Ergod. Th. & Dynam. Sys. (2004), **24**, 577–591
DOI: 10.1017/S014338570300052X
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Printed in the United Kingdom

The explosion of singular-hyperbolic attractors

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(*Received* 22 *October* 2002 *and accepted in revised form* 19 *September* 2003)

Abstract. A *singular-hyperbolic attractor* for vector fields is a partially hyperbolic attractor with singularities (that are hyperbolic) and volume expanding central direction. The geometric Lorenz attractor is the most representative example of a singular-hyperbolic attractor. In this paper, we prove that if Λ is a singular-hyperbolic attractor of a three-dimensional vector field X, then there is a neighborhood U of Λ in M such that every attractor *in* U of a C^r vector field C^r close to X is singular, i.e. it contains a singularity. With this result we prove the following corollaries. There are neighborhoods U of Λ (in M) and $\mathcal{U}$ of X (in the space of C^r vector fields) such that if n denotes the number of singularities of X in Λ, then $\#\{A \subset U : A \text{ is an attractor of } Y \in \mathcal{U}\} \leq n$. Every three-dimensional vector field C^r close to one exhibiting a singular-hyperbolic attractor has a singularity non-isolated in the non-wandering set. A singularity of a three-dimensional C^r vector field Y is stably non-isolated in the non-wandering set if it is the unique singularity of a singular-hyperbolic attractor of Y. These results generalize well-known properties of the geometric Lorenz attractor.

1. *Introduction*

A *singular-hyperbolic attractor* for vector fields is a partially hyperbolic attractor with singularities (which are hyperbolic) and volume expanding central direction [**MPP1**]. The geometric Lorenz attractor [**GW**] is the most representative example of a singular-hyperbolic attractor. In [**MPu**] we observed that singular-hyperbolic attractors are not robust in general. This motivates the study of the C^r perturbations of a singular-hyperbolic attractor, $r \geq 1$. In this paper, we prove that if Λ is a singular-hyperbolic attractor of a three-dimensional vector field X, then there is a neighborhood U of Λ in M such that every attractor *in* U of a C^r vector field C^r close to X is singular, i.e. it contains a singularity. With this result we prove the following corollaries. There are neighborhoods U of Λ (in M) and $\mathcal{U}$ of X (in the space of C^r vector fields) such that if n denotes the number of singularities of X in Λ, then $\#\{A \subset U : A \text{ is an attractor of } Y \in \mathcal{U}\} \leq n$. Every three-dimensional vector field C^r close to one exhibiting a singular-hyperbolic attractor has a

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singularity non-isolated in the non-wandering set. A singularity of a three-dimensional C^r vector field X is stably non-isolated in the non-wandering set if it is the unique singularity of a singular-hyperbolic attractor of X. These results generalize well-known properties of the Lorenz attractor.

1.1. *Statement of results.* Let M be a compact 3-manifold and let $\mathcal{X}^r(M)$ be the space of C^r vector fields in M endowed with the C^r topology, $r \geq 1$. Fix $X \in \mathcal{X}^r(M)$ and denote by X_t the flow of X, $t \in \mathbb{R}$.

The *non-wandering set* of X is the set $\Omega(X)$ of points $p \in M$ such that for every neighborhood U of p and every $T > 0$ there is $t > T$ such that $X_t(U) \cap U \neq \emptyset$. The ω-limit set of $p \in M$ is the set $\omega_X(p)$ formed by those $q \in M$ such that $q = \lim_{n\to\infty} X_{t_n}(p)$ for some sequence $t_n \to \infty$. We have $\omega_X(p) \subset \Omega(X)$ for every $p \in M$. We define the α-limit set $\alpha_X(p)$ of p in a similar way (replacing ∞ by $-\infty$). We also have $\alpha_X(p) \subset \Omega(X)$ for every p. The *positive limit set* $L^+(X)$ of X is the closure of $\bigcup_{p\in M} \omega_X(p)$. It is clear that $L^+(X) \subset \Omega(X)$.

Let Λ be an invariant set of X, i.e. $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$. We say that Λ is *transitive* if $\Lambda = \omega_X(p)$ for some $p \in \Lambda$. We say that Λ is *singular* if it contains a singularity of X. We say that Λ is *attracting* if $\Lambda = \bigcap_{t>0} X_t(U)$ for some compact neighborhood U of Λ. This neighborhood is called an *isolating block* of Λ. It is well known that the isolating block U can be chosen to be positively invariant, namely $X_t(U) \subset U$ for all $t > 0$. An *attractor* is a transitive attracting set. An attractor is *non-trivial* if it is not a closed orbit.

Next we state the main definitions of this work. First we denote by $m(L)$ the minimum norm of a linear operator L.

Definition 1.1. A compact invariant set Λ of X is *partially hyperbolic* if there is a continuous invariant splitting

$$T_\Lambda M = E^s \oplus E^c$$

such that the following properties hold for some positive constants C, λ:

(1) E^s is *contracting*, i.e.

$$\|DX_t(x)|E^s_x\| \leq Ce^{-\lambda t},$$

for all $x \in \Lambda$ and $t > 0$;

(2) E^s *dominates* E^c, i.e.

$$\frac{\|DX_t(x)|E^s_x\|}{m(DX_t(x)|E^c_x)} \leq Ce^{-\lambda t},$$

for all $x \in \Lambda$ and $t > 0$.

We say that the central direction E^c of Λ is *volume expanding* if

$$|\det DX_t(x)|E^c_x| \geq C^{-1}e^{\lambda t},$$

for all $x \in \Lambda$ and $t > 0$. Here $\det DX_t(x)|E^c_x$ denotes the jacobian of $DX_t(x)$ along E^c_x.

Recall that a singularity of a vector field is hyperbolic if the eigenvalues of its linear part have non-zero real part.

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Definition 1.2. A *singular-hyperbolic set* is a partially hyperbolic set with singularities (which are hyperbolic) and volume expanding central direction. An attractor is *singular-hyperbolic* if it is also a singular-hyperbolic set.

