Proc. Camb. Phil. Soc. (1968), **64**, 87
PCPS 64–15
Printed in Great Britain

On the coefficient problem for univalent polynomials

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(*Received* 29 *March* 1967)

1. *Introduction and statement of results.* This paper is concerned with estimates of the coefficients C_n for normalized polynomials, inverse to univalent functions in the unit disc, i.e. univalent polynomials of the form

$$P_n(w) = w + C_2 w^2 + \ldots + C_n w^n \quad (C \neq 0) \tag{1}$$

considered in the largest domain D such that $0 \in D$ and $|P_n(w)| < 1$ for $w \in D$; this domain will be denoted by μ_{P_n}. Polynomials of the type (1) have been introduced by Charzyński in (1), where it was proved among other results that the domain μ_{P_n} is simply connected.

Let $\mathscr{P}_m$ denote the class of all such polynomials of the form (1) with $n \leqslant m$ (m a positive integer). Charzyński and Ławrynowicz (2) have established the following results.

Theorem A. *In the class $\mathscr{P}_2$ the sharp estimate $|C_2| \leqslant \frac{1}{4}$ holds. All the extremal polynomials are given by the formula $P_2^*(w) = w - \frac{1}{4} e^{i\theta} w^2$, where θ is real.*

Theorem B. *In the class $\mathscr{P}_3$ the sharp estimate $|C_3| \leqslant \frac{4}{27}$ holds. All the extremal polynomials are given by the formula $P_3^*(w) = w - \frac{4}{27} e^{i\theta} w^3$, where θ is real.*

Let M_{mk} denote the supremum of $|C_k|$ taken over $P_n \in \mathscr{P}_m$ ($n = k, \ldots, m$). According to Löwner (3), $M_{mk} \leqslant 1.3.5 \ldots .(2k-1)\, 2^k/(k+1)!$ for $m = 2, 3, \ldots$ and $k = 2, \ldots, m$.

In the first part of this paper I prove some theorems concerning M_{mm}.

Theorem 1. *Suppose that $m > 1$ is an integer. A polynomial of the form $P_m(w) = w + C_m w^m$ belongs to the class $\mathscr{P}_m$ if and only if $|C_m| \leqslant (m-1)^{m-1}/m^m$.*

This implies immediately:

Theorem 2. *In the class $\mathscr{P}_m$ we have*

$$M_{mm} \geqslant (m-1)^{m-1}/m^m \quad (m = 2, 3, \ldots). \tag{2}$$

The equality $|C_m| = (m-1)^{m-1}/m^m$ holds for all polynomials of the form

$$P_m^*(w) = w - \{(m-1)^{m-1}/m^m\} e^{i\theta} w^m \quad (\theta \text{ real}). \tag{3}$$

Theorem 2 may be proved in a much more natural way which shows how the polynomials (3) can be deduced; and the method used has further applications. The second proof, found earlier, is based on the following results of mine.

Theorem 3. *Let $m > 2$ be an integer. Suppose that in the class $\mathscr{P}_m$ there exists a polynomial*

$$P_m^*(w) = w + C_2^* w^2 + \ldots + C_m^* w^m \quad (C_m^* \neq 0) \tag{4}$$

https://doi.org/10.1017/S0305004100042602 Published online by Cambridge University Press

for which the functional $|C_m|$ *attains its maximum, such that the derived polynomial has* $m-1$ *distinct zeros,* $w_1, \ldots, w_{m-1}$, *say. Suppose next* $P_m^*(w_{m-1}) = 1$. *Then* †

$$\sum_{\nu=1}^{m} \frac{(-1)^\nu}{\nu} \tau_{m-\nu}(\delta_1, \ldots, \delta_{m-2}, 1) \left\{ \delta_\lambda^{\nu-2} \prod_{\substack{\kappa=1 \\ \kappa \neq \lambda}}^{m-2} (1-\delta_\kappa) + \mu_\lambda \prod_{\substack{\kappa=1 \\ \kappa \neq \lambda}}^{m-2} (\delta_\lambda - \delta_\kappa) \right\} = 0 \quad (\lambda = 1, \ldots, m-2), \tag{5}$$

where $\delta_\lambda = w_\lambda / w_{m-1}$ $(\lambda = 1, \ldots, m-2)$,

$$\mu_\lambda = \left| \prod_{\substack{\kappa=1 \\ \kappa \neq \lambda}}^{m-2} (1-\delta_\kappa) \right| \Big/ \left| \delta_\lambda^2 \prod_{\substack{\kappa=1 \\ \kappa \neq \lambda}}^{m-2} (\delta_\lambda - \delta_\kappa) \right| \quad (\lambda = 1, \ldots, m-2), \tag{6}$$

