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INDETERMINACY AND INVESTMENT ADJUSTMENT COSTS: AN ANALYTIC RESULT

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It has been widely known that neoclassical growth models with sufficient increasing returns in production may feature indeterminacy. This note analytically shows that investment adjustment costs increase the required degree of increasing returns for indeterminacy to arise. Under empirically plausible levels of investment adjustment costs, we need an implausibly large degree of increasing returns to generate indeterminacy.

Keywords: Indeterminacy, Investment Adjustment Costs

1. INTRODUCTION

Neoclassical growth models with sufficient increasing returns to scale may feature indeterminate dynamics, in the sense that there is a continuum of equilibrium paths that converge to the unique steady state. The presence of indeterminacy is an interesting phenomenon since models with indeterminacy can generate self-fulfilling business-cycle fluctuations. The presence and the role of indeterminacy have been studied by Benhabib and Farmer (1994); see also the references in Benhabib and Farmer (1999).

The concept of adjustment costs has been widely used in the literature on investment.[1] This note shows how investment adjustment costs interact with increasing returns in generating indeterminacy. The introduction of investment adjustment costs makes it difficult for indeterminacy to occur. That is, the required degree of increasing returns is higher in the presence of investment adjustment costs. Plausible levels of investment adjustment costs mean that indeterminacy arises only if there is an implausible degree of returns to scale. Furthermore, models with large enough adjustment costs would never feature indeterminacy.

Georges (1995) also explained analytically how other types of adjustment costs influence the presence of indeterminacy. The relationship between indeterminacy and investment adjustment costs has also been analyzed by Guo and Lansing (in press) and Wen (1998) in a numerical way. They show that adjustment costs

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would make it less likely that indeterminacy will occur. The unique contribution of this note is to present this relationship in a simple analytic way and to discuss the empirical plausibility of indeterminacy under investment adjustment costs.

This note proceeds as follows. Section 2 describes the model with investment adjustment costs. In section 3, we analyze the dynamics of the model and derive a necessary and sufficient condition for indeterminacy. Intuition is given for this condition, and its empirical plausibility is discussed. Section 4 concludes.

2. THE ECONOMY

This section introduces investment adjustment costs into the continuous-time model of Benhabib and Farmer (1994). Economic environment for firms and households is described, and their optimal behavior is derived.

2.1. Firms

In our model, increasing returns to scale arise due to production externalities.[2] Even though the individual-level private technology features constant returns to scale,

$$Y_i = \Xi K_i^a L_i^b, \qquad a > 0, \qquad b > 0, \qquad a + b = 1, \tag{1}$$

externalities of the form

$$\Xi = (K^a L^b)^\xi, \qquad \xi \geq 0 \tag{2}$$

allow the aggregate production function to feature increasing returns to scale,

$$Y = K^\alpha L^\beta, \qquad \alpha > 0, \qquad \beta > 0, \qquad \alpha + \beta \geq 1, \tag{3}$$

where

$$\alpha = (1 + \xi)a, \qquad \beta = (1 + \xi)b.$$

Variables Y, K, and L represent total output, the aggregate stock of capital, and aggregate labor hours, respectively. The subscript i denotes the individual firm, and the term Ξ represents production externalities exogenous to an individual firm. We assume that $\alpha < 1$.

Aggregation also allows us to rewrite the profit maximization conditions ($ZK_i = aY_i$, $WL_i = bY_i$) in terms of aggregate quantities,

$$Z = aY/K, \tag{4}$$

$$W = bY/L, \tag{5}$$

where Z is the rental rate of capital and W is the real wage. These two equations are interpreted as factor demand schedules.

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2.2. Households

The representative household maximizes

$$\int_0^\infty \left(\log C - \frac{L^{1+\chi}}{1+\chi}\right) e^{-\rho t}\, dt, \qquad \chi \geq 0, \qquad \rho > 0, \tag{6}$$

where C is consumption. Parameters χ represents the inverse of the intertemporal elasticity of substitution for labor supply, and ρ is the discount rate. The household budget constraint is

$$C + I = ZK + WL, \tag{7}$$

where I is the gross investment. The factor prices are taken as given.

