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On piecewise linear embeddings

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0. *Notation.* Let M and Q be manifolds. The embedding $f\colon M \to Q$ is said to be *proper* if $f^{-1}\partial Q = \partial M$.

Let K be a simplicial complex and Q a PL manifold. Let $f\colon |K| \to Q$ be a PL map. A *Stallings Complex*, $\tilde{K}$, for f is a simplicial complex such that

(i) $f|K| \subset |\tilde{K}| \subset Q$,

(ii) $f\colon |K| \to |\tilde{K}|$ is a simple homotopy equivalence.

Let $f\colon Q \to Y$ be a simplicial map of simplicial complexes. We form a simplicial complex M_f, the *mapping cylinder* of f from the space $X \times I \cup Y/(x, 0) \sim fx$ by inductively subdividing the prisms $\sigma \times I \cup f\sigma/\sim$ for $\sigma \in X$.

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1. *Statement of results.* Our main results are Theorems 2 and 3 below. Theorem 1 is needed to establish these.

THEOREM 1. *Let K^k be a compact simplicial complex, Q^q a PL manifold, and $f\colon |K| \to Q$ an embedding which need not be PL. Let ϕ be a PL approximation to f, such that (for some metric on Q) $d(fx, \varphi x) \leqslant \delta$ for all $x \in |K|$. Then provided $q - k \geqslant 3$ and δ is sufficiently small, there exists a Stallings Complex $\tilde{K}$ for φ. Moreover, if $f^{-1}\partial Q$ is a subpolyhedron, with underlying complex L, say, we may require that $|\tilde{L}| = |\tilde{K}| \cap \partial Q$ be a Stallings Complex for $\varphi|K$. Furthermore, we may require that $|\tilde{K}|$ lie within the ϵ-neighbourhood of $f|K|$, for any given ϵ.*

Our argument for Theorem 1 is inspired by that in Stallings' paper (6); with our somewhat different hypotheses we can still obtain, essentially, his main theorem. Using the Casson–Sullivan embedding theorem (5), we may deduce from Theorem 1 the following:

THEOREM 2. *Let M^m and Q^q be PL manifolds, and $f\colon M \to Q$ a proper embedding. Let M be compact and $q - m \geqslant 3$. Then f is homotopic to a PL embedding. Moreover, if $f|\partial M$ is already a PL embedding, we need not alter $f|\partial M$ in the homotopy. Furthermore, we may require that the PL embedding have image within the ϵ-neighbourhood of fM, for any given ϵ.*

THEOREM 3. *If $f_i\colon M \to Q$ are PL embeddings of PL manifolds, $i = 0, 1$, M is compact and $q - m \geqslant 3$, then if f_0 and f_1 are isotopic then they are PL isotopic.*

It has recently been shown by Miller (4), using Kirby's proof (3) of the Annulus Conjecture, that if f is also locally flat, then Theorem 2 can be strengthened to give an ϵ-ambient isotopy of f to a PL embedding.

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2. *Proof of Theorems* 2 *and* 3 *assuming Theorem* 1. We quote theorems of Stallings and of Casson and Sullivan:

THEOREM (Casson–Sullivan)(5). *Let* $f\colon M^m \to W^w$ *be a simple homotopy equivalence of PL manifolds. Let* M *be compact, and* $w-m \geqslant 3$. *If* $\partial M \neq \phi$, *let* $f|\partial M$ *be a PL embedding. Then* f *is homotopic* rel ∂M *to a PL embedding.*

THEOREM (Stallings)(6). *Let* K^k *be a simplicial complex and* Q^q *a PL manifold. Let* $f\colon |K| \to Q$ *be a map inducing isomorphisms on the first* $(2k-q)$ *homotopy groups and an epimorphism on* π_{2k-q+1}. *Let* $q-k \geqslant 3$. *Then there exists a Stallings Complex for* f.

Proof of Theorem 2.