The most representative example of a singular-hyperbolic attractor is the *geometric Lorenz attractor* [**GW**].

Our main result is the following.

THEOREM A. *Let Λ be a singular-hyperbolic attractor of a three-dimensional C^r vector fields X, $r \geq 1$. Then, there is a neighborhood U of Λ such that every attractor in U of a C^r vector field C^r close to X is singular.*

Let us present some corollaries of Theorem A.

COROLLARY 1.3. *Let Λ be a singular-hyperbolic attractor of a three-dimensional C^r vector field X, $r \geq 1$. Then, there are neighborhoods U of Λ (in M) and $\mathcal{U}$ of X (in $\mathcal{X}^r(M)$) such that if n denotes the number of singularities of X in Λ, then*

$$\#\{A \subset U : A \text{ is an attractor of } Y \in \mathcal{U}\} \leq n.$$

A singularity σ of a vector field X is *non-isolated in the non-wandering set* if $\Omega(X)\backslash\{\sigma\}$ is not closed in M.

COROLLARY 1.4. *Every three-dimensional C^r vector field close to one exhibiting a singular-hyperbolic attractor has a singularity non-isolated in the non-wandering set, $r \geq 1$.*

Recall that a subset of $\mathcal{X}^r(M)$ is *residual* if it contains a countable intersection of open-dense subsets of $\mathcal{X}^r(M)$.

COROLLARY 1.5. *There is a residual subset $\mathcal{R}$ of $\mathcal{X}^1(M)$ such that if $X \in \mathcal{R}$ exhibits a non-trivial singular attractor, then every C^1 vector field close to X exhibits a singularity non-isolated in the non-wandering set.*

It is well known that a hyperbolic singularity σ of a C^r vector field X has a continuation $\sigma(Y)$ for every C^r vector field Y close to X [**dMP**]. This motivates the following definition. A hyperbolic singularity σ of a C^1 vector field X is *C^r stably non-isolated in the non-wandering set* if $\Omega(Y) \setminus \{\sigma(Y)\}$ is not closed for every C^r vector field Y close to X. An example of a singularity stably non-isolated in the non-wandering set is that exhibited by the geometric Lorenz attractor. The following result is a generalization of this example. See [**MPa2**] for a sort of converse when $r = 1$.

COROLLARY 1.6. *Let σ be a hyperbolic singularity of a three-dimensional C^r vector field X, $r \geq 1$. If σ is the* unique *singularity of a singular-hyperbolic attractor of X, then σ is C^r stably non-isolated in the non-wandering set.*

1.2. *Outline of the proof.* We prove Theorem A by contradiction. Let Λ be a singular-hyperbolic attractor of a three-dimensional C^r vector field X, $r \geq 1$. Suppose that there is a sequence X^n of C^r vector fields, converging to X in the C^r topology, such that every X^n

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has a non-singular attractor A^n *arbitrarily close to* Λ (e.g. in an $1/n$-neighborhood of Λ). Of course, we can assume that A^n belongs to a fixed isolating block U of Λ for all n.

Definition 1.7. A compact invariant set H of X is *hyperbolic* if there are a continuous tangent bundle invariant decomposition $T_H M = E^s \oplus E^X \oplus E^u$ and positive constants C, λ such that:

- E^X is the vector field's direction over H;
- E^s is *contracting*, i.e.

$$\|DX_t(x)|E^s_x\| \leq Ce^{-\lambda t}$$

for all $x \in H$ and $t > 0$.

- E^u is *expanding*, i.e.

$$\|DX_{-t}(x)|E^u_x\| \leq Ce^{-\lambda t},$$

for all $x \in H$ and $t > 0$.

A closed orbit is hyperbolic if it is also a hyperbolic set. An attractor is hyperbolic if it is also a hyperbolic set. A hyperbolic attractor is *strange* if $E^u \neq 0$.

Observe that a hyperbolic attractor is strange if and only if it is non-trivial. It will follow from Lemma 2.2 that A^n is a hyperbolic strange attractor of X^n for all n. On the other hand, the following property is obvious since A^n is non-singular for all n:

$$A^n \text{ is not a singularity of } X^n \text{ for all } n. \tag{P}$$

In Lemma 3.2 we prove that the compact set sequence $\{A^n : n \in \mathbb{N}\}$ *accumulates on the singularities of* X (note that although the attractors A^n are *arbitrarily close to* Λ, it is still possible that such attractors stay *far from* $\mathrm{Sing}(X) \cap \Lambda$, see also Remark 3.1). Then we prove that there is n such that A^n contains a singularity of X^n. This is done as follows. By Lemma 3.2 we can fix $\sigma \in \mathrm{Sing}(X) \cap \mathrm{Cl}\left(\bigcup_n A^n\right)$. By analyzing the relative position of the stable and central direction of the unperturbed attractor close to σ (which depends continuously on the flow), we prove that A^n intersects the stable manifold of $\sigma(X^n)$, the continuation of σ. This easily implies that $\sigma(X^n) \in A^n$ proving that A^n contains a singularity for n large. Finally, we arrive at a contradiction as follows: *since* A^n *is a hyperbolic attractor we conclude that* A^n *is a singularity of* X^n. This together with (P) yields the contradiction. We derive Corollaries 1.3, 1.4 and 1.5 from Theorem A in the final section.

2. *Preliminary facts*

In this section we establish some definitions and preliminary results. Hereafter, M denotes a compact 3-manifold and X denotes a C^r vector field in M, $r \geq 1$. We denote by $\mathrm{Sing}(X)$ the set of singularities of Y. Given $A \subset M$ we denote by $\mathrm{Cl}(A)$ the closure of A. If $\delta > 0$ we denote

$$B_\delta(A) = \{x \in M : d(x, A) < \delta\},$$

where $d(\cdot, \cdot)$ is the metric in M.