The literature on investment adjustment costs has introduced the costs in various forms. In this note, we incorporate the presence of investment adjustment costs in the capital accumulation equation in a way similar to Lucas and Prescott (1971),[3]

$$\dot{K}/K = \Psi(I/K), \qquad K(0) \text{ given.} \tag{8}$$

A variable with a dot above it represents its time derivative.

For later analysis of the local dynamics, we make three assumptions on the form of the adjustment costs function:

$$\Psi(\delta) = 0, \qquad \Psi'(\delta) = 1, \qquad \Psi''(\delta) \leq 0,$$

and we use a parameter $\psi\,(\equiv -\delta\Psi''(\delta)/\Psi'(\delta))$ to express the degree of investment adjustment costs.[4] In a model in which firms accumulate capital, this parameter is the inverse of the elasticity of (I/K) with respect to Tobin's q.[5]

In our specification, adjustment costs occur when the level of the capital stock moves, that is, the level of investment is different from the depreciated capital. However, in Wen (1998), adjustment costs occur when the *change* of investment is different from zero. This difference allows our model to yield an analytic expression for indeterminacy and his model to preserve the oscillation mechanism without indeterminacy.

3. INDETERMINACY

In this section, we summarize the dynamics of the model and derive a necessary and sufficient condition for indeterminacy. We give an intuition for this condition of indeterminacy and discuss the empirical plausibility of indeterminacy. Approximating the model behavior by linearization, we reduce the dynamics to the following bivariate first-order system[6]:

$$\begin{bmatrix} \dot{k} \\ \dot{c} \end{bmatrix} = \frac{1}{\delta - \psi\theta} \begin{bmatrix} \tau(\delta - \psi\theta) & \theta(\delta - \psi\theta) \\ \delta(\rho+\delta)\kappa - \psi\tau(\rho - \tau) & \delta(\rho+\delta)\gamma - \psi\theta(\rho - \tau) \end{bmatrix} \begin{bmatrix} k - \bar{k} \\ c - \bar{c} \end{bmatrix},$$

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where

$$\theta = -\frac{(\rho + b\delta)\chi + a\delta(\beta - 1) + (\rho + \delta)}{a[\chi - (\beta - 1)]},$$

$$\tau = \frac{\rho(1 + \chi)(1 + \xi) + \delta[\beta + (1 + \chi)\xi]}{\chi - (\beta - 1)},$$

$$\kappa = \frac{\alpha(1 + \chi)}{\chi - (\beta - 1)} - 1,$$

$$\gamma = \frac{-\beta}{\chi - (\beta - 1)}.$$

Lowercase letters denote the logarithmic of the variables, and variables with a bar represent their steady state.

3.1. Condition for Indeterminacy

In the standard growth model without externalities or increasing returns, there is a unique equilibrium path. Locally near the steady state, uniqueness implies that one of the two eigenvalues is negative and the other is positive.[7] Indeterminacy in the growth model with variable labor and production externalities requires two negative eigenvalues. Since the trace and the determinant of the Jacobian are the sum and the product of the eigenvalues, the necessary and sufficient condition for indeterminacy is that the trace be negative and the determinant be positive.

In our model, indeterminacy arises if and only if the parameter values satisfy the following two conditions [8]:

$$\psi < 1, \tag{9}$$

and

$$\beta - 1 > \chi + (1 + \chi)\left(\frac{\rho + \delta}{a\delta}\right)\left(\frac{\psi}{1 - \psi}\right). \tag{10}$$

The first condition says that the degree of adjustment costs should be less than unity. In terms of q regressions, the elasticity of (I/K) with respect to q should be larger than unity. This condition puts an upper bound on the degree of adjustment costs for indeterminacy to arise. If the degree of adjustment costs is as big as unity, indeterminacy will never occur regardless of the degree of externalities. The case of unitary degree of adjustment costs in our continuous-time model corresponds to the loglinear capital accumulation equation in a discrete-time model, as in Hercowitz and Sampson (1991). Guo and Lansing (1999) show that indeterminacy does not arise under this loglinear capital accumulation equation.