(*a*) $\partial M = \phi$, or else $f|\partial M$ is a PL embedding. We may take $\varphi = f$ on ∂M, and assume $\tilde{M} \cap \partial Q = \partial M$, using collars of ∂M and ∂Q to push $\tilde{M}$ off the boundary elsewhere. If the collars are small, we are safe from ϵ; more precisely, taking an $\frac{1}{3}\epsilon$-PL approximation, as $\tilde{M}$ within $\frac{1}{6}\epsilon$, and using $\frac{1}{6}\epsilon$-collars and then a regular neighbourhood W of $\tilde{M}$ in Q meeting ∂Q regularly and lying within the $\frac{1}{3}\epsilon$-neighbourhood of $\tilde{M}$, we obtain a PL manifold W lying within the ϵ-neighbourhood of fM. Moreover $W \searrow \tilde{M}$, so that $f\colon M \to W$ is a simple homotopy equivalence. The Casson–Sullivan theorem now provides us with a PL embedding homotopic to φ (and hence to f) rel ∂M.

(*b*) We still have a Stallings Complex for φ and it restricts to one for $\varphi|\partial M$. After an application of the Casson–Sullivan theorem we have, in the notation of (*a*), ∂M embedded in $W \cap \partial Q$. We now have a homotopy equivalence of M into W, but not necessarily a simple one. Stallings' Theorem, however, allows us to construct a Stallings Complex in W meeting ∂W in the embedded image of ∂M. We then once more take a regular neighbourhood and apply the Casson–Sullivan Theorem to obtain the final embedding of M in Q.

Proof of Theorem 3.

We may take $\varphi = F$ on $M \times \partial I$. Applying Theorem 1 to obtain a Stallings Complex for φ, and using collars of ∂I in I, we may assume that $\tilde{M} \cap (Q \times \partial I) = F(M \times \partial I)$. A similar argument to that of Theorem 2(*a*) lets us construct a PL concordance on $\partial M \times I$, and then a similar one to that of Theorem 2(*b*) lets us extend it to a PL concordance of $M \times I$ in $Q \times I$. Hudson's Theorem(2) now allows us to assert the existence of a PL isotopy from f_0 to f_1.

3. *Proof of Theorem* 1. We may, without loss of generality, suppose that Q is compact; for otherwise we may replace it by a simplicial neighbourhood of fM, which will be compact.

Let N be the least positive integer for which

$$2k-q-(N-1)(q-k-2) < 0.$$

Let $D_n = 2k-q-n(q-k-2)$, $M_1 = 0$ and $M_n = \Sigma_0^{n-2} D_i$.

We have a metric on Q; this induces via f a metric on $|K|$.

Now $|K|$ is compact, so that given $\epsilon > 0$, we can find $\delta > 0$ such that any set in $|K|$ of diameter $< \delta$ lies within a cone of diameter $< \epsilon$. (To prove this, subdivide K repeatedly until the mesh of the covering by open vertex stars falls below ϵ; then take

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δ to be the Lebesgue number of this open cover—recall that the vertex stars are cones.) The corresponding remark for Q clearly also holds good, with balls instead of cones.

It follows that we may choose, inductively on decreasing n and m, sequences (μ_n), (η_{n+1}), (α_{n+1}), $(\beta_{n,m})$ and $(\gamma_{n,m})$, $1 \leqslant n \leqslant N-1$, $0 \leqslant m \leqslant 2D_{n-1}$, such that

(i) any set in $|K|$ of diameter $< \beta_{n,2m-1}$ lies in a cone of diameter $< \gamma_{n,2m-1}$;
(ii) any set in Q of diameter $< \beta_{n,2m}$ lies in a ball of diameter $< \gamma_{n,2m}$;
(iii) $\mu_n + 2\gamma_{n,2m-2} + 2\alpha_n < \beta_{n,2m-1}$,
(iv) $6\alpha_n + 4\delta + \gamma_{n,2m-1} < \beta_{n,2m}$,
$\Big\}\ 0 \leqslant m \leqslant D_{n-1},\ 1 \leqslant n \leqslant N-1;$
(v) $\eta_{n+1} + \gamma_{n,2D_{n-1}} + \alpha_n < \alpha_{n+1}, \quad 1 \leqslant n \leqslant N-1;$
(vi) $\alpha_N < \epsilon$;

where $\alpha_1 > \delta$, and $\gamma_{n,-2} = 0$.

