J. Plasma Physics (2015), *vol.* 81, 905810111 © Cambridge University Press 2014
doi:10.1017/S0022377814000531

Quasineutrality and parallel force balance in kinetic magnetohydrodynamics

J. J. Ramos†

Plasma Science and Fusion Center, Massachusetts Institute of Technology, Cambridge, MA, USA

(Received 31 March 2014; revised 20 June 2014; accepted 4 July 2014;
first published online 1 August 2014)

Kinetic magnetohydrodynamics refers usually to the hybrid fluid and kinetic description of a zero-Larmor-radius collisionless plasma, originally formulated in the classic papers of Kruskal and Oberman (1958) (Kruskal, M. D. and Oberman, C. R. 1958 *Phys. Fluids* **1**, 275), Rosenbluth and Rostoker (1959) (Rosenbluth, M. N. and Rostoker, N. 1959 *Phys. Fluids* **2**, 23), and Kulsrud (1962) (Kulsrud, R. 1962 *Phys. Fluids* **5**, 192). Such a theory is revisited here, as a special limit of the more general description put forward in Ramos (2010, 2011) (Ramos, J. J. 2010 *Phys. Plasmas*, **17**, 082502; Ramos, J. J. 2011 *Phys. Plasmas*, **18**, 102506). The present approach has the advantage of fulfilling the quasineutrality condition and avoiding the redundancy between the fluid and kinetic parallel force balance conditions with a built-in, rigorous account of the parallel electric field, thus affording a clear-cut handling of these issues. At zero-frequency marginal stability, the Rosenbluth–Rostoker fluid closures for the parallel and perpendicular pressures are obtained, in a solution with vanishing parallel electric field and non-zero parallel fluid displacement that satisfies exactly the desired quasineutrality and parallel force balance.

1. Introduction

A fluid and drift-kinetic model for magnetic confinement plasmas, applicable to macroscopic physics with length scales larger than the ion Larmor radius, has been developed in Ramos (2010, 2011). This is a rather general electromagnetic theory, that takes into account finite-Larmor-radius (FLR) effects to first order for the electrons and second order for the ions, as well as collisional effects under a long mean-free-path ordering relevant to fusion conditions. As a special application, it has been shown to yield the neoclassical results of the tokamak banana regime (Ramos 2010; Lyons et al. 2012). The aim of the present work is to explore, as another special case of the theory, its zero-Larmor-radius and collisionless limit. Such a limit corresponds to the kinetic magnetohydrodynamics (KMHD) originally formulated in Kruskal and Oberman (1958), Rosenbluth and Rostoker (1959), and Kulsrud (1962a, b). The present approach has a number of distinguishing features which make for an interesting comparison with those classic works.

The work of Kruskal and Oberman (1958) concerned the derivation of a sufficient condition for stability (characterized by a positive definite potential energy functional) without enforcing the quasineutrality condition. The present work will concentrate on the linear equations for quasineutral normal modes, including their solution at

† Email address for correspondence: ramos@psfc.mit.edu

https://doi.org/10.1017/S0022377814000531 Published online by Cambridge University Press

zero-frequency marginal stability, which is another aspect of the theory and was part of the Rosenbluth–Rostoker analysis (Rosenbluth and Rostoker 1959). However, the method to be followed here does not rely on the initial value solution of the Vlasov equation by integration along unperturbed particle orbits used in Rosenbluth and Rostoker (1959). It is based on a drift-kinetic equation and formulates a standard normal-mode problem, with boundary conditions that include in a mathematically precise fashion the case of passing particles on ergodic magnetic field lines. The use of a drift-kinetic equation makes it closer to the Kulsrud approach (Kulsrud 1962a, b) but there are additional differences here too. These have to do with the fact that the velocity variables of Kulsrud's drift-kinetic equation are in the reference frame of the $\mathbf{E} \times \mathbf{B}$ drift, whereas the drift-kinetic equations of Ramos (2010, 2011) are referenced to the frame of the complete mean flow velocity of the species under consideration. Solving for a distribution function in the mean flow reference frame allows a direct evaluation of the fluid closure moments without having to subtract the parallel component of the macroscopic flow velocity. Besides, working in the mean flow reference frame allows to eliminate the electric field by exact algebraic transformations so that the correct accounting of its tricky parallel component is automatic. The other noteworthy property of the approach in Ramos (2010, 2011) is that the distribution function dynamical evolution is such that its density and random kinetic energy moments remain identical to those of a Maxwellian part whose density and temperature are determined by a quasineutral set of fluid equations. Therefore, the kinetic quasineutrality condition (electron and ion distribution functions yielding charge density moments that add up to zero) is always guaranteed in a time-evolving system, without the need to impose any additional constraint other than a quasineutral initial condition. Likewise, the parallel velocity moment of the considered drift-kinetic equation in the mean flow reference frame yields just the required condition that the random parallel velocity moment of the distribution function is zero, hence the redundancy between the drift-kinetic equation and the parallel component of the fluid momentum conservation equation is avoided.

Some comments on two technical points should be added. First, the formalism of Ramos (2010, 2011) assumes a slow dynamics ordering with mean flow velocities comparable to the diamagnetic drifts, whereas KMHD assumes fast dynamics with mean flows of the order of the ion thermal speed. However, since Ramos (2010, 2011) include FLR effects, all the flow terms needed in KMHD are available because they are retained as comparable to the FLR terms. These flow terms can be uplifted one order and kept together with the zero-Larmor radius part when the intrinsically FLR part is dropped in the KMHD limit. Second, the formalism of Ramos (2010, 2011) assumes near-Maxwellian distribution functions with non-Maxwellian parts ordered as smaller than the Maxwellians. This restriction is of no consequence once the systems are linearized, provided the equilibrium is Maxwellian as is normally the case.

