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Simultaneous rational approximations to certain algebraic numbers

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1. *Introduction.* It is generally conjectured that if $\alpha_1, \alpha_2, \ldots, \alpha_k$ are algebraic numbers for which no equation of the form

$$r_0 + r_1\alpha_1 + r_2\alpha_2 + \ldots + r_k\alpha_k = 0$$

is satisfied with rational r_i, not all zero, and if $\kappa > 1 + 1/k$, then there are only finitely many sets of integers $p_1, p_2, \ldots, p_k$, $q, q > 0$, such that

$$\left|\alpha_j - \frac{p_j}{q}\right| < \frac{1}{q^\kappa} \quad (j = 1, 2, \ldots, k). \tag{1}$$

This result would be best possible, for it is well known that (1) has infinitely many solutions when $\kappa = 1 + 1/k$.† If $\alpha_1, \alpha_2, \ldots, \alpha_k$ are elements of an algebraic number field of degree $k+1$ the result can be deduced easily (see Perron (11)). The famous theorem of Roth (13) asserts the truth of the conjecture in the case $k = 1$ and this implies that for any positive integer k, (1) certainly has only finitely many solutions if $\kappa > 2$. Nothing further in this direction however has hitherto been proved.‡

In two recent papers (1, 2) I obtained inequalities of the above type in the case $k = 1$ for certain algebraic numbers α, given essentially by fractional powers of rationals. It is the main purpose of the present paper to generalize the results of the two previous papers and so deduce the finiteness of the number of solutions of (1) for certain sets of algebraic numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ and for certain $\kappa < 2$. Indeed given any $\kappa > 1 + 1/k$ we are able to specify a corresponding set $\alpha_1, \alpha_2, \ldots, \alpha_k$, generating an algebraic number field with degree exceeding $k+1$, such that (1) has only finitely many solutions. All the results here are effective, in that they establish an explicit upper bound for the integers q occurring in any solution of (1). It was in fact the object of obtaining such results in the case $k = 1$ which originated the investigations in (1) and (2).

† The proof goes back to Dirichlet. For other interesting work in this connexion see (5) and (10).

‡ Hasse (6) investigated similar problems in 1939 but his results are now implied by the work of Roth.

Since this paper was written, some important work by Wolfgang M. Schmidt has been published (14). Schmidt has proved, in particular, that if, for $k = 2$ and $\kappa > \frac{3}{2}$, (1) possesses an infinite sequence of solutions then the denominators q, when arranged in ascending order, must increase very rapidly. *Added in proof:* Schmidt has now proved the conjecture in the case $k = 2$.

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We shall begin by proving the following

THEOREM. *Suppose that n, k are integers such that $1 \leqslant k < n$ and $n \geqslant 3$. Let $m_0, m_1, \ldots, m_k$ be distinct integers between 0 and $n-1$ inclusive. Suppose that a, b are positive integers such that*

$$\lambda = a\{4^{nk}(a-b)\}^{-k-1} > 1. \tag{2}$$

Then $\alpha_0, \alpha_1, \ldots, \alpha_k$, given by $\alpha_j = (b/a)^{m_j/n}$ $(j = 0, 1, \ldots, k)$, satisfy

$$|x_0\alpha_0 + x_1\alpha_1 + \ldots + x_k\alpha_k| > cX^{-\chi} \tag{3}$$

for all sets of integers $x_0, x_1, \ldots, x_k$, not all zero, where

$$X = \max(|x_0|, |x_1|, \ldots, |x_k|), \tag{4}$$

and χ, c are given by $\lambda\chi = (4^{n(k+1)}a)^k$, $c = (8k^{k+1}\lambda^{55})^{-\chi-1}$. (5)

The main result of the paper follows as a direct deduction:

COROLLARY 1. *Suppose the hypotheses of the theorem hold with $m_0 = 0$. If $\kappa > (\chi+1)/k$ then all solutions of (1) in integers $p_1, p_2, \ldots, p_k, q, q > 0$, satisfy*

$$q < (8^{\chi+1}kc^{-1})^{k!(\kappa k-\chi-1)}. \tag{6}$$

Our earlier remark regarding the possible values assumed by κ is now clear; for given any $\kappa > 1 + 1/k$, Corollary 1 asserts that (1) has only finitely many solutions provided that

$$\lambda > \{4^{n(k+1)}a\}^{k!(\kappa k-1)},$$

and this certainly holds if a is sufficiently large compared with $a-b$.