We first choose α_N, by (vi), and then $\gamma_{N,2m}$, $\beta_{N,2m-1}$, $\gamma_{N,2m-1}$ and $\beta_{N,2m-1}$ inductively downwards, using (v), (ii), (iv) and (i) cyclically, ending with $\beta_{N,-1}$ and $\gamma_{N,-2} = 0$.

We now return to equations (iii), (iv) and (v), and choose values for μ_N, α_{N-1} and η_N; given α_{N-1} we can now repeat the cycle for $\gamma_{N-1,2m}$, $\beta_{N-1,2m}$, $\gamma_{N-1,2m-1}$ and $\beta_{N-1,2m-1}$, and so on, ending with choice of δ to satisfy all the equations (iv).

Let $\varphi: |K| \to Q$ be a $\delta - PL$ approximation to f, restricting on $|L|$ to a PL approximation to $f|K$, and $\phi^{-1}(\partial Q) = |L|$.

We shall now apply a double induction argument. We define below certain propositions $P(n)$ and $Q(n,m)$, and prove that

(*a*) $P(1)$ is true;
(*b*) $Q(n,-1)$ is true, $1 \leqslant n \leqslant N-1$;
(*c*) $P(n-1)$ and $Q(n-1,m-1) \Rightarrow Q(n-1,m)$, $0 \leqslant m \leqslant D_{n-2}$;
(*d*) $Q(n-1,D_{n-2}) \Rightarrow P(n)$, $2 \leqslant n \leqslant N$;
(*e*) $P(N) \Rightarrow$ Theorem 1.

The Theorem will then follow by (double) induction.

$P(n)$: $1 \leqslant n \leqslant N$.

There exists a simplicial complex K_n, with $|K_n| \supset \varphi_{n-1}(K_{n-1})$, and a PL map $\varphi_n: |K_n| \to Q$, extending the inclusion of $\varphi_{n-1}|K_{n-1}|$, with singular set of dimension at most D_{n-1}.

Moreover, ϕ_1 is a homotopy equivalence of $|K|$ with $|K_n|$, with homotopy inverse s_n and homotopies S_n and T_n, from 1 to $\varphi_1 s_n$ and from 1 to $s_n \varphi_1$, respectively, such that

(i) $d(fs_n(x), \varphi_n(x)) < \alpha_n \quad \forall\, x \in |K_n|$,
(ii) $d(s_n(x,t), x) < \alpha_n \quad \forall\, (x,t) \in |K_n| \times I$,
(iii) $d(T_n(x,t), x) < \alpha_n \quad \forall\, (x,t) \in |K| \times I$.

(*a*) $P(1)$ *is true.*

Let $K_0 = \phi$, $K_1 = K$, $\varphi_1 = \varphi$, and s_1, S_1 and T_1 be the identities on the appropriate spaces. Then $P(1)$ is satisfied.

Now suppose that we have established $P(n)$. Subdivide K_n and Q so that φ_n becomes simplicial, and the meshes of the triangulations are $< \mu_n$. Let C be the image in Q of the singular points of φ_n; then $\dim C \leqslant D_{n-1}$.

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$Q(n,m)$: $-1 \leqslant m \leqslant D_{n-1}$.

For $0 \leqslant i \leqslant m$, and $\sigma^i \in C$, there exists a polyhedron $L(\sigma)$, PL maps h_σ and j_σ, a homotopy equivalence s_σ and homotopies S_σ and T_σ, such that (writing $\varphi_\sigma = \varphi_n | \varphi_n^{-1}\sigma \to \sigma$).