Using the relation that $\beta = (1 + \xi)b$, we rewrite the second condition in terms of the degree of externalities:

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$$\xi > \left(\frac{1+\chi}{b}\right)\left[1+\left(\frac{\rho+\delta}{a\delta}\right)\left(\frac{\psi}{1-\psi}\right)\right]-1. \quad (11)$$

For convenience of the discussion, we define the right-hand side as the degree of externalities required for indeterminacy to occur. The required degree of externalities is an increasing function of the degree of adjustment costs (ψ). Furthermore, the required degree increases very fast since it is a rational function with respect to the degree of adjustment costs. As the degree of adjustment costs approaches unity, the required degree of externalities diverges to infinity.[9]

3.2. Intuition for Indeterminacy

In this subsection, we give an intuition for the condition of indeterminacy in a model with investment adjustment costs. The best way to approach our intuition is to start from the condition of indeterminacy in a model without adjustment costs and its intuition. If there are no adjustment costs (i.e., $\psi = 0$), then the first condition (9) is automatically satisfied and the second condition (10) is simplified as follows:

$$\beta - 1 > \chi, \quad (12)$$

as derived in Benhabib and Farmer (1994).

An economic meaning of this condition becomes very intuitive if we rewrite the labor-market equilibrium condition as follows:

$$bK^{\alpha}L^{\beta-1} = W = CL^{\chi}. \quad (13)$$

Now, the intuition of indeterminacy is straightforward. The condition (12) means that the labor demand schedule slopes upward as a function of the real wage and is steeper than the labor supply curve. This intuition from the labor-market equilibrium has been used widely in the literature on indeterminacy.[10]

However, it is easy to see that there are pitfalls in this interpretation. This note shows that, in a model with investment adjustment costs, the labor-market equilibrium condition (13) remains unchanged but the condition for indeterminacy is different from (12). That is, the intuition from the labor market does not hold in our model with investment adjustment costs.

In contrast, this note gives an intuition for indeterminacy from the market for capital. Indeterminacy arises because shifts in investor beliefs about future investment returns are self-reinforcing: If investment increases in response to high prospective returns, the resulting capital formation brings about the necessary high returns. However, investment adjustment costs weaken this linkage and require a higher degree of increasing returns for indeterminacy to arise.

Intuitively speaking, less flexibility in investment returns due to adjustment costs should be offset by more flexibility in increasing returns for optimism to be self-fulfilling. This intuition is similar to the one given by Guo and Lansing (1998) in a model of stabilization policy. Government's stabilization policy, in a form of

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progressive taxation, taxes away the high returns from belief-driven investment. As a result, it is difficult for indeterminacy to arise.

3.3. Plausibility of Indeterminacy

To provide an idea on the plausibility of indeterminacy, we analyze the sensitivity of the required degree of externalities with respect to the degree of adjustment costs. For concreteness of this analysis, we calibrate all the parameters besides the degree of adjustment costs (ψ) and the degree of externalities (ξ). We follow the benchmark values of Benhabib and Farmer (1996), except for χ:

Parameter	a	b	χ	ρ	δ
Calibrated value	0.3	0.7	0	0.05	0.1

Our calibration of χ makes it favorable for indeterminacy to arise, since $\chi = 0$ minimizes the required degree of externalities represented by the right-hand side of (11).[11]

Now we analyze the space of ψ and ξ for indeterminacy to occur. As reviewed in Hamermesh and Pfann (1996), the estimates of the degree of investment adjustment costs vary widely. The lower bound of empirical estimates for ψ is about 0.2. Here, we consider the range of ψ between 0 and 0.2 to be conservative in concluding that investment adjustment costs make it implausible for indeterminacy to arise.[12]

With the calibrated parameter values in the preceding table, the mapping from the degree of adjustment costs to the required degree of externalities is as follows:

Degree of adjustment costs (ψ)	0	0.05	0.1	0.15	0.2
Required degree of externalities	0.43	0.80	1.22	1.69	2.21

It is easy to see that the required degree of externalities increases very fast as the degree of adjustment costs increases. Furthermore, since the required degree of externalities increases so fast that indeterminacy cannot arise under a plausible degree of externalities. Even when the degree of adjustment costs is very low at 0.05, the required degree of externalities is 0.80, which is too large to rationalize from empirical grounds. Note that this case implies that the degree of returns to scale is 1.80.

4. CONCLUSION

We have shown how the presence of positive investment adjustment costs makes it less likely that indeterminacy will occur. The larger the adjustment costs, the larger the required degree of increasing returns. Under empirically plausible parameter values for investment adjustment costs, we need an implausibly large degree of increasing returns to generate indeterminacy.

