(A_n) + f(A_n), \quad f(A_n) \neq (A_n)_1.$$

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Equivalently,

$$\mathrm{wl}_{A_n}(y^+z^+) = 2 \quad \text{for } n = 4, 5.$$

Our proof is constructive: we compute the products $(A_n)_0^+(A_n)_1^+$ explicitly and provide two-value decompositions for every element of the odd component. Methodologically, rather than relying on full identity bases which grow rapidly in complexity, we use the orbit structure of the super-reflection acting on matrix units. This structural approach effectively isolates the multiplicative compatibility conditions between adjacent superdiagonals, revealing precisely why the image fails to be a vector space yet saturates additively in two steps. Thus, the additive defect of the first superinvolutive L'vov–Kaplansky counterexample is shown to be minimal in dimensions four and five.

REMARK 1.1. These dimensional results are of foundational significance: they represent the first instances where the superinvolutive counterexample emerges, allowing us to isolate the precise mechanisms of the obstruction. While the Waring-length problem for $n \geq 6$ remains open, our constructive approach provides the necessary structural model to address the increasing complexity of higher dimensions. Specifically, as n grows, the radical filtration couples multiple orbits of matrix units across the super-reflection, generating an expanding system of row-compatibility equations. By resolving the minimal cases, we establish the baseline additive defect and identify the core algebraic constraints that must be generalised for the broader $n \geq 6$ problem.

The paper is organised as follows. Section 2 recalls the notation on superalgebras, superinvolutions, multilinear $*$-polynomials and Waring length. Section 3 proves the exact Waring-length theorem for $\mathrm{UT}_4(F)$. Section 4 proves the corresponding theorem for $\mathrm{UT}_5(F)$. Finally, Section 5 records consequences and open questions.

2. Preliminaries

This section fixes the notation used throughout the paper and recalls the superinvolutive objects needed for the computations. The precise decomposition into symmetric and skew homogeneous components is essential because the polynomial y^+z^+ is evaluated only on $A_0^+ \times A_1^+$.

Let F be an algebraically closed field of characteristic zero. A superalgebra is an associative F-algebra A with a vector space decomposition

$$A = A_0 \oplus A_1$$

such that $A_iA_j \subseteq A_{i+j}$, where indices are taken modulo 2. A homogeneous element $a \in A_i$ has degree $|a| = i$. A superinvolution on A is an F-linear map $* : A \to A$ preserving the grading, and satisfying $(a^*)^* = a$ and

$$(ab)^* = (-1)^{|a||b|}b^*a^*$$

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for homogeneous $a, b \in A$. Since char$F = 0$, every superalgebra with superinvolution decomposes as

$$A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-,$$

where

$$A_i^+ = \{a \in A_i \mid a^* = a\} \quad \text{and} \quad A_i^- = \{a \in A_i \mid a^* = -a\}.$$

Let $F\langle Y \cup Z, *\rangle$ denote the free superalgebra with superinvolution, where variables from Y are even and variables from Z are odd. We write y^+, y^-, z^+ and z^- for even symmetric, even skew, odd symmetric and odd skew variables, respectively. If A is a superalgebra with superinvolution, a polynomial

$$f(y_1^+, \dots, y_r^+, y_1^-, \dots, y_s^-, z_1^+, \dots, z_t^+, z_1^-, \dots, z_u^-)$$

is evaluated by substituting elements of A_0^+, A_0^-, A_1^+ and A_1^- in the corresponding variables. Its image is the set of all such values. A $*$-identity is a polynomial whose image is $\{0\}$. The set of all $*$-identities of A is a T_2^*-ideal, meaning a two-sided ideal invariant under graded endomorphisms commuting with the formal superinvolution.