$\tau_1(s_1, \ldots, s_r), \ldots, \tau_r(s_1, \ldots, s_r)$ *denote the elementary symmetric polynomials of the variables* $s_1, \ldots, s_r$, *and* $\tau_0(s_1, \ldots, s_r) = 1$ *identically.*

Remark 1. By Theorem C below in the class $\mathscr{P}_m$ there exists a polynomial (4) for which the functional $\Re \log C_m$ (and thus also $|C_m|$) attains its maximum. By Remark B below we may assume $P_m^*(w_{m-1}) = 1$ without any loss of generality.

Remark 2. In the case where $P_m^{*\prime}$ has $m-1$ distinct zeros, Theorem 3 reduces the problem of finding M_{mm} for $m > 2$ to the problem of solving a system of $m-2$ algebraic equations with $m-2$ unknowns.

THEOREM 4. *Suppose that a polynomial* P_m^* *of the form* (4), *where* $m > 2$, *is such that* $P_m^{*\prime}$ *has* $m-1$ *distinct zeros* $w_1, \ldots, w_{m-1}$ *satisfying the relations* (5) *and* $P_m^*(w_{m-1}) = 1$, *where* δ_λ $(\lambda = 1, \ldots, m-2)$, μ_λ $(\lambda = 1, \ldots, m-2)$ *and* τ_λ $(\lambda = 0, \ldots, m-1)$ *are defined in Theorem* 3. *Then* P_m^* *belongs to the class* $\mathscr{P}_m$.

THEOREM 5. *If* $m > 2$ *is an integer, the polynomial*

$$P_m^*(w) = w - \{(m-1)^{m-1}/m^m\} w^m$$

satisfies the hypotheses of Theorem 4 *with* $\mu_1 = \mu_2 = \ldots = \mu_{m-2} = 1$.

Theorem 2 is an immediate consequence of Theorems A, 4 and 5.

In the second part of this paper I apply Theorem 3 to prove the following result:

THEOREM 6. *In the class* $\mathscr{P}_4$ *the sharp estimate* $|C_4| \leqslant \frac{27}{256}$ *holds. All the extremal polynomials are given by the formula* $P_4^*(w) = w - \frac{27}{256} e^{i\theta} w^4$, *where* θ *is real.*

The same method can be also used to prove Theorem B, and it is much simpler than the method applied in (2).

In view of Theorems A, B and 6 it seems to me natural to pose the following:

CONJECTURE. *In the class* $\mathscr{P}_m$ *we have*

$$M_{mm} = (m-1)^{m-1}/m^m \quad (m = 2, 3, \ldots). \tag{7}$$

Equality holds if and only if P_m *is of the form* (3).

Here I should like to express my sincere thanks to Profs J. Clunie and W. K. Hayman, and to Mr D. Brannan (Imperial College, London) for reading the manuscript of this paper and for their helpful remarks. During the preparation of this paper the author held a Scholarship of the Polish Academy of Sciences at the Imperial College, London, and the University of Cambridge; in London, under the guidance of Prof. W. K. Hayman.

† Throughout this paper the empty product is interpreted as unity.

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Part I

2. We proceed to prove Theorem 1. It is known (see (1), p. 8) that a polynomial P_n of the form (1) belongs to $\mathscr{P}_m$ ($m \geqslant n$) if and only if all the roots of P_n' lie outside or on the boundary of μ_{P_n}. As for $P_m(w) = w + C_m w^m$ the roots w_μ ($\mu = 1, \ldots, m-1$) of P_m' satisfy $w_\mu^{m-1} = -1/mC_m$, then $P_m(w_\mu) = (1-1/m)w_\mu$, and, consequently, the polynomial considered belongs to $\mathscr{P}_m$ if and only if $|(1-1/m)w_\mu| \geqslant 1$. As $C_m = -1/mw_\mu^{m-1}$ ($\mu = 1, \ldots, m$), the condition obtained may be written in the equivalent form $|C_m| \leqslant (1/m)(1-1/m)^{m-1} = (m-1)^{m-1}/m^m$.