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NOTES

1. In a continuous-time partial-equilibrium model without investment adjustment costs, the decision on the capital stock is static, and so, investment becomes infinitely volatile. General-equilibrium models have introduced various forms of investment adjustment costs to improve the properties of the model, such as persistence. Kim (in press) reviews the literature on investment adjustment costs from a macroeconomic perspective.

2. Unlike standard neoclassical growth models, we allow for increasing returns to scale. Benhabib and Farmer (1994) present two economic environments that are consistent with increasing returns. This note adopts the model with production externalities. Another environment is a monopolistically competitive economy. Kim (1997) reviews three types of increasing returns in this type of models from a critical perspective.

3. In the context of our model, the budget constraint (7) may incorporate investment adjustment costs in such a way that investment is less than savings, as in Abel and Blanchard (1983), or that consumption plus savings is less than factor incomes. Kim (in press) shows that these alternative specifications generate the dynamics equivalent to our specification.

4. The first assumption defines the depreciation rate as the ratio between investment and capital at the steady state. The second assumption makes the steady state of our model with adjustment costs not different from that of a model without adjustment costs. The standard linear capital accumulation without adjustment costs, $\dot{K} = I - \delta K$, corresponds to $\psi = 0$. In other words, we consider adjustment costs with only the second-order effects relative to the linear capital accumulation equation, without reference to the level or the first-order effects. See Kim (in press) for more on this point. A functional form of $\Psi(\cdot)$ that satisfies these three assumptions is $\delta[(I/\delta K)^{1-\psi} - 1]/(1-\psi)$.

5. See the Appendix A.1 for the derivation of this relation and the two optimality conditions for households.

6. See Appendix A.2 for detailed derivation of this linear bivariate system.

7. It should be emphasized that local uniqueness does not imply global uniquesness. In Guo and Lansing (in press), a set of parameters implying local uniquess gives rise to a flip bifurcation, which allows for cycles or sunspots away from the steady state.

8. Appendix A.2 proves this statement.

9. Guo and Lansing (in press) and Wen (1998) also show the positive relationship between the required degree of externalities and the degree of investment adjustment costs. However, this relationship is not analytically derived as in (10). For example, Wen (1998) does not comment on the upper bound, such as (9), on the degree of adjustment costs for indeterminacy to arise. Furthermore, his conclusion that the oscillation mechanism is preserved even under a unique equilibrium depends on the specification that adjustment costs are a function of the first difference in investment. In our model where the level of investment determines the costs, the oscillation mechanism disappears together with indeterminacy.

10. See Benhabib and Farmer (1994) and some references in Benhabib and Farmer (1999). According to this intution, increasing returns justify households' optimism of market returns on labor, which would then induce both higher employment and a higher wage rate.

11. The benchmark value of χ is 1 in the two-sector model of Benhabib and Farmer (1996). On the basis of Kim's (in press) results proving functional equivalence between intertemporal and multisectoral adjusment costs, we can show that it is hard to get indeterminacy in multisectoral models. See Kim (1998) and Herrendorf and Valentinyi (2000) for more on this.

12. Recent estimates of ψ are much bigger. Eberly (1997, Table 4) uses data for 11 countries, whose estimates for ψ range between 0.5 and 1.5. More recent work by Abel and Eberly (2002) using Compustat data yields estimates between 1.1 and 4.2. Jermann's (1998) estimate with the U.S. aggregate data is 4.3.