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Given $p \in M$ and $\epsilon > 0$ we define

$$W_X^{ss}(p) = \{x : d(X_t(x), X_t(p)) \to 0, t \to \infty\},$$
$$W_X^{uu}(p) = \{x : d(X_t(x), X_t(p)) \to 0, t \to -\infty\},$$
$$W_X^{ss}(p, \epsilon) = \{x : d(X_t(x), X_t(p)) \leq \epsilon, \forall t \geq 0\}$$

and

$$W_X^{uu}(p, \epsilon) = \{x : d(X_t(x), X_t(p)) \leq \epsilon, \forall t \leq 0\}.$$

These sets are called the stable, unstable, local stable and local unstable set of p, respectively. We also define

$$W_X^s(p) = \bigcup_{t \in \mathbb{R}} W_X^{ss}(X_t(p))$$

and

$$W_X^u(p) = \bigcup_{t \in \mathbb{R}} W_X^{uu}(X_t(p)).$$

If H is a hyperbolic set of X and $p \in H$, then $W_X^{ss}(p)$ and $W_X^s(p)$ are C^r-immersed submanifolds of M. This is the Stable Manifold Theorem [**HPS**]. Similarly $W_X^{uu}(p)$, $W_X^u(p)$, $W_X^{ss}(p, \epsilon)$ and $W_X^{uu}(p, \epsilon)$ are C^r submanifolds. In this case $W_X^{ss}(p)$ and $W_X^{uu}(p)$ are called the *strong stable* and the *strong unstable* manifold of p, respectively. Note that $\dim(W_X^{ss}(p)) = \dim(E^s)$ and $\dim(W_X^{uu}(p)) = \dim(E^u)$. If O is a closed orbit of X then we denote by $O(Y)$ the continuation of O for Y, C^r close to X [**dMP**].

Remark 2.1.

(1) A singular-hyperbolic attractor cannot be a hyperbolic set.

(2) Let T be a transitive, non-trivial, singular-hyperbolic set of $X \in \mathcal{X}^r(M)$. Denote by $E^s \oplus E^c$ the singular-hyperbolic splitting of T. Then $\dim(E^s) = 1$, $\dim(E^c) = 2$ and E^X (the subspace generated by X in TM) is contained in E^c.

An useful property of singular-hyperbolic sets is given below.

LEMMA 2.2. [**MPP2**] *Let Λ be a singular-hyperbolic set of a C^r vector field X in M, $r \geq 1$. Then there is a neighborhood U of Λ such that if Y is C^r close to X, then every non-empty, compact, non-singular, invariant set H of Y in U is hyperbolic* saddle-type *(i.e. $E^s \neq 0$ and $E^u \neq 0$).*

LEMMA 2.3. *Let Λ be a singular-hyperbolic attractor of a C^r vector field X in M, $r \geq 1$. If $O \subset \Lambda$ is a periodic orbit of X, then O is a hyperbolic saddle-type periodic orbit. Henceforth, $W_X^{uu}(p)$ is a one-dimensional submanifold. In addition, if $p \in O$ then the set*

$$\{q \in W_X^{uu}(p) : \Lambda = \omega_X(q)\}$$

is dense in $W_X^{uu}(p)$.

Proof. First observe that the contracting direction E^s of Λ is integrable [**HPS**]. Denote by $W_X^{ss}(p)$ the leave of the resulting foliation containing $p \in \Lambda$. Note that the forward orbit of every $x \in W_X^{ss}(p)$ is asymptotic to the forward orbit of p. If follows from Lemma 2.2

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that O is hyperbolic and saddle-type. In particular, $\dim(W^{uu}_X(p)) = 1$ for all $p \in O$. Let I be an open interval in $W^{uu}_X(p)$. It follows that

$$B = \bigcup_{0\le t\le 1} X_t(I)$$

is a two-dimensional manifold of M. So,

$$B' = \bigcup_{x\in B} W^{ss}_X(x)$$

contains an open set V with $B \cap V \neq \emptyset$. Clearly $B \cap V \subset \Lambda$ since Λ is an attractor, $B \subset W^u_X(p)$ and $p \in O$. Let $q \in \Lambda$ such that $\Lambda = \omega_X(q)$. Then the forward orbit of q intersects V and so it intersects B' too. It follows from the definition of B' that the positive orbit of q is asymptotic to the forward orbit of some $q' \in B$. In particular, $\Lambda = \omega_X(q) = \omega_X(q')$. This proves that $\{q \in W^{uu}_X(p) : \Lambda = \omega_X(q)\}$ is dense in Λ as desired. □

2.1. *Lorenz-like singularities.* A singularity σ of a three-dimensional vector field X is *Lorenz-like* if its eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are real and satisfy

$$\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1.$$

It follows from the Invariant Manifold Theory [**HPS**] that the eigenspace associated with λ_2 is tangent to a one-dimensional invariant manifold of X denoted by $W^{ss}_X(\sigma)$. As usual we denote by $W^s_{\rm loc}(\sigma)$ a small ball in $W^s_X(\sigma)$ centered at σ. Similarly we define $W^u_{\rm loc}(\sigma)$. All these invariant manifolds persist and vary continuously under small C^r perturbations Y of X.

The classical Hartman–Grobman's theorem [**dMP**] gives a description of the flow around a Lorenz-like singularity σ. This description is shown in Figure 1.

Indeed, σ has a two-dimensional stable manifold $W^s_X(\sigma)$, a strong stable manifold $W^{ss}_X(\sigma)$ and an unstable manifold $W^u_X(\sigma)$, both one-dimensional. It follows that $W^{ss}_X(\sigma)$ separates $W^s_{\rm loc}(\sigma)$ into two connected components: the top and the bottom.