(i) h_σ: $M\varphi_\sigma \to |K_n|$ extends the inclusion of $\varphi_{n-1}^{-1}(\sigma)$. Let $K(\sigma) = h_\sigma M_{\varphi_\sigma}$, and $h_\sigma^* = h_\sigma$: $M_{\varphi_\sigma} \to K(\sigma)$. Let $M(\sigma)$ be a polyhedron constructed similarly to $M_{k^*\sigma}$, but with the unit interval involved shrunk to $[0, \gamma_{n,2D_{n-1}}]$. Let us define an identification relation on $M(\sigma)$ by

$$((x,s), \gamma_{n,2D_{n-1}}) \sim \varphi_n x \sim ((x,1),t),$$

for $(x,s) \in \varphi_n^{-1}\sigma \times I \subset M_{\varphi_\sigma}$.

(ii) We shall construct $L(\sigma)$ by 'simplicializing' $M(\sigma)/\sim$, for *a priori* it is no more than a cell-complex. As in Lemma 2·1 of (6), we may, inductively up the skeleta of $M(\sigma)/\sim$, PL-approximate the attaching maps of the cells to obtain a polyhedron $L(\sigma)$ simple-homotopically equivalent to $M(\sigma)/\sim$. Let q_σ: $M(\sigma) \to M(\sigma)/\sim$ denote the identification map. Since $q_\sigma | K(\sigma)$ merely identifies certain simplexes on which individually it is an embedding, we may subdivide and then work outside $q_\sigma K(\sigma)$, so as to ensure that $L(\sigma)$ contains a subcomplex which is a subdivision of $q_\sigma K(\sigma)$. Let p_σ be the map from $M(\sigma)/\sim$ to $L(\sigma)$—then we may take $p_\sigma = 1$ on $q_\sigma K(\sigma)$, so that the homotopy equivalence $p_\sigma q_\sigma$: $K(\sigma) \simeq L(\sigma)$ is just q_σ.

(iii) s_σ: $L(\sigma) \to K(\sigma)$ is a homotopy inverse to $q_\sigma | K(\sigma)$.

(iv) j_σ: $L(\sigma) \to Q$ extends the inclusion of $\phi_n K(\sigma)$; i.e. $j_\sigma q_\sigma = \varphi_n$ on $K(\sigma)$.

(v) S_σ is a homotopy from $q_\sigma s_\sigma$: $L(\sigma) \to K(\sigma) \to L(\sigma)$ to the identity.

(vi) T_σ is a homotopy from $s_\sigma q_\sigma$ to the identity on $K(\sigma)$.

(vii) if $\tau \prec \sigma$, $h_\tau = h_\sigma | M_{\varphi_\tau}$, $j_\tau = j_\sigma | \{L(\tau) \subset L(\sigma)\}$, and similarly for $s_\tau, S_\tau, T_\tau, q_\tau$ and p_τ.

(viii) $\operatorname{diam}((K(\sigma)) < \gamma_{n,2m}$.

(ix) $\operatorname{diam}(j_\sigma L(\sigma)) < \gamma_{n,2m}$.

(*b*) $Q(n-1)$ *is true.*

Vacuously

(*c*) $Q(n-1, m-1) \Rightarrow Q(n-1, m)$

Suppose $Q(n-1, m-1)$ established; this means that $P(n-1)$ is also true. Let $\sigma^m \in C$. Then $\partial\sigma = \tau_1 \cup \ldots \cup \tau_m$, for certain $(m-1)$-simplexes $\tau_i \in C$.