2. Zero-Larmor-radius, collisionless system

The zero-Larmor-radius collisionless system to be considered in this work is derived from the general system of Ramos (2010, 2011) by taking the limit of infinite Larmor frequencies and zero collision frequencies. A single ion species of unit charge is assumed for simplicity. The smallness of the electron-to-ion mass ratio does not play any essential role in a collisionless theory, hence no mass ratio expansions are to be made here and the two species masses enter in symmetrical fashion. The kinetic pressure of each species is formally taken as comparable to the magnetic pressure.

https://doi.org/10.1017/S0022377814000531 Published online by Cambridge University Press

Therefore, the skin depths tend to zero along with the Larmor radii and the two species share a common, single-fluid macroscopic velocity **u**. Under these conditions, the following magnetofluid system is obtained:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}), \tag{1}$$

$$\mathbf{j} = \nabla \times \mathbf{B}, \tag{2}$$

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0, \tag{3}$$

$$(m_i + m_e)n\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] - \mathbf{j} \times \mathbf{B} + \sum_{s=i,e} \nabla \cdot \left[nT_s \mathsf{I} + (p_{s\parallel} - p_{s\perp})(\mathbf{bb} - \mathsf{I}/3)\right] = 0, \tag{4}$$

$$\frac{3n}{2}\left(\frac{\partial T_s}{\partial t} + \mathbf{u} \cdot \nabla T_s\right) + \left[nT_s \mathsf{I} + (p_{s\parallel} - p_{s\perp})(\mathbf{bb} - \mathsf{I}/3)\right] : (\nabla \mathbf{u}) + \nabla \cdot (q_{s\parallel}\mathbf{b}) = 0, \tag{5}$$

where $\mathbf{b} = \mathbf{B}/B$ is the magnetic unit vector and I is the identity tensor. The non-Maxwellian gyrotropic moments needed to close this system, which must be determined kinetically, are the pressure anisotropies

$$(p_{s\parallel} - p_{s\perp}) = \pi m_s \int_0^\infty dv' v'^4 \int_0^\pi d\chi \sin\chi (3\cos^2\chi - 1)\, \bar{f}_{NMs}(v', \chi, \mathbf{x}, t) \tag{6}$$

and the parallel heat fluxes

$$q_{s\parallel} = \pi m_s \int_0^\infty dv' v'^5 \int_0^\pi d\chi \sin\chi \cos\chi\, \bar{f}_{NMs}(v', \chi, \mathbf{x}, t). \tag{7}$$

Here $\mathbf{v}'$ represents the random velocity in the moving frame of the macroscopic flow, the angular variables χ and α being respectively its pitch angle and gyrophase:

$$\mathbf{v} - \mathbf{u}(\mathbf{x}, t) = \mathbf{v}' = v'\{\cos\chi \mathbf{b}(\mathbf{x}, t) + \sin\chi\, [\cos\alpha \mathbf{e}_1(\mathbf{x}, t) + \sin\alpha \mathbf{e}_2(\mathbf{x}, t)]\}, \tag{8}$$

and $\bar{f}_{NMs}$ is the gyroaverage of the difference between the actual distribution function of species s and a moving Maxwellian with the fluid density and mean temperature:

$$\bar{f}_{NMs}(v', \chi, \mathbf{x}, t) = (2\pi)^{-1} \int_0^{2\pi} d\alpha [f_s(v', \chi, \alpha, \mathbf{x}, t) - f_{Ms}(v', \mathbf{x}, t)], \tag{9}$$

$$f_{Ms}(v', \mathbf{x}, t) = \left(\frac{m_s}{2\pi}\right)^{3/2} \frac{n}{T_s^{3/2}} \exp\left(-\frac{m_s v'^2}{2T_s}\right). \tag{10}$$

The dynamical evolution of $\bar{f}_{NMs}$ is governed by the drift-kinetic equation

$$\begin{aligned}
&\frac{\partial \bar{f}_{NMs}}{\partial t} + \cos\chi \left(v'\mathbf{b} \cdot \frac{\partial \bar{f}_{NMs}}{\partial \mathbf{x}} + \frac{T_s}{m_s}\mathbf{b} \cdot \nabla \ln n \frac{\partial \bar{f}_{NMs}}{\partial v'}\right) \\
&\quad - \frac{\sin\chi}{v'}\left(\frac{T_s}{m_s}\mathbf{b} \cdot \nabla \ln n - \frac{v'^2}{2}\mathbf{b} \cdot \nabla \ln B\right)\frac{\partial \bar{f}_{NMs}}{\partial \chi} \\
&= \left\{\cos\chi \frac{v'}{2T_s}\left(5 - \frac{m_s v'^2}{T_s}\right)\mathbf{b} \cdot \nabla T_s \right. \\
&\quad \left. + \cos\chi \frac{v'}{nT_s}\mathbf{b} \cdot \left[\frac{2}{3}\nabla(p_{s\parallel} - p_{s\perp}) - (p_{s\parallel} - p_{s\perp})\nabla \ln B\right]\right.
\end{aligned}$$

https://doi.org/10.1017/S0022377814000531 Published online by Cambridge University Press

$$+(3\cos^2\chi-1)\frac{m_s v'^2}{2T_s}(\mathsf{I}/3-\mathbf{bb}):(\nabla\mathbf{u})+\frac{1}{3nT_s}\left(\frac{m_s v'^2}{T_s}-3\right)\nabla\cdot(q_{s\parallel}\mathbf{b})\Bigg\}f_{Ms}. \tag{11}$$

In deriving this moving frame drift-kinetic equation (Ramos 2010, 2011), the electric field was eliminated algebraically using the exact momentum conservation equation of the species under consideration. As a result, the complete system (1–11) is written without any explicit reference to the electric field, which is nevertheless taken into account rigorously. The 1, $v'_\parallel$, and v'^2 moments of (11) yield simply the Chapman–Enskog-like result that the constraints $\int d^3\mathbf{v}'(1, v'_\parallel, v'^2)\bar{f}_{NMs}=0$ are preserved through the time evolution of $\bar{f}_{NMs}$, provided they are satisfied by the initial condition. Therefore, quasineutrality is always satisfied and the redundancy of the drift-kinetic equation with parts of the fluid system is avoided.

3. Linear normal modes

The above zero-Larmor-radius, collisionless system is to be linearized about a Maxwellian equilibrium without mean flow, assuming normal-mode perturbations with the exponential time dependence $\partial/\partial t=-i\omega$. Using the 0 and 1 subscripts to denote respectively equilibrium and perturbation quantities and defining the displacement vector $\boldsymbol{\xi}=i\mathbf{u}_1/\omega$, one gets the equilibrium conditions

$$\mathbf{u}_0=0,\quad \mathbf{B}_0\cdot\nabla n_0=0,\quad \mathbf{B}_0\cdot\nabla T_{s0}=0,\quad \mathbf{j}_0\times\mathbf{B}_0=(\nabla\times\mathbf{B}_0)\times\mathbf{B}_0=\nabla[n_0(T_{i0}+T_{e0})], \tag{12}$$

the linearized magnetofluid system

$$\mathbf{B}_1=\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0), \tag{13}$$

$$\mathbf{j}_1=\nabla\times\mathbf{B}_1, \tag{14}$$

$$n_1=-\boldsymbol{\xi}\cdot\nabla n_0-n_0\nabla\cdot\boldsymbol{\xi}, \tag{15}$$