Definition 2.4. A *singular cross section* of a Lorenz-like singularity σ will be a pair of cross sections $S^{\rm t}$, $S^{\rm b}$ as indicated at Figure 1. Note that every singular cross section is endowed with a pair of curves $l^{\rm t}, l^{\rm b}$ in the intersection of the local stable manifold of σ with $S^{\rm t}$, $S^{\rm b}$, respectively (these curves are also indicated in the figure). If $* = {\rm t}, {\rm b}$ then S^* is a rectangle, i.e. diffeomorphic to $[-1, 1] \times [-1, 1]$. Hence the boundary of S^* is formed by four curves, two of them being transverse to l^* and two of them parallel to l^*. We denote by $\partial^{\rm h} S^*$ the union of the boundary curves which are transverse to l^* (h stands for 'horizontal'). We denote by $\partial^{\rm v} S^*$ the union of the boundary curves which are parallel to l^* (v stands for 'vertical'). The interior of S^* as a submanifold of M is denoted by $\mathrm{int}(S^*)$.

Remark 2.5. Clearly $S^{\rm t}$, $S^{\rm b}$ can be chosen to be close to σ. The positive flow lines of X starting at $S^{\rm t} \cup S^{\rm b} \setminus (l^{\rm t} \cup l^{\rm b})$ exit a small neighborhood of σ passing through the cusp region as indicated in Figure 1. The positive orbits starting at $l^{\rm t} \cup l^{\rm b}$ go directly to σ.

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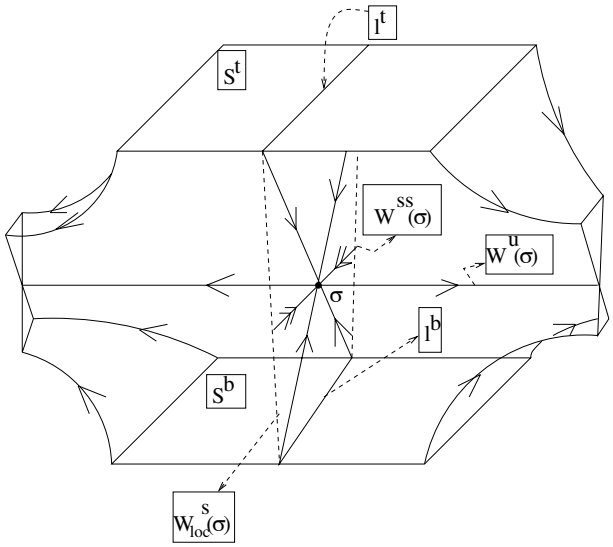


FIGURE 1. The flow around σ.

Lorenz-like singularities are considered below.

THEOREM 2.6. [**MPP2**] *Let Λ be a singular-hyperbolic attractor of a three-dimensional C^r vector field X, $r \geq 1$. Then, every $\sigma \in \mathrm{Sing}(X) \cap \Lambda$ is Lorenz-like and satisfies*

$$\Lambda \cap W_X^{ss}(\sigma) = \{\sigma\}.$$

This theorem implies the following two useful properties.

PROPOSITION 2.7. *Let Λ be a singular-hyperbolic attractor of a three-dimensional C^r vector field X, $r \geq 1$. Let σ be a singularity of X in Λ (so σ is Lorenz-like by Theorem 2.6). Then, for every isolating block U of Λ, there is a singular cross section S^{t}, S^{b} of σ in U such that*

$$\left(\bigcap_{t>0} Y_t(U)\right) \cap (\partial^{\mathrm{h}} S^{\mathrm{t}} \cup \partial^{\mathrm{h}} S^{\mathrm{b}}) = \emptyset,$$

for every C^r vector field Y close to X.

Proof. Fix an isolating block U of Λ. The equality in Theorem 2.6 implies that the negative orbit of every point in $W_X^{ss}(\sigma)\backslash\{\sigma\}$ leaves U. Hence we can arrange a singular cross section S^{t}, S^{b} nearby σ such that

$$\Lambda \cap (\partial^{\mathrm{h}} S^{\mathrm{t}} \cup \partial^{\mathrm{h}} S^{\mathrm{b}}) = \emptyset.$$

Since U is an isolating block of Λ and the boundary of S^{t}, S^{b} is compact we can find $T > 0$ such that

$$X_T(U) \cap (\partial^{\mathrm{h}} S^{\mathrm{t}} \cup \partial^{\mathrm{h}} S^{\mathrm{b}}) = \emptyset.$$

Hence,

$$Y_T(U) \cap (\partial^{\mathrm{h}} S^{\mathrm{t}} \cup \partial^{\mathrm{h}} S^{\mathrm{b}}) = \emptyset$$

for all C^r vector field close to X. The result follows since $\bigcap_{t>0} Y_t(U) \subset Y_T(U)$. □

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Let S^{t}, S^{b} be a singular-cross section of a Lorenz-like singularity σ of a three-dimensional C^r vector field X, $r \geq 1$. We refine S^{t}, S^{b} in the following way: Fix $* = \mathrm{t}, \mathrm{b}$ and fix a coordinate system (x^*, y^*) in S^* such that

$$S^* = [-1, 1] \times [-1, 1], \quad l^i = \{0\} \times [-1, 1]$$

with respect to (x^*, y^*). Given $\Delta > 0$ small we define

$$S^{*,\Delta} = [-\Delta, \Delta] \times [-1, 1].$$

Note that $S^{\mathrm{t},\Delta}$, $S^{\mathrm{b},\Delta}$ is a singular-cross section of σ for all $0 < \Delta \leq 1$ and that $S^* = S^{*,1}$. This notation will be used in the proof of Theorem A in the next section. The following lemma is a direct consequence of Theorem 2.6 and the Hartman–Grobman theorem [**dMP**].