For $n \geq 2$, let $A_n = \mathrm{UT}_n(F)$ and let e_{ij} denote the usual matrix unit, with $1 \leq i \leq j \leq n$. We endow A_n with the alternating elementary $\mathbb{Z}_2$-grading, where the even homogeneous component $(A_n)_0$ (respectively, the odd homogeneous component $(A_n)_1$) consists of upper triangular matrices whose nonzero entries lie on even (respectively, odd) upper diagonals. Equivalently, for matrix units,

$$e_{ij} \in (A_n)_0 \iff j - i \equiv 0 \pmod 2, \quad e_{ij} \in (A_n)_1 \iff j - i \equiv 1 \pmod 2.$$

Put $\rho(i) = n + 1 - i$. The ordinary reflection sends e_{ij} to $e_{\rho(j),\rho(i)}$. On the alternating grading, the super-reflection has the following effect on matrix units:

$$e^*_{i,i+k} = \varepsilon_k e_{n+1-i-k,n+1-i}, \quad \varepsilon_k = \begin{cases} 1, & k \equiv 0, 1 \pmod 4, \\ -1, & k \equiv 2, 3 \pmod 4. \end{cases}$$

This is the matrix-unit form of the super-reflection obtained by composing the reflection with the canonical sign automorphism described by Ioppolo and Martino [11]. It will be used only in dimensions four and five, where the symmetric subspaces are written explicitly.

DEFINITION 2.1. Let S be a subset of a vector space over F. For $r \geq 1$, set

$$rS = \{s_1 + \cdots + s_r \mid s_i \in S\}.$$

If $0 \in S$, then membership in rS is the same as being a sum of at most r elements of S. Let f be a polynomial on an algebra A. If $f(A) \neq \{0\}$, the Waring length of f on A is

$$\mathrm{wl}_A(f) = \min\{r \geq 1 \mid \mathrm{span}_F f(A) \subseteq r f(A)\},$$

provided such an r exists. If $f(A) = \{0\}$, we set $\mathrm{wl}_A(f) = 0$.

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The polynomial studied in this paper is

$$f(y^+, z^+) = y^+ z^+.$$

Since y^+ is evaluated in A_0^+ and z^+ is evaluated in A_1^+, all values of f lie in A_1. Thus,

$$f(A_n) = (A_n)_0^+ (A_n)_1^+ := \{ab \mid a \in (A_n)_0^+,\ b \in (A_n)_1^+\}.$$

The problem is therefore a factorisation problem inside the odd homogeneous component.

REMARK 2.2. The formulation of the additive problem addressed below relies on the qualitative nonlinearity theorem of Campedel, Fagundes and Ioppolo [5, Theorem 20]. They proved that for $n \geq 4$, the image of $f(y^+, z^+) = y^+ z^+$ on $\mathrm{UT}_n(F)$ with the alternating grading and super-reflection is not a vector space. Specifically, the obstruction arises because e_{12} and e_{23} occur as values of f, whereas $e_{12} + e_{23}$ cannot occur as a single value. Consequently, when $\mathrm{span}_F f(A_n)$ is nonzero, the Waring length is strictly greater than one. To determine the exact length, it remains to compute the linear hull and establish a sharp upper bound by proving that two values always suffice. The subsequent sections carry this out directly for $n = 4$ and $n = 5$ by evaluating the product $(A_n)_0^+ (A_n)_1^+$ explicitly on matrix units.

3. The four-dimensional computation

This section gives the exact Waring length in the first dimension where the super-reflection counterexample occurs. We first identify the symmetric even and symmetric odd variables, then compute the image of $y^+ z^+$, and finally prove that its additive square is the whole odd component.

Let $A_4 = \mathrm{UT}_4(F)$ be endowed with the alternating elementary grading and the super-reflection $*$. From the formula for the super-reflection on matrix units,

$$(A_4)_0^+ = \mathrm{span}_F\{d_1, d_2, q\}, \quad d_1 = e_{11} + e_{44}, \quad d_2 = e_{22} + e_{33}, \quad q = e_{13} - e_{24},$$

and

$$(A_4)_1^+ = \mathrm{span}_F\{p, r\}, \quad p = e_{12} + e_{34}, \quad r = e_{23}.$$

The odd component itself is $(A_4)_1 = \mathrm{span}_F\{e_{12}, e_{23}, e_{34}, e_{14}\}$.

LEMMA 3.1. *Let $a = \alpha d_1 + \beta d_2 + \gamma q \in (A_4)_0^+$, $b = up + vr \in (A_4)_1^+$. Then,*

$$ab = \alpha u e_{12} + \beta v e_{23} + \beta u e_{34} + \gamma u e_{14}.$$

Consequently, $f(A_4) = \{\alpha u e_{12} + \beta v e_{23} + \beta u e_{34} + \gamma u e_{14} \mid \alpha, \beta, \gamma, u, v \in F\}$.