$$\omega^2(m_i+m_e)n_0\boldsymbol{\xi}=-\mathbf{j}_0\times\mathbf{B}_1-\mathbf{j}_1\times\mathbf{B}_0+\sum_{s=i,e}\left\{\nabla(n_1T_{s0}+n_0T_{s1})+\nabla\cdot\left[(p_{s\parallel}-p_{s\perp})(\mathbf{b}_0\mathbf{b}_0-\mathsf{I}/3)\right]\right\}, \tag{16}$$

$$T_{s1}=-\boldsymbol{\xi}\cdot\nabla T_{s0}-\frac{2}{3}T_{s0}\nabla\cdot\boldsymbol{\xi}-\frac{2i}{3\omega n_0}\nabla\cdot(q_{s\parallel}\mathbf{b}_0), \tag{17}$$

and the linearized drift-kinetic equations

$$-i\omega\bar{f}_{NMs}+v'\cos\chi\,\mathbf{b}_0\cdot\frac{\partial\bar{f}_{NMs}}{\partial\mathbf{x}}+\frac{v'}{2}\sin\chi\,\mathbf{b}_0\cdot\nabla\ln B_0\,\frac{\partial\bar{f}_{NMs}}{\partial\chi}$$
$$=\Bigg\{\cos\chi\,\frac{v'}{2T_{s0}}\left(5-\frac{m_s v'^2}{T_{s0}}\right)(\mathbf{b}_0\cdot\nabla T_{s1}+\mathbf{b}_1\cdot\nabla T_{s0})$$
$$+\cos\chi\,\frac{v'}{nT_{s0}}\mathbf{b}_0\cdot\left[\frac{2}{3}\nabla(p_{s\parallel}-p_{s\perp})-(p_{s\parallel}-p_{s\perp})\nabla\ln B_0\right]$$
$$-i\omega(3\cos^2\chi-1)\frac{m_s v'^2}{2T_s}(\mathsf{I}/3-\mathbf{b}_0\mathbf{b}_0):(\nabla\boldsymbol{\xi})+\frac{1}{3n_0T_{s0}}\left(\frac{m_s v'^2}{T_{s0}}-3\right)\nabla\cdot(q_{s\parallel}\mathbf{b}_0)\Bigg\}f_{Ms0}. \tag{18}$$

https://doi.org/10.1017/S0022377814000531 Published online by Cambridge University Press

Since the equilibrium distribution functions are Maxwellian, the equilibrium values of $\bar{f}_{NMs}$ and their moments $(p_{s\parallel} - p_{s\perp})$ and $q_{s\parallel}$ are zero. These quantities appear only in the perturbation and the subscript 1 has been omitted for them.

The next step is to make a change of phase space independent variables, from $(v', \chi, \mathbf{x})$ to $(v', \lambda, \mathbf{x})$ where $\lambda = \sin^2\chi / B_0(\mathbf{x})$ is the ratio of the magnetic moment to the kinetic energy in the moving frame. The dependent variables are also changed in order to work with the functions

$$\hat{f}_s = \bar{f}_{NMs} - \frac{1}{3}\left[\frac{m_s v'^2}{T_{s0}}\nabla\cdot\boldsymbol{\xi} + \frac{i}{\omega n_0 T_{s0}}\left(\frac{m_s v'^2}{T_{s0}} - 3\right)\nabla\cdot(q_{s\parallel}\mathbf{b}_0)\right] f_{Ms0} \tag{19}$$

that represent the non-convective part of the total perturbed distribution functions (non-Maxwellian plus perturbed Maxwellian),

$$\hat{f}_s = \bar{f}_{NMs} + f_{Ms1} + \boldsymbol{\xi}\cdot\frac{\partial f_{Ms0}}{\partial \mathbf{x}}. \tag{20}$$

The convective part of the perturbation can be accounted for entirely within the fluid formalism and the functions $\hat{f}_s$ represent the intrinsically kinetic part. Setting $\hat{f}_s = 0$ would result in a truncated fluid model with isotropic total pressure and linearized pressure equation $p_1 = -\boldsymbol{\xi}\cdot\nabla p_0$, i.e. the so-called perpendicular ideal-MHD that neglects the compressional effect on the pressure and involves only the perpendicular displacement $\boldsymbol{\xi}_\perp$. In the present kinetic model, the resulting drift-kinetic equation for the function $\hat{f}_s$ of each species is

$$-i\omega\hat{f}_s + v'\varsigma(1-\lambda B_0)^{1/2}\,\mathbf{b}_0\cdot\frac{\partial\hat{f}_s}{\partial\mathbf{x}} = \left[\frac{v'\varsigma}{n_0 T_{s0}}(1-\lambda B_0)^{1/2}\hat{F}_{s\parallel} + \frac{i\omega m_s v'^2}{T_{s0}}Q\right] f_{Ms0}, \tag{21}$$

where $\varsigma = \mathrm{sign}(\cos\chi) = \mathrm{sign}(v'_\parallel)$, the parallel gradient $\mathbf{b}_0\cdot\partial/\partial\mathbf{x}$ is now taken at constant λ, and the coefficient functions $\hat{F}_{s\parallel}(\mathbf{x})$ and $Q(\lambda, \mathbf{x})$ are

$$\hat{F}_{s\parallel}(\mathbf{x}) = \mathbf{b}_0\cdot\left\{\nabla\left[-\frac{5}{3}n_0 T_{s0}\nabla\cdot\boldsymbol{\xi} - \frac{2i}{3\omega}\nabla\cdot(q_{s\parallel}\mathbf{b}_0) + \frac{2}{3}(p_{s\parallel} - p_{s\perp})\right] - (p_{s\parallel} - p_{s\perp})\nabla\ln B_0\right\} \tag{22}$$

and

$$Q(\lambda, \mathbf{x}) = \frac{1}{2}\lambda B_0\nabla\cdot\boldsymbol{\xi} + \left(1 - \frac{3}{2}\lambda B_0\right)\mathbf{b}_0\cdot[(\mathbf{b}_0\cdot\nabla)\boldsymbol{\xi}]. \tag{23}$$

The functions $\hat{F}_{s\parallel}(\mathbf{x})$ represent the parallel magnetofluid force density, such that the parallel component of (16) is

$$\omega^2(m_i + m_e)n_0\xi_\parallel = \hat{F}_{i\parallel} + \hat{F}_{e\parallel}. \tag{24}$$