The work of (1) and (2) was based on certain properties of Gauss's hypergeometric function. Here we shall use functions of a more general nature, represented most easily by means of contour integrals, and which reduce in special cases to functions of hypergeometric type. They were used in a similar context by Mahler in 1931 (see (8)).

The theorem admits several applications. First, as is well known, an inequality of the type (3) corresponds to an estimate for the accuracy of approximation to α_j by algebraic numbers β of bounded degree. We prove

COROLLARY 2. *Suppose that $n \geqslant 3$ and $k \geqslant 1$ are integers, and that a, b are positive integers such that (2) holds. Let m be an integer between 1 and $n-1$ inclusive and let $\alpha = (b/a)^{m/n}$. If $\mu > \chi+1$, where χ is defined by (5), then there are only finitely many algebraic numbers β with degree $\leqslant k$ such that*

$$|\alpha - \beta| < (H(\beta))^{-\mu}, \tag{7}$$

where $H(\beta)$ denotes the height of β.

We note, as a consequence, that for any value of $\mu > k+1$ there is an algebraic number α with degree exceeding μ such that (7) has only finitely many solutions. The best result in this direction so far established for a general algebraic number α is that of Siegel (see (16)) and here the exponent μ is of order $2nk^{\frac{1}{2}}$, n, k denoting the degrees of α, β respectively.†

† *Added in proof:* Wirsing has recently obtained a much stronger result.

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Secondly, the theorem enables a bound to be calculated for size of the solutions of certain Diophantine equations in three unknowns. To illustrate the type of result we prove

COROLLARY 3. *Suppose that l, m are integers such that $l = m^5 - 1$ and $m > 10^{11}$. Then for any integer n, all solutions in integers x, y, z of the equation*

$$x^5 + ly^5 + l^2z^5 + 5lxyz(xz - y^2) = n \tag{8}$$

satisfy $|x| < M$, $|y| < M$, $|z| < M$, where $M = l^{500}n^2$.

Equations of the type (8) were investigated by Skolem (17) in 1933.† His p-adic method has subsequently been developed, notably by Chabauty, and it is now known that a wide class of such equations exists each possessing only finitely many solutions. Although in general the method gives only a bound for the number of solutions (and not for their size), this has proved sufficient in a few cases to solve the equation completely.

Finally, we remark that by specializing the integers $x_0, x_1, \ldots, x_k$ indicated in the theorem we obtain effective results, not included in (1) or (2), concerning rational approximations to a single algebraic number in the field generated by an α_j.

2. *Lemmas.* We suppose that all the hypotheses of the theorem stated above are satisfied. We put $\omega_i = m_i/n$ for $i = 0, 1, \ldots, k$ and denote by δ_{ij} the Kronecker symbol, that is $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

LEMMA 1. *Let $\rho_0, \rho_1, \ldots, \rho_k$ be positive integers, let ρ denote their sum, and let C be a circle in the complex plane, described in the positive sense, with radius exceeding ρ. For every complex number z such that $0 < |z| < 1$, let*

$$\phi(z) = \prod_{h=0}^{k} \prod_{\sigma=0}^{\rho_h - 1} (z - \omega_h - \sigma), \tag{9}$$

$$A_i(z) = \sum_{l=0}^{\rho_i - 1} (1-z)^l (\phi'(\omega_i + l))^{-1} \quad (i = 0, 1, \ldots, k), \tag{10}$$

and

$$E(z) = (2\pi i)^{-1} z^{-\rho+1} \int_C (1-z)^\zeta (\phi(\zeta))^{-1} d\zeta.‡ \tag{11}$$

Then

$$\sum_{i=0}^{k} (1-z)^{\omega_i} A_i(z) = z^{\rho-1} E(z). \tag{12}$$

Proof. See Mahler (8). The equation (12) is deduced by a simple application of Cauchy's residue theorem.