PROOF. We use $e_{ij} e_{kl} = \delta_{jk} e_{il}$. First,

$$d_1 p = (e_{11} + e_{44})(e_{12} + e_{34}) = e_{12},$$

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because $e_{11}e_{12} = e_{12}$ and the other three products have incompatible middle indices. In addition,

$$d_1 r = (e_{11} + e_{44})e_{23} = 0.$$

Next,

$$d_2 p = (e_{22} + e_{33})(e_{12} + e_{34}) = e_{34}, \quad d_2 r = (e_{22} + e_{33})e_{23} = e_{23}.$$

Indeed, in the first equality, the only nonzero product is $e_{33}e_{34} = e_{34}$ and in the second equality, the only nonzero product is $e_{22}e_{23} = e_{23}$. Finally,

$$qp = (e_{13} - e_{24})(e_{12} + e_{34}) = e_{14}, \quad qr = (e_{13} - e_{24})e_{23} = 0,$$

because $e_{13}e_{34} = e_{14}$ is the only nonzero product appearing in qp, while neither $e_{13}e_{23}$ nor $e_{24}e_{23}$ is nonzero. Therefore,

$$\begin{aligned} ab &= (\alpha d_1 + \beta d_2 + \gamma q)(up + vr) \\ &= \alpha u d_1 p + \alpha v d_1 r + \beta u d_2 p + \beta v d_2 r + \gamma u q p + \gamma v q r \\ &= \alpha u e_{12} + \beta v e_{23} + \beta u e_{34} + \gamma u e_{14}. \end{aligned}$$

The displayed description of $f(A_4)$ follows because every value of f is exactly such a product and every choice of the five scalars determines elements $a \in (A_4)_0^+$ and $b \in (A_4)_1^+$. □

PROPOSITION 3.2. *For A_4 as above,* $\operatorname{span}_F f(A_4) = (A_4)_1$.

PROOF. Lemma 3.1 shows that every value of f lies in $(A_4)_1$; hence,

$$\operatorname{span}_F f(A_4) \subseteq (A_4)_1.$$

We prove the reverse inclusion by realising each basis element of $(A_4)_1$ as a value. Taking $\alpha = 1$, $u = 1$ and $\beta = \gamma = v = 0$ in Lemma 3.1 gives e_{12}. Taking $\beta = 1$, $v = 1$ and $\alpha = \gamma = u = 0$ gives e_{23}. Taking $\beta = 1$, $u = 1$ and $\alpha = \gamma = v = 0$ gives e_{34}. Taking $\gamma = 1$, $u = 1$ and $\alpha = \beta = v = 0$ gives e_{14}. Hence, every element of the basis $\{e_{12}, e_{23}, e_{34}, e_{14}\}$ belongs to $f(A_4)$, so $(A_4)_1 \subseteq \operatorname{span}_F f(A_4)$. □

PROPOSITION 3.3. *Every element of $(A_4)_1$ is a sum of two values of f. Therefore,*

$$(A_4)_1 = f(A_4) + f(A_4).$$

PROOF. Let

$$X = x_{12}e_{12} + x_{23}e_{23} + x_{34}e_{34} + x_{14}e_{14} \in (A_4)_1.$$

Set

$$X_1 = (x_{12}d_1 + x_{34}d_2 + x_{14}q)p, \quad X_2 = d_2(x_{23}r).$$

Since $x_{12}d_1 + x_{34}d_2 + x_{14}q \in (A_4)_0^+$, $p \in (A_4)_1^+$, $d_2 \in (A_4)_0^+$ and $x_{23}r \in (A_4)_1^+$, both X_1 and X_2 are values of f. By Lemma 3.1,

$$X_1 = x_{12}e_{12} + x_{34}e_{34} + x_{14}e_{14}, \quad X_2 = x_{23}e_{23}.$$

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Adding these two equalities gives $X = X_1 + X_2$. Thus, $(A_4)_1 \subseteq f(A_4) + f(A_4)$. The reverse inclusion is automatic because $f(A_4) \subseteq (A_4)_1$ and $(A_4)_1$ is a vector space. □