LEMMA 2.8. *Let Λ be a singular-hyperbolic attractor of a three-dimensional C^r vector field X. Let σ be a singularity of X in Λ. Let Y^n be a sequence of vector fields converging to X in the C^r topology. Let O_n be a periodic orbit of Y^n such that the sequence $\{O_n : n \in \mathbb{N}\}$ accumulates on σ. If $0 < \Delta \leq 1$ and S^{t}, S^{b} is a singular-cross section of σ, then there is n such that either*

$$O_n \cap \mathrm{int}(S^{\mathrm{t},\Delta}) \neq \emptyset \quad \textit{or} \quad O_n \cap \mathrm{int}(S^{\mathrm{b},\Delta}) \neq \emptyset.$$

Proof. Fix an isolating block U of Λ. Since O_n accumulates on $\sigma \in \Lambda$ and U is an isolating block we have that $O_n \subset U$ for all n large (recall $Y^n \to X$ as $n \to \infty$). Fix a fundamental domain D of the vector field's flow X_t restricted to the local stable manifold $W^s_{\mathrm{loc}}(\sigma)$ [**dMP**]. Since $W^s_{\mathrm{loc}}(\sigma)$ is two-dimensional, we have that D is a circle which intersects $W^{ss}_X(\sigma)$ in two points a, b. Note that the orbits of a, b together with σ yield $W^{ss}_X(\sigma)$. In particular, $a, b \notin \Lambda$ by Theorem 2.6. Since D is a circle we can put an angular coordinate system θ such that $a = 0$ and $b = \pi$ (say). Let D^{ext} be a cross section of X such that $W^s_{\mathrm{loc}}(\sigma) \cap D^{\mathrm{ext}} = D$. It follows that D^{ext} is a cylinder and so we can put cylindrical coordinates (θ, s) with $\theta \in D$ and $s \in [-1, 1]$ say. Under such a coordinate system we have $D = \{(\theta, 0) : s \in [-1, 1]\}$ and also $a = (0, 0)$, $b = (\pi, 0)$. With this coordinate system we can construct a family of singular-cross sections S^{t}_δ, S^{b}_δ (for all $\delta \in [-1, 1]$) by setting

$$S^{\mathrm{t}}_\delta = \{(\theta, s) \in D^{\mathrm{ext}} : s \in [-1, 1], \theta \in [\delta, \pi - \delta]\}$$

and

$$S^{\mathrm{b}}_\delta = \{(\theta, s) \in D^{\mathrm{ext}} : s \in [-1, 1], \theta \in [\pi + \delta, 2\pi - \delta]\}.$$

Due to the smooth variation of $W^{ss}_Y(\sigma(Y))$ with respect to Y close to X we can assume that $\sigma(Y) = \sigma$ and that $W^{ss}_{\mathrm{loc},Y}(\sigma(Y)) = W^{ss}_{\mathrm{loc}}(\sigma)$ for every Y close to X. By choosing D so close to σ we can further assume that D^{ext} is a cross section of Y, for all Y close to X. In particular, the singular cross sections S^{t}_δ, S^{b}_δ are in U. We claim that there is $\delta > 0$ such that the conclusion of the lemma holds for $S^{\mathrm{t}} = S^{\mathrm{t}}_\delta$ and $S^{\mathrm{b}} = S^{\mathrm{b}}_\delta$. Indeed, we first note that under the cylindrical coordinate system (θ, s) we have $S^{*,\Delta} = S^*_\Delta$ for all $0 < \Delta \leq \delta$ (where $* = \mathrm{t}, \mathrm{b}$). If the conclusion of the claim fails, then the phase portrait of Y^n given by the Hartman–Grobman theorem [**dMP**] implies that O_n intersects $D^{\mathrm{ext}} \setminus (S^{\mathrm{t}}_\Delta \cup S^{\mathrm{b}}_\Delta)$ for all $\Delta > 0$ small. In other words, there would exist $p_n \in O_n$ (for all n large) such that

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$p_n = (\theta_n, s_n)$ with $\theta_n \in [\pi - \Delta, \pi + \Delta] \cup [-\Delta, \Delta]$ and $s_n \to 0$ as $n \to \infty$. Since Δ is arbitrary and $s_n \to 0$, we conclude that p_n converges either to $a = (0, 0)$ or $b = (\pi, 0)$ by passing to a subsequence if necessary. The latter would imply either $a \in \Lambda$ or $b \in \Lambda$ since $O_n \subset U$ and $Y^n \to X$ (recall that U is an isolating block of Λ). This contradicts Theorem 2.6 and the proof follows. □

3. *Proof of Theorem A*

We prove the theorem by contradiction. Let Λ be a singular-hyperbolic attractor of a three-dimensional C^r vector field X, $r \geq 1$. Suppose that there is a sequence $X^n \overset{C^r}{\to} X$ such that every X^n exhibits a non-singular attractor A^n arbitrarily close to Λ. Note that although such attractors are *arbitrarily close to* Λ it is still possible that such attractors stay *far from* $\mathrm{Sing}(X) \cap \Lambda$ (see Remark 3.1). Fix the neighborhood U of Λ as in Lemma 2.2. Obviously we can assume that U is an isolating block of Λ. Since A^n is arbitrarily close to Λ we can assume that A^n belongs to U for all n. Then the Lemmas 2.2 and 2.3 imply

$$A^n \text{ is a hyperbolic strange attractor of } X^n \text{ for all } n. \tag{*}$$

On the other hand, the following property is obvious since A^n is non-singular for all n:

$$A^n \text{ is not a singularity of } X^n \text{ for all } n. \tag{P}$$

Remark 3.1. Note that $A^n \subset U$ and $A^n \neq \bigcap_{t>0} X_t^n(U)$ for every $n \in \mathbb{N}$. In addition, *every* limit point of every sequence $x_n \in A^n$ belong to Λ. However, it is possible that *none* of such limit points belong to $\mathrm{Sing}(X) \cap \Lambda$.

The above remark is the reason why we prove the following lemma.

LEMMA 3.2. *The attractors A^n accumulate on* $\mathrm{Sing}(X)$*, i.e.*

$$\mathrm{Sing}(X) \cap \mathrm{Cl}\left(\bigcup_{n\in\mathbb{N}} A^n\right) \neq \emptyset.$$

Proof. Suppose by contradiction that the conclusion fails. Then, there is $\delta > 0$ such that

$$B_\delta(\mathrm{Sing}(X)) \cap \left(\bigcup_{n\in\mathbb{N}} A^n\right) = \emptyset. \tag{1}$$

Define

$$H = \bigcap_{t\in\mathbb{R}} X_t(U \setminus B_{\delta/2}(\mathrm{Sing}(X))).$$

Obviously $\mathrm{Sing}(X) \cap H = \emptyset$. Let us prove that $H \neq \emptyset$. For all n we choose $x^n \in A^n$. By passing to a subsequence if necessary we can assume that x^n converges to some $x \in M$. Clearly $x \in H$, otherwise the X^n-orbit of x^n would intersect $B_\delta(\mathrm{Sing}(X))$ by [**dMP**, Tubular Flow-Box Theorem]. This is impossible by (1), proving $H \neq \emptyset$ as desired.