Separating the contributions of the parallel and perpendicular components of the displacement, $\xi_\parallel$ and $\boldsymbol{\xi}_\perp$, (23) can be rewritten as

$$Q(\lambda, \mathbf{x}) = (1-\lambda B_0)^{1/2}\mathbf{b}_0\cdot\frac{\partial}{\partial\mathbf{x}}\left[(1-\lambda B_0)^{1/2}\xi_\parallel\right] + \frac{1}{2}\lambda B_0\nabla\cdot\boldsymbol{\xi}_\perp - \left(1 - \frac{3}{2}\lambda B_0\right)\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}_0, \tag{25}$$

where $\boldsymbol{\kappa}_0 = (\mathbf{b}_0\cdot\nabla)\mathbf{b}_0$ is the equilibrium magnetic curvature.

https://doi.org/10.1017/S0022377814000531 Published online by Cambridge University Press

It is convenient to work separately with the parts of $\hat{f}_s$ that are even and odd with respect to $v'_\parallel$, so that (21) splits into the system

$$-i\omega \hat{f}_s^{\text{even}} + v'\varsigma(1-\lambda B_0)^{1/2}\ \mathbf{b}_0 \cdot \frac{\partial \hat{f}_s^{\text{odd}}}{\partial \mathbf{x}} = \frac{i\omega m_s v'^2}{T_{s0}} Q f_{Ms0} \tag{26}$$

$$-i\omega \hat{f}_s^{\text{odd}} + v'\varsigma(1-\lambda B_0)^{1/2}\ \mathbf{b}_0 \cdot \frac{\partial \hat{f}_s^{\text{even}}}{\partial \mathbf{x}} = \frac{v'\varsigma}{n_0 T_{s0}}(1-\lambda B_0)^{1/2} \hat{F}_{s\parallel} f_{Ms0}. \tag{27}$$

Then, $\hat{f}_s^{\text{odd}}$ can be eliminated, to obtain the normal-mode drift-kinetic equation for $\hat{f}_s^{\text{even}}$

$$\omega^2 \hat{f}_s^{\text{even}} + v'^2(1-\lambda B_0)^{1/2}\ \mathbf{b}_0 \cdot \frac{\partial}{\partial \mathbf{x}} \left[(1-\lambda B_0)^{1/2}\ \mathbf{b}_0 \cdot \frac{\partial \hat{f}_s^{\text{even}}}{\partial \mathbf{x}} \right]$$
$$= \left\{ \frac{v'^2}{n_0 T_{s0}}(1-\lambda B_0)^{1/2}\ \mathbf{b}_0 \cdot \frac{\partial}{\partial \mathbf{x}} \left[(1-\lambda B_0)^{1/2}\ \hat{F}_{s\parallel} \right] - \frac{\omega^2 m_s v'^2}{T_{s0}} Q \right\} f_{Ms0}. \tag{28}$$

4. Moment relations

A number of interesting relations can be obtained by considering moments of the previous equations. First, changing the parallel heat flux definition (7) to the new variables (v', λ) and recalling the fact that the difference between $\bar{f}_{NMs}$ and $\hat{f}_s$ (19) is even with respect to $v'_\parallel$, one gets

$$q_{s\parallel} = \pi m_s B_0 \int_0^\infty dv' v'^5 \int_0^{1/B_0} d\lambda\ \varsigma \hat{f}_s^{\text{odd}} \tag{29}$$

from which one can evaluate

$$\nabla \cdot (q_{s\parallel} \mathbf{b}_0) = \mathbf{b}_0 \cdot \nabla q_{s\parallel} - q_{s\parallel} \mathbf{b}_0 \cdot \nabla \ln B_0 = \pi m_s B_0 \int_0^\infty dv' v'^5 \int_0^{1/B_0} d\lambda\ \varsigma \mathbf{b}_0 \cdot \frac{\partial \hat{f}_s^{\text{odd}}}{\partial \mathbf{x}}. \tag{30}$$

Substituting now equation (26) for $\mathbf{b}_0 \cdot \partial \hat{f}_s^{\text{odd}}/\partial \mathbf{x}$ and equation (23) for Q, and carrying out the integral of the term proportional to f_{Ms0}, it follows that

$$\frac{i}{\omega} \nabla \cdot (q_{s\parallel} \mathbf{b}_0) = -\frac{5}{2} n_0 T_{s0} \nabla \cdot \boldsymbol{\xi} - \pi m_s B_0 \int_0^\infty dv' v'^4 \int_0^{1/B_0} d\lambda\ (1-\lambda B_0)^{-1/2} \hat{f}_s^{\text{even}} \tag{31}$$

which, compared with the v'^2 moment of equation (19), can be verified to be equivalent to the expected condition

$$\int d^3\mathbf{v}' v'^2 \bar{f}_{NMs} = 0. \tag{32}$$

Next, one can change the definition of the pressure anisotropy equation (6) to the (v', λ) variables and recall the fact that the difference between $\bar{f}_{NMs}$ and $\hat{f}_s$ equation (19) is isotropic in velocity space, to get the expression

$$(p_{s\parallel} - p_{s\perp}) = 2\pi m_s B_0 \int_0^\infty dv' v'^4 \int_0^{1/B_0} d\lambda (1-\lambda B_0)^{-1/2} \left(1 - \frac{3}{2}\lambda B_0 \right) \hat{f}_s^{\text{even}} \tag{33}$$

https://doi.org/10.1017/S0022377814000531 Published online by Cambridge University Press

which, combined with equation (31), yields

$$\frac{2}{3}(p_{s\parallel}-p_{s\perp})-\frac{2i}{3\omega}\nabla\cdot(q_{s\parallel}\mathbf{b}_0)-\frac{5}{3}n_0T_{s0}\nabla\cdot\boldsymbol{\xi}=2\pi m_sB_0\int_0^\infty dv'v'^4\int_0^{1/B_0}d\lambda(1-\lambda B_0)^{1/2}\hat{f}_s^{\text{even}}. \tag{34}$$

Bringing these results to the definition of $\hat{F}_{s\parallel}$ equation (22), one obtains

$$\hat{F}_{s\parallel}=\ 2\pi m_sB_0\int_0^\infty dv'v'^4\int_0^{1/B_0}d\lambda(1-\lambda B_0)^{1/2}\ \mathbf{b}_0\cdot\frac{\partial\hat{f}_s^{\text{even}}}{\partial\mathbf{x}}. \tag{35}$$