PROPOSITION 3.4. *The image $f(A_4)$ is not a vector space. More precisely,*

$$e_{12} \in f(A_4), \quad e_{23} \in f(A_4), \quad e_{12} + e_{23} \notin f(A_4).$$

PROOF. The memberships $e_{12}, e_{23} \in f(A_4)$ were proved in Proposition 3.2. Suppose, towards a contradiction, that $e_{12} + e_{23} \in f(A_4)$. By Lemma 3.1, there exist $\alpha, \beta, \gamma, u, v \in F$ such that

$$e_{12} + e_{23} = \alpha u e_{12} + \beta v e_{23} + \beta u e_{34} + \gamma u e_{14}.$$

The matrix units $e_{12}, e_{23}, e_{34}, e_{14}$ are linearly independent, so comparison of coefficients gives

$$\alpha u = 1, \quad \beta v = 1, \quad \beta u = 0, \quad \gamma u = 0.$$

From $\alpha u = 1$, we get $u \neq 0$ and from $\beta v = 1$, we get $\beta \neq 0$. Hence, $\beta u \neq 0$, which contradicts $\beta u = 0$. Therefore, $e_{12} + e_{23} \notin f(A_4)$ and $f(A_4)$ is not a vector space. □

THEOREM 3.5. *Let $A_4 = \mathrm{UT}_4(F)$ be endowed with the alternating elementary grading and the super-reflection superinvolution. For $f(y^+, z^+) = y^+ z^+$,*

$$\operatorname{span}_F f(A_4) = (A_4)_1 \quad \textit{and} \quad \mathrm{wl}_{A_4}(f) = 2.$$

PROOF. Proposition 3.2 gives $\operatorname{span}_F f(A_4) = (A_4)_1$. Proposition 3.3 gives $(A_4)_1 \subseteq 2f(A_4)$, so $\mathrm{wl}_{A_4}(f) \leq 2$. Proposition 3.4 shows that $f(A_4)$ is not equal to its linear span, so $\mathrm{wl}_{A_4}(f) \neq 1$. Since $e_{12} \in f(A_4)$, the image is nonzero. Hence, $\mathrm{wl}_{A_4}(f) = 2$. □

4. The five-dimensional computation

This section treats the next alternating super-reflection algebra. The product formula has additional terms on the third superdiagonal, but the additive width remains two. The proof separates the scalar cases in the two-value decomposition so that every division is by a scalar already assumed to be nonzero.

Let $A_5 = \mathrm{UT}_5(F)$ be endowed with the alternating elementary grading and the super-reflection $*$. The symmetric even component is $(A_5)_0^+ = \operatorname{span}_F\{D_1, D_2, D_3, Q, R\}$, where

$$D_1 = e_{11} + e_{55}, \quad D_2 = e_{22} + e_{44}, \quad D_3 = e_{33}, \quad Q = e_{13} - e_{35}, \quad R = e_{15}.$$

The symmetric odd component is $(A_5)_1^+ = \operatorname{span}_F\{P, T, H\}$, where

$$P = e_{12} + e_{45}, \quad T = e_{23} + e_{34}, \quad H = e_{14} - e_{25}.$$

Finally, $(A_5)_1 = \operatorname{span}_F\{e_{12}, e_{23}, e_{34}, e_{45}, e_{14}, e_{25}\}$.

LEMMA 4.1. *Let*

$$a = \alpha D_1 + \beta D_2 + \gamma D_3 + \delta Q + \varepsilon R \in (A_5)_0^+$$

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and

$$b = uP + vT + wH \in (A_5)_1^+.$$

Then, $ab = \alpha u e_{12} + \beta v e_{23} + \gamma v e_{34} + \beta u e_{45} + (\alpha w + \delta v) e_{14} - \beta w e_{25}$.

Consequently, $f(A_5)$ *is the set of all matrices of the displayed form as* $\alpha, \beta, \gamma, \delta, \varepsilon, u, v, w$ *vary in* F. *The scalar* ε *does not affect the product.*

PROOF. We again use $e_{ij}e_{kl} = \delta_{jk}e_{il}$. First,

$$D_1P = (e_{11} + e_{55})(e_{12} + e_{45}) = e_{12}, \quad D_1T = (e_{11} + e_{55})(e_{23} + e_{34}) = 0$$

and

$$D_1H = (e_{11} + e_{55})(e_{14} - e_{25}) = e_{14}.$$

The term e_{55} produces no nonzero product in these three computations because no nonzero matrix unit in P, T, H begins in row 5.