LEMMA 2. *We have*

$$E(z) = (-1)^{\rho-1} \left(\prod_{i=0}^{k} (\rho_i - 1)! \right)^{-1} \int_0^1 \int_0^{u_0} \int_0^{u_1} \cdots \int_0^{u_{k-1}} f(u_0, \ldots, u_{k-1}, z)\, du_0 \ldots du_{k-1}, \tag{13}$$

where

$$f(u_0, \ldots, u_{k-1}, z) = (1-z)^{\omega_0} (1-u_0)^{\rho_0 - 1} u_{k-1}^{\rho_k - 1} \times \left\{ \prod_{j=1}^{k-1} (u_{j-1} - u_j)^{\rho_j - 1} \right\} \left\{ \prod_{i=0}^{k-1} (1 - zu_i)^{\omega_{i+1} - \omega_i - \rho_i} \right\}. \tag{14}$$

† For further references see (18).

‡ We are using i both as a variable suffix and to denote the number $(-1)^{\frac{1}{2}}$, but this should cause no confusion.

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Proof. See Mahler (8). By expanding the factors of the integrand on the right-hand side of (11) as power series in ζ, uniformly convergent on C, and integrating term by term, we see that $E(z)$ is representable by a power series in z beginning with the constant $(-1)^{\rho-1}((\rho-1)!)^{-1}$. Then operating on (12) successively by

$$\frac{d^{\rho_0}}{dz^{\rho_0}}(1-z)^{-\omega_0}, \quad \frac{d^{\rho_i}}{dz^{\rho_i}}(1-z)^{\rho_{i-1}+\omega_{i-1}-\omega_i} \quad (i=1,2,\ldots,k-1),$$

the left-hand side becomes $(1-z)^{\omega_k-\omega_{k-1}-\rho_{k-1}}p(z)$, where $p(z)$ is a polynomial of degree at most ρ_k-1. A comparison with the right-hand side shows that

$$p(z) = (-1)^{\rho-1}((\rho_k-1)!)^{-1}z^{\rho_k-1},$$

and the equation (13) then follows by the reverse integration operations.

LEMMA 3. *Let r be a positive integer and for each $j = 0, 1, \ldots, k$ let*

$$\phi_j(z), \quad A_{ij}(z)\ (i=0,1,\ldots,k) \quad \text{and} \quad E_j(z)$$

be the functions given by (9), (10) *and* (11) *of Lemma* 1 *corresponding to the integers $\rho_i = r+\delta_{ij}\,(i=0,1,\ldots,k)$. Then if $0<|z|<1$, the determinant $\Delta(z)$ with $A_{ij}(z)$ in the i-th row and j-th column $(i,j=0,1,\ldots,k)$ is not zero.*

Proof. See Mahler (8). From the definition of the $A_{ij}(z)$ it is clear that $\Delta(z)$ represents a polynomial with degree exactly $(k+1)r$. On the other hand, the $k+1$ equations corresponding to (12) imply that $\Delta(z)$ has a zero at $z=0$ of order at least $(k+1)r$ and this gives the required result.

LEMMA 4. *Suppose the hypotheses of Lemma* 3 *hold and that $r \geqslant 54$. Let N be the least common multiple of the integers $1, 2, \ldots, n(r+1)$. Then $q_{ij}\,(i,j=0,1,\ldots,k)$, given by*

$$q_{ij} = \{(r+1)!N\}^{k+1}a^rA_{ij}(1-b/a), \tag{15}$$

are integers with absolute values at most $4^{nr(k+1)}a^r$.