The $v'_\parallel$ moment of equation (19) is

$$\int d^3\mathbf{v}'v'_\parallel\bar{f}_{NMs}=\ \int d^3\mathbf{v}'v'_\parallel\hat{f}_s=\ 2\pi B_0\int_0^\infty dv'v'^3\int_0^{1/B_0}d\lambda\varsigma\,\hat{f}_s^{\text{odd}} \tag{36}$$

and, substituting equation (27) for $\hat{f}_s^{\text{odd}}$, provided $\omega\neq0$ it becomes

$$\int d^3\mathbf{v}'v'_\parallel\bar{f}_{NMs}=\ \frac{2\pi B_0}{i\omega}\int_0^\infty dv'v'^4\int_0^{1/B_0}d\lambda(1-\lambda B_0)^{1/2}\left(\mathbf{b}_0\cdot\frac{\partial\hat{f}_s^{\text{even}}}{\partial\mathbf{x}}-\ \frac{\hat{F}_{s\parallel}}{n_0T_{s0}}f_{Ms0}\right). \tag{37}$$

Now, after integrating the term proportional to f_{Ms0}, it is verified that (35) and (37) yield the expected condition

$$\int d^3\mathbf{v}'v'_\parallel\bar{f}_{NMs}=0. \tag{38}$$

Finally, from the parallel gradient of (36), (38) divided by B_0, it follows that

$$\int_0^\infty dv'v'^3\int_0^{1/B_0}d\lambda\varsigma\ \mathbf{b}_0\cdot\frac{\partial\hat{f}_s^{\text{odd}}}{\partial\mathbf{x}}=0. \tag{39}$$

The density moment of $\hat{f}_s$ is

$$\int d^3\mathbf{v}'\hat{f}_s=\ 2\pi B_0\int_0^\infty dv'v'^2\int_0^{1/B_0}d\lambda(1-\lambda B_0)^{-1/2}\hat{f}_s^{\text{even}} \tag{40}$$

and, taking into account (26) and (39), provided $\omega\neq0$ it becomes

$$\int d^3\mathbf{v}'\hat{f}_s=\ -2\pi m_sB_0\int_0^\infty dv'v'^4\int_0^{1/B_0}d\lambda(1-\lambda B_0)^{-1/2}\ \frac{Q}{T_{s0}}f_{Ms0}. \tag{41}$$

With expression (23) for Q, this can be integrated to get

$$\int d^3\mathbf{v}'\hat{f}_s=\ -n_0\nabla\cdot\boldsymbol{\xi} \tag{42}$$

which, compared with the density moment of equation (19), is equivalent to the expected condition

$$\int d^3\mathbf{v}'\bar{f}_{NMs}=0. \tag{43}$$

Equations (17), (31), (40), and (42) can be combined to express the perturbed mean temperature as the sum of its convective part plus a single moment of $\hat{f}_s^{\text{even}}$,

$$T_{s1}=-\boldsymbol{\xi}\cdot\nabla T_{s0}-\ \frac{2\pi}{3n_0}B_0\int_0^\infty dv'v'^2(3T_{s0}-m_sv'^2)\int_0^{1/B_0}d\lambda(1-\lambda B_0)^{-1/2}\hat{f}_s^{\text{even}}. \tag{44}$$

https://doi.org/10.1017/S0022377814000531 Published online by Cambridge University Press

This, together with (33), (35) that give $(p_{s\parallel} - p_{s\perp})$ and $\hat{F}_{s\parallel}$ also as single moments of $\hat{f}_s^{\text{even}}$ and its parallel gradient respectively, allows to close the normal-mode magnetofluid set of (13)–(16) with the solution of just the normal-mode drift-kinetic equation (28) for $\hat{f}_s^{\text{even}}$.

5. Solution of the normal-mode drift-kinetic equation and marginal stability closure

The only spatial derivative involved in the normal-mode drift-kinetic equation is the parallel gradient $\mathbf{b}_0 \cdot \partial/\partial \mathbf{x}$. Therefore, it is natural to use the arc length along the magnetic field lines ℓ as one of the three spatial coordinates, the other two (ψ, ζ) being Clebsch coordinates that label each magnetic line $\mathbf{B}_0 = \nabla\psi \times \nabla\zeta$. Further, in order to solve (28), the arc length is conveniently replaced by the time-dimensioned variable

$$\tau = \frac{1}{v'} \int_{\ell_0}^{\ell} d\ell' \ \left[1 - \lambda B_0(\ell')\right]^{-1/2}, \tag{45}$$

with the line integral taken at constant v' and λ along the magnetic field line from an appropriately chosen origin ℓ_0 (at the bottom of the magnetic wells) to make τ real. Then, one has

$$v' \left(1 - \lambda B_0\right)^{1/2} \mathbf{b}_0 \cdot \frac{\partial}{\partial \mathbf{x}} = \frac{\partial}{\partial \tau} \tag{46}$$

so that (28) becomes

$$\omega^2 \hat{f}_s^{\text{even}} + \frac{\partial^2 \hat{f}_s^{\text{even}}}{\partial \tau^2} = \left\{ \frac{v'}{n_0 T_{s0}} \frac{\partial}{\partial \tau} \left[\left(1 - \lambda B_0\right)^{1/2} \hat{F}_{s\parallel} \right] - \frac{\omega^2 m_s v'^2}{T_{s0}} Q \right\} f_{Ms0}, \tag{47}$$

and the integral relation (35) that links $\hat{F}_{s\parallel}$ to $\hat{f}_s^{\text{even}}$ becomes

$$\hat{F}_{s\parallel} = 2\pi m_s B_0 \int_0^{\infty} dv' v'^3 \int_0^{1/B_0} d\lambda \frac{\partial \hat{f}_s^{\text{even}}}{\partial \tau}. \tag{48}$$

Equation (47) has the form of a driven simple harmonic oscillator. Thus, a general expression that solves formally for it, but that still has to be coupled to (48), is

$$\begin{aligned} \hat{f}_s^{\text{even}} = {} & \frac{v'}{n_0 T_{s0}} f_{Ms0} \int_0^{\tau} d\tau' \ \cos[\omega(\tau - \tau')] \left[\left(1 - \lambda B_0\right)^{1/2} \hat{F}_{s\parallel} \right](\tau') \\ & - \frac{\omega^2 m_s v'^2}{T_{s0}} f_{Ms0} \int_0^{\tau} d\tau' \ \exp[-i\omega(\tau - \tau')] \int_0^{\tau'} d\tau'' \exp[i\omega(\tau' - \tau'')] \ Q(\tau'') \\ & + C_s^{\omega}(v', \lambda, \psi, \zeta) \cos\omega\tau + D_s^{\omega}(v', \lambda, \psi, \zeta) \sin\omega\tau. \end{aligned} \tag{49}$$