For $D_2 = e_{22} + e_{44}$,

$$D_2P = (e_{22} + e_{44})(e_{12} + e_{45}) = e_{45},$$
$$D_2T = (e_{22} + e_{44})(e_{23} + e_{34}) = e_{23}$$

and

$$D_2H = (e_{22} + e_{44})(e_{14} - e_{25}) = -e_{25}.$$

Indeed, the nonzero products are $e_{44}e_{45} = e_{45}$, $e_{22}e_{23} = e_{23}$ and $e_{22}e_{25} = e_{25}$, with the last product carrying the minus sign from $H = e_{14} - e_{25}$.

For $D_3 = e_{33}$,

$$D_3P = 0, \quad D_3T = e_{34}, \quad D_3H = 0,$$

and for $Q = e_{13} - e_{35}$,

$$QP = 0, \quad QT = e_{14}, \quad QH = 0.$$

The equality $QT = e_{14}$ follows because $(e_{13} - e_{35})(e_{23} + e_{34}) = e_{13}e_{34} = e_{14}$, and all other products in that expression have incompatible middle indices. Finally,

$$RP = RT = RH = 0,$$

because $R = e_{15}$ has terminal index 5, while all nonzero matrix units in P, T, H begin in rows $1, 2, 3, 4$.

Multiplying

$$a = \alpha D_1 + \beta D_2 + \gamma D_3 + \delta Q + \varepsilon R$$

by

$$b = uP + vT + wH$$

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and substituting the products just computed gives

$$\begin{aligned} ab &= \alpha u D_1 P + \alpha v D_1 T + \alpha w D_1 H + \beta u D_2 P + \beta v D_2 T + \beta w D_2 H \\ &\quad + \gamma u D_3 P + \gamma v D_3 T + \gamma w D_3 H + \delta u Q P + \delta v Q T + \delta w Q H \\ &\quad + \varepsilon u R P + \varepsilon v R T + \varepsilon w R H \\ &= \alpha u e_{12} + \beta v e_{23} + \gamma v e_{34} + \beta u e_{45} + (\alpha w + \delta v) e_{14} - \beta w e_{25}. \end{aligned}$$

This proves the formula and the description of $f(A_5)$. □

PROPOSITION 4.2. *For A_5 as above,* $\operatorname{span}_F f(A_5) = (A_5)_1$.

PROOF. Lemma 4.1 shows that $f(A_5) \subseteq (A_5)_1$, and hence, $\operatorname{span}_F f(A_5) \subseteq (A_5)_1$.

We prove the reverse inclusion by realising the six displayed basis elements of $(A_5)_1$ as values. Taking $\alpha = 1$, $u = 1$, and all other relevant parameters equal to 0, gives e_{12}. Taking $\beta = 1$, $v = 1$, and all other relevant parameters equal to 0, gives e_{23}. Taking $\gamma = 1$, $v = 1$, and all other relevant parameters equal to 0, gives e_{34}. Taking $\beta = 1$, $u = 1$, and all other relevant parameters equal to 0, gives e_{45}. Taking $\delta = 1$, $v = 1$, and all other relevant parameters equal to 0, gives e_{14}. Finally, taking $\beta = -1$, $w = 1$, and all other relevant parameters equal to 0, gives e_{25}. Thus, every element of the basis $\{e_{12}, e_{23}, e_{34}, e_{45}, e_{14}, e_{25}\}$ belongs to $f(A_5)$ and the reverse inclusion follows. □

PROPOSITION 4.3. *Every element of $(A_5)_1$ is a sum of two values of f. Therefore,*

$$(A_5)_1 = f(A_5) + f(A_5).$$

PROOF. Let $X = x_{12}e_{12} + x_{23}e_{23} + x_{34}e_{34} + x_{45}e_{45} + x_{14}e_{14} + x_{25}e_{25} \in (A_5)_1$. We split into three disjoint and exhaustive cases. The only divisions below are by scalars explicitly assumed to be nonzero.