Proof. We note first that by a result of Rosser† (see (12) and the footnote on p. 378 of (9)) N is less than $2^{3n(r+1)/2}$. Also we observe that if u, v are relatively prime integers and each of $u, u+v, \ldots, u+rv$ has modulus less than $n(r+1)$ then the product P of these $r+1$ integers divides $(r+1)!\,N$. For if p is a prime factor of P then the exponent to which p divides P is clearly at most

$$\sum_{j=1}^{J}([(r+1)/p^j]+1) < \sum_{j=1}^{\infty}[(r+1)/p^j]+J,$$

where J is the greatest integer satisfying $p^J < n(r+1)$. The first number on the right is exactly the exponent to which p divides $(r+1)!$ and the second is that to which p divides N.

For each pair of integers $i,j = 0,1,\ldots,k$ we put $\eta_{ij} = r+\delta_{ij}-1$ and then define $\psi(h,i,j,l)$ for all sets of integers h,i,j,l satisfying $0 \leqslant h,i,j \leqslant k, 0 \leqslant l \leqslant \eta_{ij}$ by

$$\psi(h,i,j,l) = \prod_{\sigma=0}^{\eta_{hj}}(\omega_i-\omega_h+l-\sigma).$$

† Rosser's result depends on a large amount of computation. We can in principle use the more elementary work of Tchebychef (see (7), p. 90) but this would give a less precise estimate.

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Since $\eta_{ij} \leqslant r$ and $\omega_i - \omega_h$, with $i \neq h$, can be written in the form u/v, where u, v are relatively prime and $0 < |u| < v \leqslant n$, it follows from the above remarks that $(r+1)!\, N(\psi(h,i,j,l))^{-1}$ is an integer for each set h, i, j, l with $i \neq h$. Now from the definition of the $\phi_j(z)$ we see that

$$\phi_j'(\omega_i + l) = (-1)^{\eta_{ij}-l}\, l!(\eta_{ij}-l)! \prod_{\substack{h=0 \\ h \neq i}}^{k} \psi(h,i,j,l) \tag{16}$$

and hence $\{(r+1)!\, N\}^{k+1} (\phi_j'(\omega_i + l))^{-1}$ is an integer. Then from the equations corresponding to (10) and the definition (15) we deduce that the q_{ij} are integers.

In order to obtain an upper bound for the absolute values of the q_{ij} we first calculate a lower bound for the absolute values of the $\psi(h,i,j,l)$ with $i \neq h$. From the inequality

$$|\omega_i - \omega_h + l - \sigma| > |l - \sigma| - 1$$

we see that the products of the factors of $\psi(h,i,j,l)$ given by $\sigma = 0, 1, \ldots, l-2$, by $\sigma = l-1,\ l,\ l+1$ and by $\sigma = l+2, \ldots, \eta_{ij}$ are greater than $(l-1)!$, n^{-2} and $(\eta_{ij}-l-1)!$ respectively. Thus

$$|\psi(h,i,j,l)| > n^{-2}(l-1)!\,(\eta_{ij}-l-1)!.\dagger$$

Then from (16) we obtain

$$|\phi_j'(\omega_i + l)| > l!(\eta_{ij}-l)!\,\{n^{-2}(l-1)!\,(\eta_{ij}-l-1)!\}^k$$

and since $\eta_{ij} \leqslant r$ it follows that

$$\{(r+1)!\, N\}^{k+1} |(\phi_j'(\omega_i + l))^{-1}| < \binom{\eta_{ij}}{l} \binom{r}{l-1}^k \{N n^2 (r+1)^2\}^{k+1}.\ddagger$$

In virtue of the estimate for N mentioned initially and the hypothesis $r \geqslant 54$, the number on the right is at most

$$\binom{\eta_{ij}}{l} \binom{r}{l-1}^k 2^{(2n-1)(k+1)r} \leqslant \binom{\eta_{ij}}{l} 2^{(2n(k+1)-1)r}.$$

Hence from the equations corresponding to (10), and noting that $a > b$, we deduce that

$$|q_{ij}| < 2^{(2n(k+1)-1)r} a^r \sum_{l=0}^{\eta_{ij}} \binom{\eta_{ij}}{l} \leqslant 4^{nr(k+1)} a^r,$$

and this proves the lemma.