13. We use the notation, $\simeq$, when we approximate a function with respect to variables around their steady state.

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REFERENCES

Abel, A.B. & O.J. Blanchard (1983) An intertemporal model of saving and investment. *Econometrica* 51, 675–692.
Abel, A.B. & J.C. Eberly (2002) Investment and q with Fixed Costs: An Empirical Analysis. Manuscript, Northwestern University.
Benhabib, J. & R.E.A. Farmer (1994) Indeterminacy and increasing returns. *Journal of Economic Theory* 63, 19–41.
Benhabib, J. & R.E.A. Farmer (1996) Indeterminacy and sector-specific externalities. *Journal of Monetary Economics* 37, 421–443.
Benhabib, J. & R.E.A. Farmer (1999) Indeterminacy and sunspots in macroeconomics. In J. Taylor & M. Woodford (eds.), *Handbook of Macroeconomics*, Vol. 1A, pp. 387–488. New York: North-Holland.
Eberly, J.C. (1997) International evidence on investment and fundamentals. *European Economic Review* 41, 1055–1078.
Georges, C. (1995) Adjustment costs and indeterminacy in perfect foresight models. *Journal of Economic Dynamics and Control* 19, 39–50.
Guo, J.-T. & K.J. Lansing (1998) Indeterminacy and stabilization policy. *Journal of Economic Theory* 82, 481–490.
Guo, J.-T. & K.J. Lansing (1999) Fiscal Policy, Increasing Returns, and Endogenous Fluctuations. Working paper 99-08, Federal Reserve Bank of San Francisco.
Guo, J.-T. & K.J. Lansing (in press) Fiscal policy, increasing returns, and endogenous fluctuations. *Macroeconomic Dynamics*.
Hamermesh, D.S. & G.A. Pfann (1996) Adjustment costs in factor demand. *Journal of Economic Literature* 34, 1264–1292.
Hercowitz, Z. & M. Sampson (1991). Output growth, the real wage, and employment fluctuations. *American Economic Review* 81, 1215–1237.
Herrendorf, B. & A. Valentinyi (2000) Determinacy with Adjustment Cost and Sector-Specific Externalities. Manuscript, Central European University.
Jermann, U.J. (1998) Asset pricing in production economies. *Journal of Monetary Economics* 41, 257–275.
Kim, J. (1997) Three Sources of Increasing Returns to Scale. Finance and Economics Discussion Series 1997-18, Federal Reserve Board.
Kim, J. (1998) Indeterminacy and Investment Adjustment Costs. Finance and Economics Discussion Series 1998-38, Federal Reserve Board.
Kim, J. (in press) Functional equivalence between intertemporal and multisectoral investment adjustment costs. *Journal of Economic Dynamics and Control*.
Lucas, R.E., Jr. & E.C. Prescott (1971) Investment under uncertainty. *Econometrica* 39, 659–681.
Wen, Y. (1998) Indeterminacy, dynamic adjustment costs, and cycles. *Economics Letters* 59, 213–216.

APPENDIX

A.1. BEHAVIOR OF HOUSEHOLDS

The representative household maximizes (6) subject to (7) and (8). The current-value Hamiltonian is

$$H = \log C - \frac{L^{1+\chi}}{1+\chi} + \eta(ZK + WL - C - I) + \omega K \Psi\left(\frac{I}{K}\right),$$

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where η and ω are the marginal utility of an extra income and capital, respectively. The first-order conditions for C, L, I, and K are

$$\frac{1}{C} = \eta, \tag{A.1}$$

$$L^{\chi} = \eta W, \tag{A.2}$$

$$\eta = \omega \Psi' \left(\frac{I}{K} \right), \tag{A.3}$$

$$\eta Z + \omega \Psi \left(\frac{I}{K} \right) - \omega \left(\frac{I}{K} \right) \Psi' \left(\frac{I}{K} \right) = \rho \omega - \dot{\omega}, \tag{A.4}$$

Combining the conditions for consumption (A.1) and labor (A.2), we have a labor supply equation:

$$C L^{\chi} = W.$$

Note that the parameter χ represents the elasticity of labor supply conditional on consumption. In our setup where capital is accumulated by households rather than by firms, Tobin's q (the marginal value of capital measured in output) would be

$$q = \omega / \eta.$$

This transforms the condition for investment (A.3) into

$$q = \frac{1}{\Psi' \left(\frac{I}{K} \right)},$$

whose linearization produces a q regression in logs[13]:

$$\log \left(\frac{I}{K} \right) \simeq \log \delta + \frac{1}{\psi} \log q.$$

This shows that the elasticity of the Tobin's q investment equation is ψ^{-1}.