First, assume $x_{23} \neq 0$. Define

$$X^{(1)} = (x_{12}D_1 + x_{45}D_2)P \quad \text{and} \quad X^{(2)} = (x_{23}D_2 + x_{34}D_3 + x_{14}Q)\Big(T - \frac{x_{25}}{x_{23}}H\Big).$$

Both are values of f. By Lemma 4.1,

$$X^{(1)} = x_{12}e_{12} + x_{45}e_{45} \quad \text{and} \quad X^{(2)} = x_{23}e_{23} + x_{34}e_{34} + x_{14}e_{14} + x_{25}e_{25}.$$

Thus, $X = X^{(1)} + X^{(2)}$.

Next, assume $x_{23} = 0$ and $x_{45} \neq 0$. Define

$$X^{(1)} = (x_{12}D_1 + x_{45}D_2)\Big(P - \frac{x_{25}}{x_{45}}H\Big) \quad \text{and} \quad X^{(2)} = \Big(x_{34}D_3 + \Big(x_{14} + \frac{x_{12}x_{25}}{x_{45}}\Big)Q\Big)T.$$

Again, both summands are values of f. Lemma 4.1 gives

$$X^{(1)} = x_{12}e_{12} + x_{45}e_{45} - \frac{x_{12}x_{25}}{x_{45}}e_{14} + x_{25}e_{25} \quad \text{and} \quad X^{(2)} = x_{34}e_{34} + \Big(x_{14} + \frac{x_{12}x_{25}}{x_{45}}\Big)e_{14}.$$

Since $x_{23} = 0$, adding these two equalities gives $X = X^{(1)} + X^{(2)}$.

Finally, assume $x_{23} = 0$ and $x_{45} = 0$. Define

$$X^{(1)} = (x_{12}D_1 + D_2 + x_{34}D_3 + x_{14}Q)(P + T) \quad \text{and} \quad X^{(2)} = D_2(-P - T - x_{25}H).$$

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These are values of f. By Lemma 4.1,

$$X^{(1)} = x_{12}e_{12} + e_{23} + x_{34}e_{34} + e_{45} + x_{14}e_{14} \quad \text{and} \quad X^{(2)} = -e_{23} - e_{45} + x_{25}e_{25}.$$

Since in the present case, $x_{23} = x_{45} = 0$, their sum is exactly X. Therefore, every element of $(A_5)_1$ belongs to $f(A_5) + f(A_5)$. The reverse inclusion follows from $f(A_5) \subseteq (A_5)_1$ and the fact that $(A_5)_1$ is a vector space. □

PROPOSITION 4.4. *The image $f(A_5)$ is not a vector space. More precisely,*

$$e_{12} \in f(A_5), \quad e_{23} \in f(A_5), \quad e_{12} + e_{23} \notin f(A_5).$$

PROOF. The elements e_{12} and e_{23} are values of f by Proposition 4.2. Suppose, towards a contradiction, that $e_{12} + e_{23} \in f(A_5)$. Lemma 4.1 gives scalars $\alpha, \beta, \gamma, \delta, \varepsilon, u, v, w \in F$ such that

$$e_{12} + e_{23} = \alpha u e_{12} + \beta v e_{23} + \gamma v e_{34} + \beta u e_{45} + (\alpha w + \delta v)e_{14} - \beta w e_{25}.$$

The matrix units $e_{12}, e_{23}, e_{34}, e_{45}, e_{14}, e_{25}$ are linearly independent, so comparison of the e_{12}, e_{23} and e_{45}-coefficients gives

$$\alpha u = 1, \quad \beta v = 1, \quad \beta u = 0.$$

From $\alpha u = 1$, we get $u \neq 0$ and from $\beta v = 1$, we get $\beta \neq 0$. Hence, $\beta u \neq 0$, which contradicts $\beta u = 0$. Therefore, $e_{12} + e_{23} \notin f(A_5)$ and $f(A_5)$ is not a vector space. □

THEOREM 4.5. *Let $A_5 = \mathrm{UT}_5(F)$ be endowed with the alternating elementary grading and the super-reflection superinvolution. For $f(y^+, z^+) = y^+z^+$,*

$$\operatorname{span}_F f(A_5) = (A_5)_1 \quad \textit{and} \quad \operatorname{wl}_{A_5}(f) = 2.$$