LEMMA 5. *Suppose the hypotheses of Lemma 4 hold. Then for each $j = 0, 1, \ldots, k$ we have*

$$\left| \sum_{i=0}^{k} q_{ij} \alpha_i \right| < 4^{nr(k+1)} (a-b)^{r(k+1)} a^{-rk}. \tag{17}$$

Proof. By (12), (15), the definition of $E_j(z)$ and the hypotheses of the theorem we see that for each $j = 0, 1, \ldots, k$

$$\sum_{i=0}^{k} q_{ij} \alpha_i = \{(r+1)!\, N\}^{k+1} a^r w^{r(k+1)} E_j(w), \tag{18}$$

where, for brevity, we have put $w = 1 - b/a$.

† It is understood that $l!$ with $l \leqslant 0$ is l.

‡ If $l < 0$ we take $\binom{r}{l}$ to be r.

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To calculate an upper bound for the absolute value of $E_j(w)$ we use the equations corresponding to (13) and (14) of Lemma 2. Let j denote an integer between 0 and k inclusive and let $\rho_i (i = 0, 1, \ldots, k)$ be defined as in Lemma 3. Noting that $w < \frac{1}{2}$ and that $r + 2 \geqslant \rho_i + \omega_i - \omega_{i+1}$ we obtain

$$\left| \prod_{i=0}^{k-1} (1 - wu_i)^{\omega_{i+1} - \omega_i - \rho_i} \right| < 2^{(r+2)k},$$

provided $0 \leqslant u_i \leqslant 1$ for each i. The number on the right is thus an upper bound for the absolute value of $f(u_0, u_1, \ldots, u_{k-1}, w)$, defined by (14), and it follows from (13) that

$$|E_j(w)| < 2^{(r+2)k} \{(r-1)!\}^{-(k+1)}. \tag{19}$$

Now using the estimate for N mentioned in the proof of Lemma 4 and noting that $r \geqslant 54$ we obtain

$$\{(r+1)!\, N\}^{k+1} 2^{(r+2)k} \{(r-1)!\}^{-(k+1)} \leqslant 2^{(r+2)k} \{(r+1)^2 N\}^{k+1} < 4^{nr(k+1)}.$$

This together with (18) and (19) gives the required inequality.

LEMMA 6. *Let k be a positive integer and let $q_{ij} (i, j = 0, 1, \ldots. k)$ be $(k+1)^2$ integers with absolute values at most Q such that the matrix (q_{ij}) is non-singular. Suppose that $\xi_0, \xi_1, \ldots, \xi_k$ are real or complex numbers and let*

$$\Phi_j = q_{0j}\xi_0 + q_{1j}\xi_1 + \ldots + q_{kj}\xi_k \tag{20}$$

for $j = 0, 1, \ldots, k$. Suppose that the Φ_j have absolute values at most Φ. If $x_0, x_1, \ldots, x_k$ are integers, not all zero, with absolute values at most X and

$$\Psi = x_0\xi_0 + x_1\xi_1 + \ldots + x_k\xi_k \tag{21}$$

then

$$|\Psi| > |\xi_0| (k^{\frac{1}{2}k} Q^k)^{-1} - k\Phi X Q^{-1}. \tag{22}$$

Proof.† Since (q_{ij}) is non-singular, there are k of the linear forms (20) in $\xi_0, \ldots, \xi_k$ which together with the linear form (21) make up a linearly independent set. Without loss of generality we can take them to be the last k forms. We have