Since $\phi_{n-1} K(\tau) \subset j_\tau L(\tau)$, condition (i) of $P(n-1)$ gives

$$\operatorname{diam}\{s_{n-1}(\phi_{n-1}^{-1}\sigma \cup \cup_1^m K(\tau_i))\} < \mu_{n-1} + 2\gamma_{n-1,2m-2} + 2\alpha_{n-1} < \beta_{n-1,2m-1}.$$

This set must then lie in a cone in $|K|$ of diameter $< \gamma_{n-1,2m-1}$; it follows that we can obtain a map g of M_{φ_τ} into this cone extending the maps $s_{n-1} h_{\tau_i}$.

By using the homotopy extension theorem on φ_{1g} and the $s_{n-1} | \varphi_{1g} M_{\varphi_{\tau_i}}$, and then the simplicial approximation theorem, we can obtain a PL map h_σ: $M_{\varphi_\sigma} \to |K_{n-1}|$ which extends the h_{σ_i}, and with

$$\operatorname{diam}(h_\sigma M_{\varphi_\sigma}) < \gamma_{n-1,2m-1} + 2\alpha_{n-1} + 2\delta.$$

But

$$\begin{aligned} \operatorname{diam}(\varphi_{n-1} K(\sigma)) &< 2\alpha_{n-1} + \operatorname{diam}(f s_{n-1} K(\sigma)) \\ &< 2\alpha_{n-1} + 2\delta + \operatorname{diam}(\varphi_1 s_{n-1} K(\sigma)) \\ &< 4\alpha_{n-1} + 2\delta + \operatorname{diam}\{K(\sigma)\} \\ &< 6\alpha_{n-1} + 4\delta + \gamma_{n-1,2m-1}, \end{aligned}$$

using (iii) of $P(n-1)$ for T_{n-1}.

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Thus diam $(\varphi_{n-1}K(\sigma)) < \beta_{n-1,2m}$, and so lies in a ball in Q of diameter $< \gamma_{n-1,2m}$. We may map $L(\sigma)$ into this ball, extending the maps j_{τ_i} and the inclusion of $\varphi_{n-1}K(\sigma)$.

Now, by choice of $L(\sigma)$, we have q_σ: $K(\sigma) \simeq L(\sigma)$. By a theorem of Brown (1) we can extend the homotopy inverse and one of the homotopies given on the $K(\tau_i)$ and $L(\tau_i)$; while the other homotopy cannot actually be extended, it can be replaced by a map taking each simplex into its original image simplex and then extended. This gives s_σ, S_σ and T_σ. Do this for each $\sigma^m \in C$.

(*d*) $Q(n-1, D_{n-2}) \Rightarrow P(n)$.

Suppose we have $Q(n-1, D_{n-2})$.

Let $|K_n|$ be formed from the union of $|K_{n-1}|$ and all the $L(\sigma)$ for $\sigma \in C$, by identifying each $K(\sigma)$ in $|K_{n-1}|$ with $q_\sigma K(\sigma)$ in $L(\sigma)$, and of course also the copies of $L(\tau)$ in the $L(\sigma)$, for $\tau \prec \sigma$.

Since the attaching maps are PL, $|K_n|$ can be given an (induced) triangulation.

Let φ_n^* be the map induced on $|K_n|$ from φ_{n-1} and the j_σ. Let i: $|K_{n-1}| \to |K_n|$ be induced from the inclusion.

It remains to find suitable homotopies to show φ_1: $|K| \simeq |K_n|$. It suffices to find $(\frac{1}{2}\eta_n + \alpha_n)$-homotopies showing that i: $|K_{n-1}| \simeq |K_n|$. We then obtain φ_n by an $(\frac{1}{2}\eta_n)$-shift of φ_n^* into general position, leaving it fixed on $i|K_{n-1}|$, on which it is already an embedding. The double points then have dimension

$$\max\{D_{n-2}(q-k-2), 2D_{n-2}-q+4\} = D_{n-2}-(q-k-2) = D_{n-1}.$$

Write $L = \cup L(\sigma)$, $K^* = \cup K(\sigma)$, over all $\sigma \in C$, with the $K(\tau) \subset K(\sigma)$ for $\tau < \sigma$ identified, and similarly for L.