The properties of q rewrite the condition for capital (A.4) as follows:

$$\frac{\dot{\omega}}{\omega} = \rho - \Psi \left(\frac{I}{K} \right) + \frac{(I/K) - Z}{q}.$$

These properties, again, transform this equation into a dynamic optimality condition for households,

$$\dot{C} / C = R - \rho,$$

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where

$$R = \frac{Z + \left[q\Psi\left(\frac{I}{K}\right) - \left(\frac{I}{K}\right)\right] + \dot{q}}{q}$$

$$= Z\Psi'\left(\frac{I}{K}\right) + \left[\Psi\left(\frac{I}{K}\right) - \left(\frac{I}{K}\right)\Psi'\left(\frac{I}{K}\right)\right] - \frac{\left(\frac{\dot{I}}{I} - \frac{\dot{K}}{K}\right)\left(\frac{I}{K}\right)\Psi''\left(\frac{I}{K}\right)}{\Psi'\left(\frac{I}{K}\right)}.$$

The variable R would represent the interest rate of bonds if they were available to households.

Recapitulating, the optimizing behavior of the representative household is characterized by two equations. One is the labor supply equation,

$$CL^{\chi} = W, \tag{A.5}$$

and the other optimality condition relates consumption growth with the rental rate and investment adjustment costs as follows:

$$\frac{\dot{C}}{C} = Z\Psi'\left(\frac{I}{K}\right) + \left[\Psi\left(\frac{I}{K}\right) - \left(\frac{I}{K}\right)\Psi'\left(\frac{I}{K}\right)\right] - \frac{\left(\frac{\dot{I}}{I} - \frac{\dot{K}}{K}\right)\left(\frac{I}{K}\right)\Psi''\left(\frac{I}{K}\right)}{\Psi'\left(\frac{I}{K}\right)} - \rho. \tag{A.6}$$

A.2. ANALYSIS OF THE DYNAMICS

The labor demand equation (5) and the labor supply equation (A.5) are combined into a labor-market equilibrium condition,

$$\frac{bY}{L} = CL^{\chi}. \tag{A.7}$$

Substituting the demand for capital (4) for the rental rate in (A.6), we have an intertemporal optimality condition,

$$\frac{\dot{C}}{C} = \frac{aY}{K}\Psi'\left(\frac{I}{K}\right) + \left[\Psi\left(\frac{I}{K}\right) - \left(\frac{I}{K}\right)\Psi'\left(\frac{I}{K}\right)\right] - \frac{\left(\frac{\dot{I}}{I} - \frac{\dot{K}}{K}\right)\left(\frac{I}{K}\right)\Psi''\left(\frac{I}{K}\right)}{\Psi'\left(\frac{I}{K}\right)} - \rho. \tag{A.8}$$

Using the two factor demand schedules, (4) and (5), we transform the budget constraint (7) into a resource constraint,

$$C + I = Y. \tag{A.9}$$

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Now the equilibrium is restricted by five equations that consist of the aggregate production function (3), the capital accumulation equation (8), and (A.7)–(A.9).

Among the five variables in the system, labor (L) is eliminated from the system by combining the aggregate production function (3) and the labor-market equilibrium condition (A.7). Denoting lowercase letters for the logarithmic of the variables, we have

$$y = \lambda + (1+\kappa)k + \gamma c,$$

where

$$\lambda = \frac{\beta \log b}{\chi - (\beta - 1)}, \qquad \kappa = \frac{\alpha(1+\chi)}{\chi - (\beta - 1)} - 1, \qquad \gamma = \frac{-\beta}{\chi - (\beta - 1)}.$$

We now have a four-variable system, but it is not easy to analyze its global dynamics. To discuss the local dynamics, we approximate this system by linearizing it with respect to the variables around their steady state. Now we move on to the derivation of the bivariate first-order system step by step.

First, we linearize the function representing investment adjustment costs around the argument's steady state ($\bar{I}/\bar{K} = \delta$),

$$\Psi\left(\frac{I}{K}\right) \simeq \Psi(\delta) + \Psi'(\delta)\left(\frac{I}{K} - \delta\right) = \frac{I}{K} - \delta.$$

This approximation transforms the capital accumulation equation (8) as follows:

$$\frac{\dot{K}}{K} = \Psi\left(\frac{I}{K}\right) \simeq \frac{I}{K} - \delta, \tag{A.10}$$

which is equivalent to the capital accumulation equation without adjustment costs, $\dot{K} = I - \delta K$. That is, adjustment costs have not been introduced in the first order.