$$\begin{vmatrix} \Psi & x_1 & \ldots & x_k \\ \Phi_1 & q_{11} & \ldots & q_{k1} \\ \ldots & & & \\ \Phi_k & q_{1k} & \ldots & q_{kk} \end{vmatrix} = \xi_0 \begin{vmatrix} x_0 & x_1 & \ldots & x_k \\ q_{01} & q_{11} & \ldots & q_{k1} \\ \ldots & & & \\ q_{0k} & q_{1k} & \ldots & q_{kk} \end{vmatrix}$$

and the determinant on the right is a non-zero integer. Expanding the determinant on the left by the first column and estimating the cofactors by Hadamard's inequality (see (4), p. 147) we obtain

$$k^{\frac{1}{2}k} Q^k |\Psi| + k(k^{\frac{1}{2}k} Q^{k-1} X \Phi) \geqslant |\xi_0|,$$

and this gives (22).

3. *Proof of theorem.* Let $x_0, x_1, \ldots, x_k$ be integers, not all zero, and let X be given by (4). Suppose that Ψ is defined by (21) of Lemma 6 with $\xi_i = \alpha_i$ for $i = 0, 1, \ldots, k$. Then the assertion of the theorem is that

$$|\Psi| > cX^{-\varkappa}. \tag{23}$$

† I have given a slightly modified form of this lemma elsewhere (see (3)).

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We suppose first that $X \geqslant \lambda^{54}$. Let r be the integer for which

$$\lambda^{r-1} \leqslant 4k^{k+1}X < \lambda^r, \tag{24}$$

so that $r \geqslant 54$, and let $q_{ij}\,(i,j = 0,1,\ldots,k)$ be the integers defined by (15) of Lemma 4. Suppose that $\Phi_j\,(j = 0,1,\ldots,k)$ are given by (20) of Lemma 6. Clearly Lemma 3 implies that the matrix (q_{ij}) is non-singular. Further, from Lemmas 4 and 5 we see that the q_{ij} have absolute values at most $Q = 4^{nr(k+1)}a^r$, and the Φ_i have absolute values at most Φ, where Φ is given by the number on the right of (17). With Q and Φ thus defined, all the hypotheses of Lemma 6 are satisfied and hence (22) holds. We proceed to calculate a lower bound for the number on the right of (22).

We note first that (2) implies that $\xi_0 > \frac{1}{2}$. Further we see that

$$\lambda^{-r} = \Phi Q^{k-1},$$

and from the right-hand inequality of (24) we obtain

$$\xi_0^{-1}k^{\frac{1}{2}(k+3)}X\Phi Q^{k-1} < \xi_0^{-1}k^{\frac{1}{2}(k+3)}(4k^{k+1})^{-1} < \tfrac{1}{2}.$$

The number on the left is a quotient of the two terms on the right of (22) and hence it follows that

$$|\Psi| > \tfrac{1}{2}\xi_0(k^{\frac{1}{2}k}Q^k)^{-1}.$$

Now from (5) we see that $Q^k = \lambda^{r\chi}$ and the left-hand inequality of (24) implies that $\lambda^r \leqslant 4k^{k+1}\lambda X$. Thus we deduce

$$|\Psi| > CX^{-\chi}, \tag{25}$$

where

$$C = \tfrac{1}{4}k^{-\frac{1}{2}k}(4k^{k+1}\lambda)^{-\chi} > c.$$

It remains only to deal with the case $X < \lambda^{54}$, and here it suffices to apply (25) with $x_i\,(i = 0,1,\ldots,k)$ and X replaced by Wx_i and WX respectively where $W = [\lambda^{54}]+1$. Since $C > W^{\chi+1}c$, it follows that (23) is satisfied, and this completes the proof of the theorem.