PROOF. Proposition 4.2 gives $\operatorname{span}_F f(A_5) = (A_5)_1$. Proposition 4.3 gives $(A_5)_1 \subseteq 2f(A_5)$, so $\operatorname{wl}_{A_5}(f) \leq 2$. Proposition 4.4 shows that $f(A_5)$ is not equal to its linear span, so $\operatorname{wl}_{A_5}(f) \neq 1$. Since $e_{12} \in f(A_5)$, the image is nonzero. Hence, $\operatorname{wl}_{A_5}(f) = 2$. □

5. Consequences and open questions

This section summarises the consequences of the two computations and indicates the next structural questions. The results show that the first superinvolutive counterexample to the linear-image property has the smallest possible nonlinear Waring length in dimensions four and five.

COROLLARY 5.1. *Let F be an algebraically closed field of characteristic zero. For $n = 4, 5$, let $A_n = \mathrm{UT}_n(F)$ be endowed with the alternating elementary $\mathbb{Z}_2$-grading and the super-reflection superinvolution. Then, the polynomial $f(y^+, z^+) = y^+z^+$ satisfies*

$$f(A_n) \neq \operatorname{span}_F f(A_n), \quad \operatorname{span}_F f(A_n) = f(A_n) + f(A_n).$$

Equivalently,

$$\operatorname{wl}_{A_n}(f) = 2.$$

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PROOF. The results for $n = 4$ and $n = 5$ follow directly from Theorems 3.5 and 4.5, respectively. □

The computations also identify the precise linear hull:

$$\operatorname{span}_F f(A_4) = (A_4)_1, \quad \operatorname{span}_F f(A_5) = (A_5)_1.$$

Thus, in the first two counterexample dimensions, the obstruction is not that the polynomial misses a homogeneous direction linearly, but rather, the obstruction is a multiplicative compatibility condition among adjacent first-superdiagonal entries. In dimension four, this condition is visible in the relation between the e_{23}- and e_{34}-coefficients in Lemma 3.1. In dimension five, it is visible in the simultaneous appearance of the e_{23}- and e_{45}-coefficients through the common parameter β in Lemma 4.1. In both cases, two summands allow these compatibility conditions to be separated.

QUESTION 5.2. Let $n \geq 6$, and let $A_n = \mathrm{UT}_n(F)$ have the alternating elementary grading and the super-reflection superinvolution. Is $\mathrm{wl}_{A_n}(y^+z^+) = 2$ for all $n \geq 4$?

The computations for $n = 4$ and $n = 5$ provide a structural model that clarifies the transition to higher dimensions. In these foundational cases, the interplay between even-symmetric orbits and odd-superdiagonal entries is manageable, and the elementary decompositions in Sections 3 and 4 suffice to cover all coefficients independently. However, a positive answer for $n \geq 6$ requires a uniform decomposition strategy capable of handling a higher degree of complexity: the radical filtration causes multiple even-symmetric orbits to map into the same odd superdiagonal entry.

This creates a coupled system of row-compatibility equations where the choice of scalars for one product restricts the possibilities for others. While our constructive method—decomposing elements via orbit-based scalar selection—remains valid, a full resolution for $n \geq 6$ requires a generalised existence proof for solutions to these over-determined systems. Thus, our $n = 4, 5$ results serve as the necessary baseline for identifying the obstruction patterns that persist and amplify as n increases.

QUESTION 5.3. For a fixed $n \geq 4$, can one classify the Waring lengths of all degree-two multilinear $*$-polynomials in one even and one odd variable on $\mathrm{UT}_n(F)$ with super-reflection?

This question serves as a natural intermediate step between the present work and a comprehensive multilinear Waring theory. Degree-two polynomials are sufficient to detect the failure of the L'vov–Kaplansky property while remaining tractable, avoiding the full normal-form complexity associated with arbitrary multilinear $*$-polynomials. A complete classification would require a systematic analysis of the products

$$A_i^{\epsilon} A_j^{\eta}, \quad i, j \in \{0, 1\},\ \epsilon, \eta \in \{+, -\},$$

and their additive widths within the corresponding homogeneous components.

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