The first two terms of this expression are a particular solution that can be verified to satisfy (47) by direct substitution. The last two terms are the general solution of the homogeneous part of (47) and contain the integration constants $C_s^{\omega}(v', \lambda, \psi, \zeta)$ and $D_s^{\omega}(v', \lambda, \psi, \zeta)$ which must be specified by imposing appropriate boundary conditions on $\hat{f}_s^{\text{even}}$. These boundary conditions depend on the region of phase space and on the topology of the magnetic field. For the sake of conciseness, only confined plasma configurations with magnetic shear will be considered here, a class that includes the most promising magnetic confinement devices such as tokamaks and stellerators. In these configurations, except for a negligible zero-measure set of closed loops, the

https://doi.org/10.1017/S0022377814000531 Published online by Cambridge University Press

magnetic field lines have infinite length and fill ergodically some bounded spatial domain. Two types of boundary conditions must be taken into account. The first one applies to trapped particle regions of phase space, where $\hat{f}_s^{\text{even}}$ should satisfy

$$\frac{\partial \hat{f}_s^{\text{even}}(\tau = \tau_{b+})}{\partial \tau} = \frac{\partial \hat{f}_s^{\text{even}}(\tau = \tau_{b-})}{\partial \tau} = 0 \tag{50}$$

with τ_{b+} and τ_{b-} as the coordinates of the turning points. This is required so that the distribution function has continuous derivative at $v'_\parallel = 0$ when expressed back in the primary coordinates. The second one applies to the passing particle regions, where the τ coordinate ranges from $-\infty$ to $+\infty$. Now, assuming an unstable mode with $\gamma = \text{Im}\,\omega > 0$, the expression (49) for $\hat{f}_s^{\text{even}}$ gives rise to exponentially growing terms as $\tau \to \pm\infty$. The appropriate boundary conditions are such that those exponentially growing terms cancel out and $\hat{f}_s^{\text{even}}$ remains bounded.

In general, implementing the above boundary conditions and solving simultaneously for (48), (49) is complicated. However, the analysis is greatly simplified if one considers only the zero-frequency, marginal stability point defined by the limit $\omega \to 0$ with $\gamma = \text{Im}\,\omega > 0$. In this limit, it is apparent that the solution

$$\hat{f}_s^{\text{even}} = C_s^0(v', \lambda, \psi, \zeta), \quad \hat{F}_{s\parallel} = 0 \tag{51}$$

satisfies (47)–(49) and the boundary conditions. Moreover, this is identically consistent with the parallel component of the fluid momentum equation (16, 24) at $\omega = 0$. The specific function $C_s^0(v', \lambda, \psi, \zeta)$ must be determined by imposing the boundary conditions first for $\omega \neq 0$ and then letting $\omega \to 0$, as shown in the Appendix. The final result is

$$\hat{f}_s^{\text{even}} = -\frac{m_s v'^2}{T_{s0}} \langle Q \rangle_\tau f_{Ms0}, \tag{52}$$

where the $\langle Q \rangle_\tau$ average is defined as follows. For trapped particles,

$$\langle Q \rangle_\tau = \frac{1}{\tau_{b+} - \tau_{b-}} \int_{\tau_{b-}}^{\tau_{b+}} d\tau\, Q, \tag{53}$$

and for passing particles,

$$\langle Q \rangle_\tau = \frac{1}{2} \lim_{\gamma \to 0} \left[\gamma \int_{-\infty}^{+\infty} d\tau \exp(-\gamma|\tau|) Q \right]. \tag{54}$$

This mathematically precise expression for passing particles on infinitely long magnetic field lines (54) follows from the previously stated boundary conditions that the exponentially growing terms of $\hat{f}_s^{\text{even}}$ cancel out. It has the properties of an average and matches continuously (53) at the trapped-passing boundary, where $\tau_{b\pm} \to \pm\infty$ on the trapped side. Given the representation (25) for Q, the contribution of the parallel displacement $\xi_\parallel$ to $\langle Q \rangle_\tau$ vanishes and one is left simply with

$$\langle Q \rangle_\tau = \left\langle \frac{1}{2} \lambda B_0 \nabla \cdot \boldsymbol{\xi}_\perp - \left(1 - \frac{3}{2} \lambda B_0\right) \boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}_0 \right\rangle_\tau. \tag{55}$$

As seen in the analysis of Sec. 4, the zero-frequency case is exceptional in that some of the derivations there, which involve divisions by ω, become singular. Specifically, (37), (38) and (42), (43) do not follow automatically from the normal-mode drift-kinetic equations (26, 27) if $\omega = 0$, and their validity at $\omega = 0$ would necessitate to invoke a continuity argument as $0 \neq \omega \to 0$. It is therefore advisable to verify

https://doi.org/10.1017/S0022377814000531 Published online by Cambridge University Press

explicitly such continuity for the present zero-frequency solution. The determination of the integration constants carried out in the Appendix shows that $D_s^\omega = O(\omega)$ as $0 \neq \omega \to 0$. Then, differentiating (49) with respect to τ, one gets

$$\mathbf{b}_0 \cdot \frac{\partial \hat{f}_s^{\text{even}}}{\partial \mathbf{x}} = \frac{1}{v'(1-\lambda B_0)^{1/2}} \frac{\partial \hat{f}_s^{\text{even}}}{\partial \tau} = \frac{\hat{F}_{s\parallel}}{n_0 T_{s0}} f_{Ms0} + O(\omega^2) \tag{56}$$

which, taken to (37), ensures that (38) remains valid at $\omega = 0$. As for (42) which implies the Chapman–Enskog-like relation (43) for the density moment and therefore kinetic quasineutrality, using the zero-frequency solution (52) to evaluate its left-hand-side yields the condition

$$\nabla \cdot \boldsymbol{\xi} = \frac{3B_0}{2} \int_0^{1/B_0} d\lambda (1-\lambda B_0)^{-1/2} \langle Q \rangle_\tau . \tag{57}$$

Since $\langle Q \rangle_\tau$ (55) depends only on the perpendicular displacement $\boldsymbol{\xi}_\perp$ and the parallel force balance condition (24) is satisfied identically with the solution (51, 52) at $\omega = 0$ for any $\xi_\parallel$, a parallel displacement can be specified such that

$$\mathbf{B}_0 \cdot \nabla \left(\frac{\xi_\parallel}{B_0} \right) = -\nabla \cdot \boldsymbol{\xi}_\perp + \frac{3B_0}{2} \int_0^{1/B_0} d\lambda (1-\lambda B_0)^{-1/2} \left\langle \frac{1}{2} \lambda B_0 \nabla \cdot \boldsymbol{\xi}_\perp - \left(1 - \frac{3}{2} \lambda B_0 \right) \boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}_0 \right\rangle_\tau \tag{58}$$

and (57) holds. Having thus verified that (52) is the proper solution of the zero-frequency normal-mode drift-kinetic equation, one can evaluate with it the variables (44, 33) that close the magnetofluid system:

$$T_{s1} = -\boldsymbol{\xi} \cdot \nabla T_{s0} - T_{s0} B_0 \int_0^{1/B_0} d\lambda (1-\lambda B_0)^{-1/2} \langle Q \rangle_\tau , \tag{59}$$

$$(p_{s\parallel} - p_{s\perp}) = -\frac{15}{2} n_0 T_{s0} B_0 \int_0^{1/B_0} d\lambda (1-\lambda B_0)^{-1/2} \left(1 - \frac{3}{2} \lambda B_0 \right) \langle Q \rangle_\tau . \tag{60}$$

It is now a straightforward exercise to double-check that, with this solution, all the moment relations of Sec. 4 become identities.