Our further considerations are based on the following known results.

Theorem C (see (1), p. 20). *If E is a holomorphic function of the complex variables $z_2, \ldots, z_k$ ($k \leqslant m$) in a sufficiently large domain Δ, then in the class $\mathscr{P}_m$ there exist extremal polynomials for which the functional $\Re E(C_2, \ldots, C_k)$ attains its maximum. Moreover, if*

$$P_n^*(w) = w + C_2^* w^2 + \ldots + C_n^* w^n \quad (C_n^* \neq 0,\ k \leqslant n \leqslant m) \tag{8}$$

is a such polynomial, $w_1, \ldots, w_l$ denote all the distinct zeros of the derived polynomial, and $\gamma_1, \ldots, \gamma_l$—the multiplicity of these zeros, respectively, then $w_1, \ldots, w_l$ lie on the boundary of the domain μ_{P^*}, and there exist numbers $\rho_\lambda > 0$ ($\lambda = 1, \ldots, l$) satisfying the relations

$$\operatorname{Res}_{w_\lambda}\{P_n^*(w)\,\phi_k(w)/w^2 P_n^{*\prime}(w)\} = \rho_\lambda \quad (\lambda = 1, \ldots, l) \tag{9}$$

where

$$\phi_k(w) = \sum_{\kappa=2}^{k} w^{1-\kappa} \sum_{\nu=\kappa}^{k} (\nu - \kappa + 1)\, C_{\nu-\kappa+1}^* E_{z_\nu}'(C_2^*, \ldots, C_k^*) \quad (C_1^* = 1). \tag{10}$$

Remark A (see (2)). Any polynomial, satisfying the necessary condition formulated in Theorem C, belongs to the class $\mathscr{P}_m$.

Remark B (see (2)). The modulus of the kth coefficient ($k \leqslant n$) of any polynomial of the form $z = e^{i\theta} P_n(e^{-i\theta} w)$, P_n being fixed, and θ real and arbitrary, is the same, and all these polynomials belong to the class $\mathscr{P}_m$ provided $P_n \in \mathscr{P}_m$.

3. In order to prove Theorem 3 we need the following

Lemma. *Let P_n^* be a polynomial of the form* (8).† *Suppose that the derived polynomial has a zero w_λ different from the others, $w_1, \ldots, w_{\lambda-1}, w_{\lambda+1}, \ldots, w_{n-1}$, say. Let ϕ_k be defined by* (10) *where $E(z_2, \ldots, z_k) = \log z_k$, let $\tau_1(s_1, \ldots, s_r), \ldots, \tau_r(s_1, \ldots, s_r)$ denote the elementary symmetric polynomials of the variables $s_1, \ldots, s_r$, and let $\tau_0(s_1, \ldots, s_r) = 1$ identically. Then*

$$\operatorname{Res}_{w_\lambda}\{P_n^*(w)\,\phi_k(w)/w^2 P_n^{*\prime}(w)\} = \left\{(-1)^n k \Big/ \prod_{\substack{\kappa=1 \\ \kappa \neq \lambda}}^{n-1} (w_\lambda - w_\kappa)\right\} \frac{P_n^*(w_\lambda)\,\tau_{n-1}(w_1, \ldots, w_{n-1})}{w_\lambda^2\, \tau_{n-k}(w_1, \ldots, w_{n-1})}$$
$$\times \tau_{n-k}(w_1, \ldots, w_{\lambda-1}, w_{\lambda+1}, \ldots, w_{n-1}). \tag{11}$$

Proof. By (10), we have for $E(z_2, \ldots, z_k) = \log z_k$

$$\phi_k(w) = (1/C_k^*) \sum_{\kappa=2}^{k} (k - \kappa + 1)\, C_{k-\kappa+1}^* w^{1-\kappa}.$$

On the other hand, $w_1, \ldots, w_{n-1}$ are zeros of $P_n^{*\prime}(w) = 0$, and hence

$$C_\kappa^* = (-1)^{\kappa-1}(1/\kappa)\tau_{n-\kappa}(w_1, \ldots, w_{n-1})/\tau_{n-1}(w_1, \ldots, w_{n-1}) \quad (\kappa = 1, \ldots, n). \tag{12}$$

† We do not assume that P_n^* fulfils the conditions of Theorem C.