Clearly the s_σ, S_σ and T_σ compose to give q: $K^* \simeq L$, with homotopy inverse s and homotopies S and T which restrict on $K(\sigma)$ or $L(\sigma)$ to s_σ, S_σ or T_σ.

Let
$$X_1 = |K_n| = |K_{n-1}| \cup_q L,$$
$$X_2 = X_1 \cup \{|K_{n-1}| \times [0, \tfrac{1}{2}\eta_n]\}/(x,0) \sim x \quad \forall\, x \in |K_{n-1}|$$
$$X_3 = L \cup \{K^* \times [0, \tfrac{1}{2}\eta_n]\} \cup |K_{n-1}|/(x,0) \sim qx, \ \&\ (x, \tfrac{1}{2}\eta_n) \sim x \,\forall\, x \in K^*.$$
$$X_4 = |K_{n-1}|.$$

We shall show that $X_i \simeq X_{i+1}$, $1 \leqslant i \leqslant 3$. It is helpful to consider the diagram (see Fig. 1).

Let f_1: $X_1 \to X_2$ be the inclusion, and g_1: $X_2 \to X_1$ send (x,t) to x but leave L fixed.

Let g_2: $X_3 \to X_2$ be the inclusion—the identity on $L \cup_q \{K^\kappa \times [0, \frac{1}{2}\eta_n]\}$ and $x \to (x, \frac{1}{2}\eta_n)$ on $|K_{n-1}|$.

Now, X_2 deformation retracts onto $g_2 X_3$; we may take a retraction ρ, such that

$$\rho_t(x) = X \quad \text{on} \quad L \cup_q \{K^* \times [0, \tfrac{1}{2}\eta_n]\},$$

$$\rho_t(x,s) = (x, \tfrac{1}{2}\eta_n\{s+t-st\}) \quad \text{for} \quad \left(x, \frac{2s}{\eta_n}\right) \in \{|K_{n-1}| - W\} \times [0, \tfrac{1}{2}\eta_n]$$

where W is some $(\frac{1}{2}\eta_n)$-simplicial neighbourhood of K^* in K_{n-1} subdivided.

$$\rho_t(\sigma) = \left\{\sigma \times \left[\frac{t}{2}\eta_n, \tfrac{1}{2}\eta_n\right]\right\} \cup \{\tau \times [0, \tfrac{1}{2}\eta_n]\},$$

where $\tau \prec \sigma$ is such that $\tau = \sigma \cap W$.

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Let $f_2 = \rho_1$.

Since q is a homotopy equivalence, M_q strongly deformation retracts onto K^*. We may (1) construct the deformation retraction inductively over the skeleton of C, so that $M_{\phi_{n-1}|K(\sigma)}$ is retracted onto $K(\sigma)$. Let r_t be this retraction.