Second, linearization of the intertemporal optimality condition (A.8) needs the following approximations:

$$\Psi'\left(\frac{I}{K}\right) \simeq 1 - \frac{\psi}{\delta}\left(\frac{I}{K} - \delta\right) \simeq 1 - \frac{\psi}{\delta}\frac{\dot{K}}{K},$$

$$\left(\frac{I}{K}\right)\Psi'\left(\frac{I}{K}\right) \simeq \delta + (1-\psi)\left(\frac{I}{K} - \delta\right) \simeq \delta + (1-\psi)\frac{\dot{K}}{K},$$

$$-\frac{\left(\frac{\dot{I}}{I} - \frac{\dot{K}}{K}\right)\left(\frac{I}{K}\right)\Psi''\left(\frac{I}{K}\right)}{\Psi'\left(\frac{I}{K}\right)} \simeq \psi\left(\frac{\dot{I}}{I} - \frac{\dot{K}}{K}\right) = \frac{\psi}{\left(\frac{I}{K}\right)}\frac{d}{dt}\left(\frac{I}{K}\right) \simeq \frac{\psi}{\delta}\frac{d}{dt}\left(\frac{\dot{K}}{K}\right).$$

Now we linearize the intertemporal optimality condition (21) as follows:

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$$\frac{\dot{C}}{C} = \frac{aY}{K}\Psi'\left(\frac{I}{K}\right) + \left[\Psi\left(\frac{I}{K}\right) - \left(\frac{I}{K}\right)\Psi'\left(\frac{I}{K}\right)\right] - \frac{\left(\frac{\dot{I}}{I} - \frac{\dot{K}}{K}\right)\left(\frac{I}{K}\right)\Psi''\left(\frac{I}{K}\right)}{\Psi'\left(\frac{I}{K}\right)} - \rho$$

$$\simeq \frac{aY}{K}\left(1 - \frac{\psi}{\delta}\frac{\dot{K}}{K}\right) - \left[\delta - \psi\frac{\dot{K}}{K}\right] + \frac{\psi}{\delta}\frac{d}{dt}\left(\frac{\dot{K}}{K}\right) - \rho. \qquad \textbf{(A.11)}$$

Third, we express the two linearized dynamic equations in terms of only capital (K) and consumption (C). The resource constraint (A.9) is used in transforming the linearized capital accumulation equation (A.10),

$$\dot{k} = \frac{\dot{K}}{K} \simeq \frac{I}{K} - \delta = \frac{Y - C}{K} - \delta = \frac{Y}{K} - \frac{C}{K} - \delta = e^{\lambda + \kappa k + \gamma c} - e^{-k+c} - \delta, \qquad \textbf{(A.12)}$$

and we rewrite the linearized intertemporal optimality condition (A.11) as follows:

$$\dot{c} = \frac{\dot{C}}{C} \simeq ae^{\lambda+\kappa k+\gamma c}\left(1 - \frac{\psi}{\delta}\dot{k}\right) - (\delta - \psi\dot{k}) + \frac{\psi}{\delta}\dot{\dot{k}} - \rho. \qquad \textbf{(A.13)}$$

Fourth, linearizing this autonomous system of k and c, we will get the bivariate first-order linear system. Linearization requires calculating the steady state, and the relevant information on the steady state is

$$e^{\lambda+\kappa\bar{k}+\gamma\bar{c}} = \frac{\bar{Y}}{\bar{K}} = \frac{\rho+\delta}{a},$$

$$e^{-\bar{k}+\bar{c}} = \frac{\bar{C}}{\bar{K}} = \frac{\rho+b\delta}{a}.$$

Now, linearizing (A.12), we have

$$\begin{aligned}
\dot{k} &\simeq e^{\lambda+\kappa k+\gamma c} - e^{-k+c} - \delta \\
&\simeq e^{\lambda+\kappa\bar{k}+\gamma\bar{c}}\left(\frac{e^{\lambda+\kappa k+\gamma c} - e^{\lambda+\kappa\bar{k}+\gamma\bar{c}}}{e^{\lambda+\kappa\bar{k}+\gamma\bar{c}}}\right) - e^{-\bar{k}+\bar{c}}\left(\frac{e^{-k+c} - e^{-\bar{k}+\bar{c}}}{e^{-\bar{k}+\bar{c}}}\right) \\
&\simeq \frac{\rho+\delta}{a}[\kappa(k-\bar{k}) + \gamma(c-\bar{c})] - \frac{\rho+b\delta}{a}[-(k-\bar{k}) + (c-\bar{c})] \\
&= \tau(k-\bar{k}) + \theta(c-\bar{c}),
\end{aligned}$$

where

$$\tau = \frac{\rho+\delta}{a}\kappa + \frac{\rho+b\delta}{a} = \frac{\rho(1+\chi)(1+\xi) + \delta[\beta + (1+\chi)\xi]}{\chi - (\beta-1)},$$

$$\theta = \frac{\rho+\delta}{a}\gamma - \frac{\rho+b\delta}{a} = -\frac{(\rho+b\delta)\chi + a\delta(\beta-1) + (\rho+\delta)}{a[\chi-(\beta-1)]}.$$