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Consequently,

$$\phi_k(w) = \{k/\tau_{n-k}(w_1, \ldots, w_{n-1})\} \sum_{\kappa=1}^{k-1} (-1)^\kappa \tau_{\kappa+n-k}(w_1, \ldots, w_{n-1}) w^{-\kappa}. \tag{13}$$

Since, obviously, $w_\lambda \neq 0$, the function $z = (1/w^2)\phi_k(w)$ has no singularity at the point $w = w_\lambda$. Hence

$$\operatorname{Res}_{w_\lambda}\{P_n^*(w)\phi_k(w)/w^2 P_n^{*\prime}(w)\} = \left\{(-1)^{n-1} \Big/ \prod_{\substack{\kappa=1 \\ \kappa\neq\lambda}}^{n-1} (w_\lambda - w_\kappa)\right\} \tau_{n-1}(w_1, \ldots, w_{n-1}) \\ w_\lambda^{-2} P_n^*(w_\lambda)\phi_k(w_\lambda), \tag{14}$$

where ϕ_k is given by (13).

In order to simplify the formula derived, we note that, by $P_n^{*\prime}(w_\lambda) = 0$,

$$\begin{aligned} &\sum_{\kappa=1}^{k-1} (-1)^\kappa \tau_{\kappa+n-k}(w_1, \ldots, w_{n-1}) w_\lambda^{-\kappa} \\ &\quad = (-1)^{n-k} w_\lambda^{-k+1} \sum_{\kappa=1}^{k-1} (-1)^{\kappa+n-k} \tau_{\kappa+n-k}(w_1, \ldots, w_{n-1}) w_\lambda^{-\kappa+k-1} \\ &\quad = (-1)^{n-k} w_\lambda^{-k+1} \sum_{\kappa=n-k+1}^{n-1} (-1)^\kappa \tau_\kappa(w_1, \ldots, w_{n-1}) w_\lambda^{n-\kappa-1} \\ &\quad = (-1)^{n-k+1} w_\lambda^{-k+1} \sum_{\kappa=0}^{n-k} (-1)^\kappa \tau_\kappa(w_1, \ldots, w_{n-1}) w_\lambda^{n-\kappa-1}. \end{aligned}$$

Now, applying for $\kappa = 0, \ldots, n-k$ the well-known identity

$$\tau_\kappa(w_1, \ldots, w_{n-1}) = \tau_{\kappa-1}(w_1, \ldots, w_{\lambda-1}, w_{\lambda+1}, \ldots, w_{n-1}) w_\lambda \\ + \tau_\kappa(w_1, \ldots, w_{\lambda-1}, w_{\lambda+1}, \ldots, w_{n-1}),$$

we obtain

$$\begin{aligned} &\sum_{\kappa=0}^{n-k} (-1)^\kappa \tau_\kappa(w_1, \ldots, w_{n-1}) w_\lambda^{n-\kappa-1} \\ &\quad = w_\lambda^{n-1} + \sum_{\kappa=1}^{n-k} (-1)^\kappa \tau_{\kappa-1}(w_1, \ldots, w_{\lambda-1}, w_{\lambda+1}, \ldots, w_{n-1}) w_\lambda^{n-\kappa} \\ &\qquad + \sum_{\kappa=1}^{n-k} (-1)^\kappa \tau_\kappa(w_1, \ldots, w_{\lambda-1}, w_{\lambda+1}, \ldots, w_{n-1}) w_\lambda^{n-\kappa-1} \\ &\quad = (-1)^{n-k} \tau_{n-k}(w_1, \ldots, w_{\lambda-1}, w_{\lambda+1}, \ldots, w_{n-1}) w_\lambda^{k-1}. \end{aligned}$$

Consequently,

$$\sum_{\kappa=1}^{k-1} (-1)^\kappa \tau_{\kappa+n-k}(w_1, \ldots, w_{n-1}) w_\lambda^{-\kappa} = -\tau_{n-k}(w_1, \ldots, w_{\lambda-1}, w_{\lambda+1}, \ldots, w_{n-1}). \tag{15}$$

Formulae (14), (13) and (15) imply (11). This completes the proof of the Lemma.