Note that H is compact since U is. It follows that H is a non-empty compact set which is clearly invariant for X. It follows that H is hyperbolic by Lemma 2.2 since $\mathrm{Sing}(X) \cap H = \emptyset$. Denote by $E^s \oplus E^X \oplus E^u$ the corresponding hyperbolic splitting (see Definition 1.7).

By the stability of hyperbolic sets we can fix a neighborhood W of H and $\epsilon > 0$ such that if Y is a vector field C^r close to X and H_Y is a compact invariant set of Y in W then:

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(H1) H_Y is hyperbolic and its hyperbolic splitting $E^{s,Y} \oplus E^Y \oplus E^{u,Y}$ satisfies

$$\dim(E^u) = \dim(E^{u,Y}), \quad \dim(E^s) = \dim(E^{s,Y});$$

(H2) the manifolds $W_Y^{uu}(x, \epsilon)$, $x \in H_Y$, are one-dimensional and have uniform size ϵ.

As $X^n \to X$ we have that

$$\bigcap_{t\in\mathbb{R}} X_t^n(U \setminus B_{\delta/2}(\mathrm{Sing}(X))) \subset W$$

for all n large. However,

$$A^n \subset U \setminus B_{\delta/2}(\mathrm{Sing}(X))$$

for all n by (1). As A^n is X^n-invariant we conclude that

$$A^n \subset W$$

for all n large.

Now, by (H2) we have that $W_{X^n}^{uu}(x^n, \epsilon)$ is one dimensional and has uniform size ϵ, for all $x^n \in A^n$ and for all n.

Choose $x^n \in A^n$ so that x^n converges to some $x \in M$. As observed above, we have that $x \in H$. Note that the tangent vectors of the curve $W_{X^n}^{uu}(x^n, \epsilon)$ at $c \in W_{X^n}^{uu}(x^n, \epsilon)$ is in E_c^{u,X^n} (E^{u,X^n} is defined by (H1) as $X^n \to X$).

As $X^n \to X$ we have that the angle between the directions E^{u,X^n} and E^u goes to zero as $n \to \infty$. Hence the manifolds $W_X^{uu}(x, \epsilon)$ and $W_{X^n}^{uu}(x^n, \epsilon)$ are almost parallel as $n \to \infty$. As $x^n \to x$, we conclude that

$$W_{X^n}^{uu}(x^n, \epsilon) \to W_X^{uu}(x, \epsilon)$$

in the sense of C^1 submanifolds [**PT**].

Fix an open interval $I \subset W_X^{uu}(x, \epsilon)$ containing x. By Lemma 2.3, as $\Lambda \cap \mathrm{Sing}(X) \neq \emptyset$, we have that there are $q \in I$ and $T > 0$ such that

$$X_T(q) \in B_{\delta/5}(\mathrm{Sing}(X)).$$

By [**dMP**, Tubular Flow Box Theorem] there is an open set V_q containing q such that

$$X_T(V_q) \subset B_{\delta/5}(\mathrm{Sing}(X)).$$

As $X^n \to X$ we have

$$X_T^n(V_q) \subset B_{\delta/4}(\mathrm{Sing}(X)) \tag{2}$$

for all n large.

However, $W_{X^n}^{uu}(x^n, \epsilon) \to W_X^{uu}(x, \epsilon)$, $q \in I \subset W_X^{uu}(p, \epsilon)$, $q \in V_q$ and V_q is open. So,

$$W_{X^n}^{uu}(x^n, \epsilon) \cap V_q \neq \emptyset$$

for all n large. Applying (2) to X^n for n large, we have

$$X_T^n(W_{X^n}^{uu}(x^n, \epsilon)) \cap B_{\delta/4}(\mathrm{Sing}(X)) \neq \emptyset.$$

As $W_{X^n}^{uu}(x^n, \epsilon) \subset W_{X^n}^u(x^n)$ the invariance of $W_{X^n}^u(x^n)$ implies that

$$W_{X^n}^u(x^n) \cap B_{\delta/2}(\mathrm{Sing}(X)) \neq \emptyset.$$

Observe that $W_X^u(x^n) \subset A^n$ since $x^n \in A^n$ and A^n is an attractor. We conclude that

$$A^n \cap B_\delta(\mathrm{Sing}(X)) \neq \emptyset.$$

This contradicts (1) and the proof follows. □

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Proof of Theorem A. By the previous lemma

$$\exists \sigma \in \mathrm{Sing}(X) \cap \mathrm{Cl}\left(\bigcup_{n\in\mathbb{N}} A^n\right).$$

Clearly $\sigma \in \Lambda$. By Theorem 2.6 we have that σ is Lorenz-like and satisfies

$$\Lambda \cap W_X^{ss}(\sigma) = \{\sigma\}.$$

Let $S^{\mathrm{t}}, S^{\mathrm{b}}$ be the singular-cross sections given by Proposition 2.7 for σ and U. In particular, $S^{\mathrm{t}} \cup S^{\mathrm{b}} \subset U$ and

$$\Lambda \cap (\partial^{\mathrm{h}} S^{\mathrm{t}} \cup \partial^{\mathrm{h}} S^{\mathrm{b}}) = \emptyset.$$

As $X^n \to X$ we have that $S^{\mathrm{t}}, S^{\mathrm{b}}$ are also singular cross sections of X^n. By using the Implicit Function Theorem we can assume that $\sigma(X^n) = \sigma$ and

$$l^{\mathrm{t}} \cup l^{\mathrm{b}} \subset W_{X^n}^s(\sigma) \tag{3}$$

for all n.