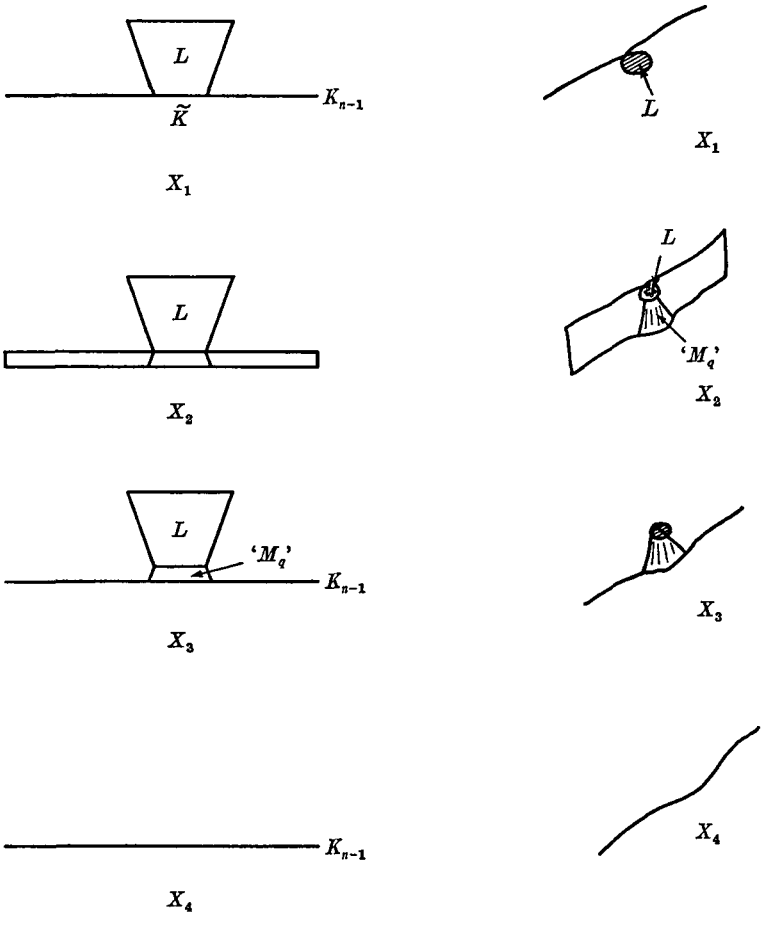


Fig. 1

Let $f_3\colon X_3 \to X_4$ be r_1 on L, and the identity elsewhere.

Let $g_3\colon X_4 \to X_3$ be inclusion.

Then $f_3f_2f_1\colon X_1 \to X_4$ and $g_3g_2g_1\colon X_4 \to X_1$ are the maps we want; by construction they are homotopy inverse to each other; we must check the δ-condition.

The homotopy from $f_3f_2f_1g_1g_2g_3$ to the identity on X_4 is itself the identity outside W, and keeps each $K(\sigma)$ within itself. Recalling the properties of ρ, we find that our homotopy moves no point further than $\frac{1}{2}\eta_n + \text{diam}\,(K(\sigma)) < \frac{1}{2}\eta_n + \gamma_{n,2D_{n-1}}$.

Similarly the homotopy on X_1 moves no point further than $\frac{1}{2}\eta_n + \text{diam}\,(L(\sigma))$ which is also $< \frac{1}{2}\eta_n + \gamma_{n,2D_{n-1}}$.

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It remains to check that $d(fs_n(x), \varphi_n(x))$ is small enough. But s_n is the composition of $f_3 f_2 f_1$ with s_{n-1}. It is thus sufficient that

$$d(\phi_{n-1} f_3 f_2 f_1(x), \varphi_n^*(x)) < \tfrac{1}{2}\eta_n + \gamma_{n,2D_{n-1}}.$$

This is true because $f_3 f_2 f_1$ maps each $L(\sigma)$ into itself, and diam $(j_\sigma L(\sigma))$ is $< \gamma_{n,2D_{n-1}}$.

(*e*) $P(N) \Rightarrow$ *Theorem* 1.

At the N-stage we have obtained a polyhedron $|K_N|$ embedded in Q, containing $\phi|K|$, and homotopy equivalent to $|K|$ by φ.

Let W be a regular neighbourhood of $\varphi_N|K_N|$ in Q; then W is PL manifold and φ maps $|K|$ to W. Since φ is a homotopy equivalence of $|K|$ with W, we may quote Stallings (6) to get $\tilde{K}$ embedded in W, and hence in Q, of the same simple homotopy type as $|K|$.

The assertions in Theorem 1 about the boundary are clearly true—the whole procedure can have been relative to the boundary.

For the ϵ-clause, recall that the δ's were such that $|K_N|$ is $< \epsilon$ from $f|K|$; then W can also be within this ϵ-neighbourhood.

Remark. Probably our homotopies are in fact simple, so that K_N is itself a candidate for $\tilde{K}$; it could be rather difficult to check up on this, however.

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