Instead of the perturbed mean temperatures T_{s1} and pressure anisotropies $(p_{s\parallel} - p_{s\perp})$, one can use as fluid closure variables the perturbed parallel and perpendicular pressures

$$p_{s\parallel 1} = n_1 T_{s0} + n_0 T_{s1} + \frac{2}{3} (p_{s\parallel} - p_{s\perp}) \tag{61}$$

and

$$p_{s\perp 1} = n_1 T_{s0} + n_0 T_{s1} - \frac{1}{3} (p_{s\parallel} - p_{s\perp}) . \tag{62}$$

In terms of these, the normal-mode momentum equation (16) at $\omega = 0$ reads

$$-\mathbf{j}_0 \times \mathbf{B}_1 - \mathbf{j}_1 \times \mathbf{B}_0 + \sum_{s=i,e} \nabla \cdot \left[p_{s\parallel 1} \mathbf{b}_0 \mathbf{b}_0 + p_{s\perp 1} (\mathsf{I} - \mathbf{b}_0 \mathbf{b}_0) \right] = 0 . \tag{63}$$

Taking the corresponding linear combinations of (59), (60) and using (15), (57) for the perturbed density, one obtains

$$p_{s\parallel 1} = -\boldsymbol{\xi} \cdot \nabla (n_0 T_{s0}) - \frac{15}{2} n_0 T_{s0} B_0 \int_0^{1/B_0} d\lambda (1-\lambda B_0)^{1/2} \langle Q \rangle_\tau \tag{64}$$

https://doi.org/10.1017/S0022377814000531 Published online by Cambridge University Press

and

$$p_{s\perp 1} = -\boldsymbol{\xi}\cdot\nabla(n_0 T_{s0}) - \frac{15}{4} n_0 T_{s0} B_0^2 \int_0^{1/B_0} d\lambda (1-\lambda B_0)^{-1/2} \lambda \langle Q\rangle_\tau, \tag{65}$$

in agreement with the result of Rosenbluth and Rostoker (1959).

One final remark can be made regarding the electric field. The formulation used in this work eliminated the electric field, so the whole analysis was carried out without any reference to it. Nonetheless, once a solution is available, the electric field can be inferred from that. In particular, in the $\omega = 0$ case for which an explicit solution was obtained, the parallel component of the linearized momentum equation of each species yields

$$E_\parallel = \frac{\hat{F}_{i\parallel}}{e n_0} = -\frac{\hat{F}_{e\parallel}}{e n_0}. \tag{66}$$

Thus, at zero-frequency, the solution $\hat{F}_{s\parallel} = 0$ implies that the parallel electric field vanishes in its $1/e$ order which is not retained by the magnetofluid part of the system, but must be accounted for in the drift-kinetic equations.

6. Summary

An alternative to the traditional formulations of KMHD has been presented, based on drift-kinetic equations in the reference frame of the complete fluid velocity. Using this framework, the fluid closures that apply at zero-frequency marginal stability have been derived for confined equilibria with ergodic magnetic field lines. The zero-frequency marginal stability condition results in an integro-differential problem for the fluid perpendicular displacement $\boldsymbol{\xi}_\perp$, given by the closed system of (13), (14), (55), (64), (65), and the two perpendicular components of (63), in agreement with Rosenbluth and Rostoker (1959). Additional pieces of information shown by the present zero-frequency solution are that the quasineutrality condition and the parallel component of (63) are satisfied identically, the non-zero parallel displacement $\xi_\parallel$ is determined after $\boldsymbol{\xi}_\perp$ by (58), the divergence of the displacement and the associated perturbed density are given by (15), (55), (57), and the parallel electric field is zero.

Acknowledgements

The author thanks J. Freidberg and R. Betti for very helpful discussions. This work was sponsored by the US Department of Energy under grants DEFG02-91ER54109 and DEFC02-08ER54969 at the Massachusetts Institute of Technology and as part of the author's participation in the Center for Extended MHD Modeling (CEMM).

Appendix. Boundary conditions for the marginal stability solution

The procedure to implement the boundary conditions on the zero-frequency, marginal stability solution of the normal-mode drift-kinetic equation is outlined here. Such boundary conditions must be imposed first on the $\omega \neq 0$ solution with $\gamma = \mathrm{Im}\,\omega > 0$, then one can take the $\omega \to 0$ limit.

https://doi.org/10.1017/S0022377814000531 Published online by Cambridge University Press

In the trapped particle case, differentiating (49) with respect to τ, imposing the boundary conditions (50) and recalling that $B_0(\tau = \tau_{b\pm}) = 1/\lambda$, one gets

$$-\frac{\omega v'}{n_0 T_{s0}} f_{Ms0} \int_0^{\tau_{b\pm}} d\tau \ \sin[\omega(\tau_{b\pm} - \tau)] \left[(1 - \lambda B_0)^{1/2} \ \hat{F}_{s\|}\right](\tau)$$
$$+ \frac{i\omega^3 m_s v'^2}{T_{s0}} f_{Ms0} \int_0^{\tau_{b\pm}} d\tau \ \exp[-i\omega(\tau_{b\pm} - \tau)] \int_0^{\tau} d\tau' \exp[i\omega(\tau - \tau')] \ Q(\tau')$$
$$- \frac{\omega^2 m_s v'^2}{T_{s0}} f_{Ms0} \int_0^{\tau_{b\pm}} d\tau \exp[i\omega(\tau_{b\pm} - \tau)] Q(\tau) - \omega C_s^{\omega} \sin \omega\tau_{b\pm} + \omega D_s^{\omega} \cos \omega\tau_{b\pm} = 0. \tag{A1}$$