We denote by $TB = T(B)$ the tangent space at B. The one-dimensional subbundle E^s of Λ extends to a contracting invariant subbundle on the whole U. Take a continuous (not necessarily invariant) extension of E^c to U. We denote by $E^s \oplus E^c$ the above-mentioned extension.

By the Invariant Manifold Theory [**HPS**], it follows that the splitting $E^s \oplus E^c$ persists by small perturbations of X. More precisely, for all n large the vector field X^n has a splitting $E^{s,n} \oplus E^{c,n}$ over U such that $E^{s,n}$ is invariant contracting, $E^{s,n} \to E^s$ and $E^{c,n} \to E^c$ as $n \to \infty$. In particular, $E^{s,n} \oplus E^{c,n}$ is defined in $S^{\mathrm{t}} \cup S^{\mathrm{b}}$ for all n large. In the following we denote by E^Y the subbundle in TM generated by a vector field Y in M.

The dominance condition at Definition 1.1(2) together with [**D**, Proposition 2.2] imply that for $i = \mathrm{t}, \mathrm{b}$, we have

$$T_x S^i \cap (E_x^s \oplus E_x^X) = T_x l^i,$$

for all $x \in l^i$.

Denote by $\angle(E, F)$ the angle between two linear subspaces. The last equality implies that there is $\rho > 0$ such that

$$\angle(T_x S^i \cap E_x^c, T_x l^i) > \rho,$$

for all $x \in l^i$ $(i = \mathrm{t}, \mathrm{b})$. However, $E^{c,n} \to E^c$ as $n \to \infty$. So for all n large we have

$$\angle(T_x S^i \cap E_x^{c,n}, T_x l^i) > \frac{\rho}{2}, \tag{4}$$

for all $x \in l^i$ $(i = \mathrm{t}, \mathrm{b})$.

Fix $* = \mathrm{t}, \mathrm{b}$. As in the previous section we fix a coordinate system $(x, y) = (x^*, y^*)$ in S^* such that

$$S^* = [-1, 1] \times [-1, 1], \quad l^* = \{0\} \times [-1, 1]$$

with respect to (x, y).

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Denote by $\Pi^* : S^* \to [-1, 1]$ the projection

$$\Pi^*(x, y) = x.$$

As before given $\Delta > 0$ we define

$$S^{*,\Delta} = [-\Delta, \Delta] \times [-1, 1].$$

Define the line field F^n in S^{*,Δ_0} by

$$F^n_x = T_x S^* \cap E^{c,n}_x, \quad x \in S^{*,\Delta_0}.$$

Remark 3.3. The continuity of $E^{c,n}$ and (4) imply that there exists $\Delta_0 > 0$ such that for all n large the line F^n is *transverse to* Π^*. By this we mean that $F^n(z)$ is *not tangent to the curves* $(\Pi^*)^{-1}(c)$, for all $c \in [-1, 1]$.

Now recall that A^n is a hyperbolic strange attractor of X^n for all n (see (*)). It follows that the periodic orbits of X^n in A^n are dense in A^n [**PT**]. Then, as $\sigma \in \mathrm{Cl}\left(\bigcup_{n\in\mathbb{N}} A^n\right)$, there is a periodic orbit sequence $O_n \in A^n$ accumulating on σ. It follows from Lemma 2.8 applied to $Y^n = X^n$ that there is $n_0 \in \mathbb{N}$ such that either

$$O_{n_0} \cap \mathrm{int}(S^{\mathrm{t},\Delta_0}) \neq \emptyset \quad \text{or} \quad O_{n_0} \cap \mathrm{int}(S^{\mathrm{b},\Delta_0}) \neq \emptyset.$$

Because $O_{n_0} \subset A_{n_0}$ we conclude that either

$$A^{n_0} \cap \mathrm{int}(S^{\mathrm{t},\Delta_0}) \neq \emptyset \quad \text{or} \quad A^{n_0} \cap \mathrm{int}(S^{\mathrm{b},\Delta_0}) \neq \emptyset.$$

We denote $Z = X^{n_0}$, $A = A^{n_0}$, $F = F^{n_0}$ for simplicity. Again recall that A^n is a hyperbolic strange attractor of X^n (see (*) at the beginning of this section). Thus we have the following property:

$$A \text{ is a hyperbolic strange attractor of } Z. \tag{**}$$

Hence the following property also holds.

$$A \text{ is not a singularity of } Z. \tag{PP}$$

We assume that $A \cap \mathrm{int}(S^{\mathrm{t},\Delta_0}) \neq \emptyset$ (the proof for $* = \mathrm{b}$ is similar). Note that $\partial^{\mathrm{h}} S^{\mathrm{t},\Delta_0} \subset \partial^{\mathrm{h}} S^{\mathrm{t}}$ by definition. Then, by Lemma 2.7 we have

$$A \cap \partial^{\mathrm{h}} S^{\mathrm{t},\Delta_0} = \emptyset.$$

We denote $S = S^{\mathrm{t},\Delta_0}$, $(x, y) = (x^{\mathrm{t}}, y^{\mathrm{t}})$ and $\Pi = \Pi^{\mathrm{t}}$ for simplicity. Note that $A \cap S$ is a compact non-empty subset of S. Hence there is $p \in S \cap A$ such that

$$\mathrm{dist}(\Pi(S^{\mathrm{t}} \cap A), 0) = \mathrm{dist}(\Pi(p), 0),$$

where dist denotes the distance in $[-\Delta_0, \Delta_0]$.