4. *Proof of Corollary* 1. Let $p_1, p_2, \ldots, p_k, q, q > 0$ denote integers satisfying (1). We note first that this implies $|p_i| < 2q$ for all i. By an application of Dirichlet's box argument (see for example (15), p. 140) there exists integers $x_0, x_1, \ldots, x_k$, not all zero, such that

$$x_0q - x_1p_1 - \ldots - x_kp_k = 0$$

and

$$X = \max(|x_0|, |x_1|, \ldots, |x_k|) < 8q^{1/k}.$$

The theorem asserts that

$$|x_0 + x_1\alpha_1 + \ldots + x_k\alpha_k| > cX^{-\chi}.$$

However, the number on the left can also be written in the form

$$|x_1(\alpha_1 - p_1/q) + \ldots + x_k(\alpha_k - p_k/q)|$$

and this is clearly less than $kXq^{-\kappa}$. Hence

$$q^\kappa < kc^{-1}X^{\chi+1} < kc^{-1}8^{\chi+1}q^{(\chi+1)/k}$$

and (6) follows.

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5. *Proof of Corollary* 2. Let $m/n = u/v$, where u, v are relatively prime positive integers. We may suppose that $k < v$ as otherwise the result is well known.† We denote by u_i the least non-negative residue of $ui \pmod{v}$ for $i = 0, 1, \ldots, k$, and put $u_i = ui - v_i v$. Clearly all the u_i are distinct.

Now let β be any algebraic number with degree at most k and let its irreducible defining polynomial with relatively prime integer coefficients be

$$y_0 + y_1 x + \ldots + y_k x^k.$$

The height $H(\beta)$ of β is the maximum of the absolute values of the y_i. We apply the theorem with $\alpha_i = (b/a)^{u_i/v}$ and with $x_i = y_i a^{k-v_i} b^{v_i}$ $(i = 0, 1, \ldots, k)$, as is possible since v divides n. It follows from (3) that

$$|y_0 + y_1\alpha + \ldots + y_k\alpha^k| > ca^{-k(1+\chi)}(H(\beta))^{-\chi}.$$

On the other hand, by Hilfssatz 15 of (15) (as cited in earlier footnote) the number on the left is less than $2n^2H(\beta)|\alpha - \beta|$, and since there are only a finite number of algebraic numbers with both degree and height bounded the required result follows.

6. *Proof of Corollary* 3. Let X, Y, Z be an integral solution of (8) and put $x = X$, $y = mY$, $z = m^2Z$. We take $\alpha = l^{\frac{1}{5}}/m$ and define

$$\phi_j = x + y\alpha e^{\frac{2}{5}\pi ij} + z\alpha^2 e^{\frac{4}{5}\pi ij},$$

for $j = 0, 1, 2, 3$ and 4. Then ϕ_1, ϕ_4 and ϕ_2, ϕ_3 are complex conjugates pairs. Also it is easily verified that

$$\phi_0\phi_1\phi_2\phi_3\phi_4 = n. \tag{26}$$

We proceed to prove that the maximum H of the absolute values of the integers x, y, z is less than M, and it is clear that the same will then hold for X, Y, Z.

We note first that if each of the ϕ_j has absolute value exceeding $10^{-3}H$ then (26) implies that $H < 10^3|n|^{\frac{1}{5}}$, and this is certainly less than M. Hence we may suppose that $|\phi_J| \leqslant 10^{-3}H$, where J is given by 0, 1 or 2. Our next object is to prove that this implies $|\phi_j| > 10^{-3}H$ for each of the other two integers $j = 0, 1$ or 2.