This implies that $D_s^{\omega} = O(\omega)$ as $\omega \to 0$, so it is convenient to set $D_s^{\omega} = \omega D_s'^{\omega}$. Then, dividing by ω^2 and taking the limit of the zero-frequency solution ($\omega \to 0$ and $\hat{F}_{s\|} \to 0$), the above reduces to

$$-\frac{m_s v'^2}{T_{s0}} f_{Ms0} \int_0^{\tau_{b\pm}} d\tau \ Q(\tau) - \ \tau_{b\pm} C_s^0 + D_s'^0 = 0. \tag{A2}$$

The result of (52), (53) follows from the difference between the τ_{b+} and τ_{b-} forms of (A2) that solves for C_s^0,

$$C_s^0 = -\frac{m_s v'^2 f_{Ms0}}{T_{s0}(\tau_{b+} - \tau_{b-})} \int_{\tau_{b-}}^{\tau_{b+}} d\tau \ Q(\tau). \tag{A3}$$

In the passing particle case and for $\gamma = \text{Im } \omega > 0$, the leading, exponentially growing behavior of the normal-mode solution (49) as $\tau \to \pm\infty$ is

$$\hat{f}_s^{\text{even}}(\tau \to +\infty) \simeq \exp(-i\omega\tau) \left\{ \frac{v'}{2n_0 T_{s0}} f_{Ms0} \int_0^{\infty} d\tau' \ \exp(i\omega\tau') \left[(1 - \lambda B_0)^{1/2} \ \hat{F}_{s\|}\right](\tau') \right.$$
$$\left. - \frac{\omega^2 m_s v'^2}{T_{s0}} f_{Ms0} \int_0^{\infty} d\tau' \ \exp(2i\omega\tau') \int_0^{\tau'} d\tau'' \exp(-i\omega\tau'') \ Q(\tau'') + \ \frac{C_s^{\omega}}{2} - \frac{D_s^{\omega}}{2i} \right\} \tag{A4}$$

and

$$\hat{f}_s^{\text{even}}(\tau \to -\infty) \simeq \exp(i\omega\tau) \left\{ - \frac{v'}{2n_0 T_{s0}} f_{Ms0} \int_{-\infty}^{0} d\tau' \exp(-i\omega\tau') \left[(1 - \lambda B_0)^{1/2} \hat{F}_{s\|}\right](\tau') \right.$$
$$\left. - \frac{i\omega m_s v'^2}{2T_{s0}} f_{Ms0} \int_{-\infty}^{0} d\tau' \ \exp(-i\omega\tau') Q(\tau') + \ \frac{C_s^{\omega}}{2} + \frac{D_s^{\omega}}{2i} \right\}. \tag{A5}$$

Setting these terms equal to zero so that $\hat{f}_s^{\text{even}}$ remains bounded, provides the system that determines the integration constants C_s^{ω} and D_s^{ω}:

$$C_s^{\omega} = \frac{v' f_{Ms0}}{2n_0 T_{s0}} \left\{ \int_{-\infty}^{0} d\tau \exp(-i\omega\tau) \left[(1 - \lambda B_0)^{1/2} \hat{F}_{s\|}\right](\tau) \right.$$
$$\left. - \int_0^{\infty} d\tau \exp(i\omega\tau) \left[(1 - \lambda B_0)^{1/2} \hat{F}_{s\|}\right](\tau) \right\}$$
$$+ \frac{m_s v'^2 f_{Ms0}}{T_{s0}} \left[\frac{i\omega}{2} \int_{-\infty}^{0} d\tau \exp(-i\omega\tau) Q(\tau) \right.$$
$$\left. + \omega^2 \int_0^{\infty} d\tau \exp(2i\omega\tau) \int_0^{\tau} d\tau' \exp(-i\omega\tau') Q(\tau') \right] \tag{A6}$$

https://doi.org/10.1017/S0022377814000531 Published online by Cambridge University Press

and

$$D_s^{\omega} = \frac{iv' f_{Ms0}}{2n_0 T_{s0}} \left\{ \int_{-\infty}^{0} d\tau \exp(-i\omega\tau) \left[(1-\lambda B_0)^{1/2} \hat{F}_{s\|} \right](\tau) + \int_{0}^{\infty} d\tau \exp(i\omega\tau) \left[(1-\lambda B_0)^{1/2} \hat{F}_{s\|} \right](\tau) \right\} + \frac{im_s v'^2 f_{Ms0}}{T_{s0}} \left[\frac{i\omega}{2} \int_{-\infty}^{0} d\tau \exp(-i\omega\tau) Q(\tau) - \omega^2 \int_{0}^{\infty} d\tau \exp(2i\omega\tau) \int_{0}^{\tau} d\tau' \exp(-i\omega\tau') Q(\tau') \right]. \quad \text{(A7)}$$

If one assumes now the limit of a marginally stable solution with $\omega = i\gamma \to 0$ and $\hat{F}_{s\|} \to 0$, after integrating by parts the double integral terms of (A6, A7), these equations yield

$$C_s^0 = -\frac{m_s v'^2 f_{Ms0}}{2T_{s0}} \lim_{\gamma\to 0} \left[\gamma \int_{-\infty}^{0} d\tau \exp(\gamma\tau) Q(\tau) + \gamma \int_{0}^{\infty} d\tau \exp(-\gamma\tau) Q(\tau) \right], \quad \text{(A8)}$$

which corresponds to the result of (52), (54), and

$$D_s^0 = -\frac{im_s v'^2 f_{Ms0}}{2T_{s0}} \lim_{\gamma\to 0} \left[\gamma \int_{-\infty}^{0} d\tau \exp(\gamma\tau) Q(\tau) - \gamma \int_{0}^{\infty} d\tau \exp(-\gamma\tau) Q(\tau) \right] = 0. \quad \text{(A9)}$$

REFERENCES

Kruskal, M. D. and Oberman, C. R. 1958 *Phys. Fluids* **1**, 275.

Kulsrud, R. 1962a *Phys. Fluids* **5**, 192.

Kulsrud, R. 1962b Course XXV Advanced Plasma Theory. In: *Proc. Int. School of Physics 'Enrico Fermi', Varenna, Italy,* (ed. M. N. Rosenbluth). North Holland, New York, p. 54.

Lyons, B. C., Jardin, S. C. and Ramos, J. J. 2012 *Phys. Plasmas* **19**, 082515.

Ramos, J. J. 2010 *Phys. Plasmas* **17**, 082502.

Ramos, J. J. 2011 *Phys. Plasmas* **18**, 102506.

Rosenbluth, M. N. and Rostoker, N. 1959 *Phys. Fluids* **2**, 23.

https://doi.org/10.1017/S0022377814000531 Published online by Cambridge University Press