Now, $p \in A$ and so $W^u_Z(p)$ is a well-defined two-dimensional submanifold. The dominance in Definition 1.1(2) implies that

$$T_z(W^u_Z(p)) = E^c_z$$

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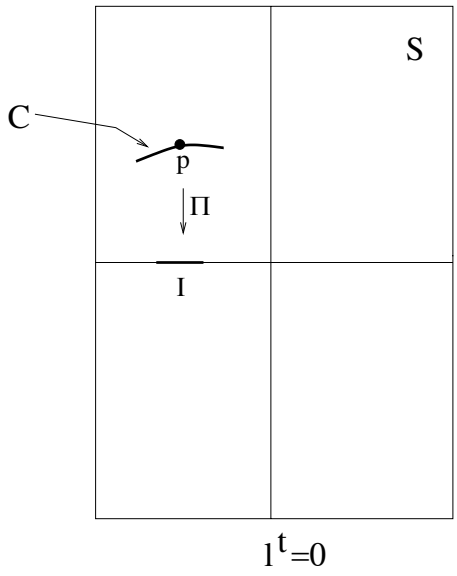


FIGURE 2. The curve C.

for every $z \in W^u_Z(p)$. Hence

$$T_z(W^u_Z(p)) \cap T_z S = E^c_z \cap T_z S = F_z$$

for every $z \in W^u_Z(p) \cap S$.

As $W^u_Z(p) \cap S$ is transversal, we have that $W^u_Z(p) \cap S$ contains a curve C whose interior contains p as in Figure 2. The last equality implies that C is tangent to F.

As F is transverse to Π, we have that C is *transverse* to Π (i.e. C is transverse to the curves $\Pi^{-1}(c)$, for every $c \in [-1, 1]$). We conclude that $\Pi(C)$ contains an open interval $I \subset [-\Delta_0, \Delta_0]$ with $\Pi(p) \in \text{int}(I)$. So, there is $z_0 \in C$ such that

$$\text{dist}(\Pi(z_0), 0) < \text{dist}(\Pi(p), 0).$$

Note that $C \subset S \cap A$ since A is an attractor of Z (see property (**)). Moreover, $p \in A$ and $C \subset W^u_Z(p)$. As $A \cap \partial^h S = \emptyset$ we conclude that

$$\text{dist}(\Pi(S \cap A), 0) = 0.$$

As A is closed, this last equality implies

$$A \cap l^t \neq \emptyset.$$

Since $l^t \subset W^s_Z(\sigma)$ and A is closed invariant for Z we conclude that $\sigma \in A$. We have proved that A contains a singularity of Z. However, A is a hyperbolic attractor of Z by (**). Henceforth, $A = \{\sigma\}$. This contradicts the property (PP) and the proof follows. □

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4. *Proofs of the corollaries*

Proof of Corollary 1.3. Fix the neighborhood U as in Theorem A. Let n be the number of singularities of X in Λ. By Theorem A there is a neighborhood $\mathcal{W}$ of X in $\mathcal{X}^r(M)$ such that every attractor in U of every $Y \in \mathcal{W}$ is singular. Shrinking $\mathcal{W}$ if necessary we can assume that the number of singularities of Y in U is also n. As the family of attractors of Y in U is pairwise disjoint, we conclude that every $Y \in \mathcal{W}$ has at most n attractors in U. This proves the result. □

Proof of Corollary 1.4. Let X be a vector field in $\mathcal{X}^r(M)$ exhibiting a singular-hyperbolic attractor Λ, $r \geq 1$. Fix the neighborhood U as in Theorem A. Then every attractor in U of every vector field C^r close to X is singular. Suppose by contradiction that the conclusion of the theorem fails. Then there is a sequence $X^n \to X$ such that every singularity of X^n is isolated in the non-wandering set. Choosing n large we have that X^n is C^r close to X. It follows that the sequence X^n satisfies the following property:

$$\text{every attractor in } U \text{ of } X^n \text{ is singular.} \qquad (***)$$

On the other hand, define

$$C^n = (\Omega(X^n) \cap U) \setminus \mathrm{Sing}(X^n).$$

Then, C^n is an invariant set of X^n which is compact since each singularity is isolated in the non-wandering set. Moreover, C^n is hyperbolic by Lemma 2.2 since it has no singularities. Now,

$$\Omega(X^n) \cap U = C^n \cup (\mathrm{Sing}(X^n) \cap U)$$

and $\mathrm{Sing}(X^n) \cap U$ is also a hyperbolic set of X^n. It follows that $\Omega(X^n) \cap U$ is a hyperbolic set of X^n. In particular, the positive limit set $L^+ = L^+(X^n) \cap U$ is hyperbolic since it is contained in $\Omega(X^n) \cap U$ (recall the definition of $L^+(X)$ in the introduction). It follows from the Shadowing Lemma that L^+ is the closure of its closed orbits. Hence, there is a spectral decomposition $L^+ = L_1 \cup \cdots \cup L_k$, where each L_i is a hyperbolic basic set of X^n (see [**PT**]). Note that

$$U \subset W^s_{X^n}(L^+ \cap U) = \bigcup_{i=1}^{k} W^s_{X^n}(L_i).$$

Because U is open, we conclude that there is i_0 such that $W^s_{X^n}(L_{i_0})$ has non-empty interior. This implies that $A = L_{i_0}$ is a hyperbolic attractor of X^n. By (***) we have that A contains a singularity of X^n. Since A is a hyperbolic attractor we conclude that A is a singularity of X^n. However this is impossible (for n large) since the singularities of X^n are precisely the continuation of those of X (which are saddle-type). This contradiction proves the result. □

Proof of Corollary 1.5. First observe that a non-trivial attractor of a flow cannot be accumulated by attracting or repelling closed orbits of the flow. Then, by applying the arguments in [**MPa1**], we can prove that there is $\mathcal{R} \subset \mathcal{X}^1(M)$ residual such that every non-trivial singular attractor of every vector field X in $\mathcal{R}$ is singular-hyperbolic. Then the conclusion follows by applying Corollary 1.4 to $r = 1$. □

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We finish with an observation. By using Corollary 1.6 it is possible to construct examples of C^r vector fields without C^r robust singular attractors exhibiting singularities C^r stably non-isolated in the non-wandering set. The non-robust singular-hyperbolic attractor described in [**MPu**, Appendix] is one such example.

Acknowledgements. This work is partially supported by CNPq, FAPERJ and PRONEX-Dyn. Systems/Brazil.

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