From the definition of the ϕ_j we see that

$$\phi_j = \phi_J + \alpha(e^{\frac{2}{5}\pi ij} - e^{\frac{2}{5}\pi iJ})\psi_j, \tag{27}$$

where

$$\psi_j = y + z\alpha(e^{\frac{2}{5}\pi ij} + e^{\frac{2}{5}\pi iJ}). \tag{28}$$

Since for $j = 0, 1, 2, j \neq J$,

$$|e^{\frac{2}{5}\pi ij} - e^{\frac{2}{5}\pi iJ}| \geqslant |\sin\{\tfrac{2}{5}\pi(J-j)\}| \geqslant \sin(\tfrac{1}{5}\pi) > \tfrac{14}{25},$$

and $\alpha > \frac{25}{28}$, it follows from (27) and our supposition concerning $|\phi_J|$ that

$$|\phi_j| > \tfrac{1}{2}|\psi_j| - 10^{-3}H. \tag{29}$$

The absolute value of the imaginary part of ψ_j is

$$|2z\alpha \sin\{\tfrac{1}{5}\pi(J+j)\}\cos\{\tfrac{1}{5}\pi(J-j)\}| > 2|z|\alpha \sin(\tfrac{1}{5}\pi)\cos(\tfrac{2}{5}\pi)$$

† The required inequality follows for example by combining Satz 3 on p. 7 and Hilfssatz 15 on p. 74 of Schneider (15).

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and hence $|\psi_j| > \frac{1}{4}|z|$. Now this implies that also $|\psi_j| > \frac{1}{10}|y|$, for otherwise we would have $|z| < \frac{2}{5}|y|$ and on the other hand, (28) would give $|z| > \frac{9}{20}|y|$, a contradiction. Thus if $H = |y|$ or $|z|$ the required inequality follows from (29). For the remaining case in which $H = |x|$ we note that our supposition $|\phi_J| \leqslant 10^{-3}H$ implies that

$$|x| < 2\max(|y|, |z|) + \tfrac{1}{2}H.$$

Hence we obtain $|\psi_j| > \frac{1}{40}H$, which again gives the required inequality.

We now distinguish two cases according as $J = 0$ or $J > 0$.

(i) We suppose that $J = 0$ and apply the theorem with

$$n = 5, \quad k = 2, \quad m_h = h \quad (h = 0, 1, 2), \quad a = m^5, \quad b = l.$$

This gives $|\phi_0| > c_1 H^{-\frac{7}{2}}$, where $c_1 = 4^{-15}(4^{30}a^{-1})^{\frac{495}{2}}$.

Thus using the bounds for $|\phi_j|$ with $j \neq J$ deduced above we obtain

$$|\phi_0|\,|\phi_1|^2\,|\phi_2|^2 > 10^{-12}c_1 H^{\frac{1}{2}}$$

and from (26) it follows that $H < 10^{24}n^2c_1^{-2}$. Then since $a < l^{1\cdot01}$ we see that $H < M$, and this proves Corollary 3 in the case $J = 0$.

(ii) We suppose that $J = 1$ or 2 and apply the theorem with

$$n = 5, \quad k = 1, \quad m_0 = 0, \quad m_1 = 2, \quad a = m^5, \quad b = l.$$

This gives $|x - z\alpha^2| > c_2 H^{-\frac{5}{4}}$, where $c_2 = 4^{-4}(4^{10}a^{-1})^{\frac{495}{4}}$.

However, it is easily seen that for $j = 1, 2$

$$x - z\alpha^2 = \operatorname{re}\phi_j - \cot(\tfrac{2}{5}\pi j)\operatorname{im}\phi_j,$$

and hence, on noting that $\cot(\frac{1}{5}\pi) < \frac{3}{2}$, we obtain

$$|\phi_j| \geqslant \max(|\operatorname{re}\phi_j|, |\operatorname{im}\phi_j|) > \tfrac{2}{5}c_2 H^{-\frac{5}{4}}.$$

Now in virtue of (26) and the inequalities deduced above for $|\phi_j|$ with $j \neq J$ it follows that

$$H < 10^{18}(\tfrac{5}{2})^4 c_2^{-4}n^2 < M,$$

and this completes the proof of Corollary 3.

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