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Replacing $\dot{\bar{k}}$ of (A.13) with the derivative of the above equation and then linearizing it, we have

$$\dot{c} \simeq ae^{\lambda+\kappa k+\gamma c}\left(1-\frac{\psi}{\delta}\dot{k}\right)-(\delta-\psi\dot{k})+\frac{\psi}{\delta}\dot{\bar{k}}-\rho$$

$$\simeq ae^{\lambda+\kappa\bar{k}+\gamma\bar{c}}\left(\frac{e^{\lambda+\kappa k+\gamma c}-e^{\lambda+\kappa\bar{k}+\gamma\bar{c}}}{e^{\lambda+\kappa\bar{k}+\gamma\bar{c}}}-\frac{\psi}{\delta}\dot{k}\right)+\psi\dot{k}+\frac{\psi}{\delta}(\tau\dot{k}+\theta\dot{c})$$

$$\simeq(\rho+\delta)\left[\kappa(k-\bar{k})+\gamma(c-\bar{c})-\frac{\psi}{\delta}\dot{k}\right]+\psi\dot{k}+\frac{\psi}{\delta}(\tau\dot{k}+\theta\dot{c})$$

$$=(\rho+\delta)[\kappa(k-\bar{k})+\gamma(c-\bar{c})]-\frac{\psi}{\delta}(\rho-\tau)\dot{k}+\frac{\psi}{\delta}\theta\dot{c}.$$

Multiplying δ on both sides, we rewrite this equation as

$$\psi(\rho-\tau)\dot{k}+(\delta-\psi\theta)\dot{c}\simeq\delta(\rho+\delta)[\kappa(k-\bar{k})+\gamma(c-\bar{c})].$$

Summarizing, our bivariate system is

$$\begin{bmatrix}1 & 0\\ \psi(\rho-\tau) & (\delta-\psi\theta)\end{bmatrix}\begin{bmatrix}\dot{k}\\ \dot{c}\end{bmatrix}=\begin{bmatrix}\tau & \theta\\ \delta(\rho+\delta)\kappa & \delta(\rho+\delta)\gamma\end{bmatrix}\begin{bmatrix}k-\bar{k}\\ c-\bar{c}\end{bmatrix},$$

The Jacobian of our first-order system is

$$\begin{bmatrix}1 & 0\\ \psi(\rho-\tau) & (\delta-\psi\theta)\end{bmatrix}^{-1}\begin{bmatrix}\tau & \theta\\ \delta(\rho+\delta)\kappa & \delta(\rho+\delta)\gamma\end{bmatrix}$$

$$=\frac{1}{\delta-\psi\theta}\begin{bmatrix}\tau(\delta-\psi\theta) & \theta(\delta-\psi\theta)\\ \delta(\rho+\delta)\kappa-\psi\tau(\rho-\tau) & \delta(\rho+\delta)\gamma-\psi\theta(\rho-\tau)\end{bmatrix}.$$

Its trace and determinant are expressed in terms of original parameters as follows:

$$\mathtt{Tr}=\frac{a\delta(\rho+\delta)(1+\chi)\xi}{a\delta[\chi-(\beta-1)]+\psi[(\rho+b\delta)\chi+a\delta(\beta-1)+(\rho+\delta)]}+\rho,$$

$$\mathtt{Det}=-\frac{\delta(1+\chi)(1-\alpha)(\rho+\delta)(\rho+b\delta)}{a\delta[\chi-(\beta-1)]+\psi[(\rho+b\delta)\chi+a\delta(\beta-1)+(\rho+\delta)]}.$$

Since the determinant passes through minus infinity to plus infinity as we increase ξ (and therefore β), the condition involving the trace is not binding.

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