4. *Proof of Theorem* 3. We apply Theorem C with $E(C_2, \ldots, C_m) = \log C_m$. Then

(i) there exist numbers $\rho_\lambda > 0$ ($\lambda = 1, \ldots, m-1$) satisfying the relations (9) where $k = n = m$, $l = m-1$ and ϕ_m is defined by (10),

(ii) we have

$$w_\lambda + C_2^* w_\lambda^2 + \ldots + C_m^* w_\lambda^m = \sigma_\lambda \quad (|\sigma_\lambda| = 1, \lambda = 1, \ldots, m-2; \sigma_{m-1} = 1) \tag{16}$$

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(cf. Remark B), and, obviously,

$$1+2C_2^*w_\lambda+\ldots+mC_m^*w_\lambda^{m-1}=0 \quad (\lambda=1,\ldots,m-1). \tag{17}$$

As $w_1,\ldots,w_{m-1}$ are assumed to be distinct, we can apply our lemma to each of them. Hence, in view of (9), we have

$$\rho_\lambda=(-1)^m m\sigma_\lambda \prod_{\kappa=1}^{m-1} w_\kappa \Big/ w_\lambda^2 \prod_{\substack{\kappa=1\\ \kappa\neq\lambda}}^{m-1}(w_\lambda-w_\kappa) \,(|\sigma_\lambda|=1, \lambda=1,\ldots,m-2; \sigma_{m-1}=1). \tag{18}$$

Comparing for $\lambda=1,\ldots,m-2$ the λth of the equations (18) with the $m-1$th one, we get

$$\mu_\lambda(1/\sigma_\lambda)\,\delta_\lambda^2(\delta_\lambda-1)\prod_{\substack{\kappa=1\\ \kappa\neq\lambda}}^{m-2}(\delta_\lambda-\delta_\kappa)=\prod_{\kappa=1}^{m-2}(1-\delta_\kappa) \quad (|\sigma_\lambda|=1, \lambda=1,\ldots,m-2),$$

where $\delta_\lambda=w_\lambda/w_{m-1}$ and $\mu_\lambda=\rho_\lambda/\rho_{m-1}$ $(\lambda=1,\ldots,m-2)$. Hence

$$\mu_\lambda\delta_\lambda^2\prod_{\substack{\kappa=1\\ \kappa\neq\lambda}}^{m-2}(\delta_\lambda-\delta_\kappa)+\sigma_\lambda\prod_{\substack{\kappa=1\\ \kappa\neq\lambda}}^{m-2}(1-\delta_\kappa)=0 \quad (|\sigma_\lambda|=1, \lambda=1,\ldots,m-2), \tag{19}$$

and, consequently, since $\rho_\lambda>0$ $(\lambda=1,\ldots,m-1)$, we obtain (6).

On the other hand, the relations (16) and (17), or, equivalently, (16) and (12) with $n=m$, yield

$$\{1/\tau_{m-1}(w_1,\ldots,w_{m-1})\}\sum_{\nu=1}^{m}\frac{(-1)^\nu}{\nu}\tau_{m-\nu}(w_1,\ldots,w_{m-1})w_\lambda^\nu+\sigma_\lambda=0 \quad (\lambda=1,\ldots,m-1). \tag{20}$$

Comparing for $\lambda=1,\ldots,m-2$ the λth of the equations (20) with the $m-1$th one, we get, in view of $\sigma_{m-1}=1$,

$$\sum_{\nu=1}^{m}\frac{(-1)^\nu}{\nu}\tau_{m-\nu}(\delta_1,\ldots,\delta_{m-2},1)\,(\delta_\lambda^\nu-\sigma_\lambda)=0 \quad (\lambda=1,\ldots,m-2). \tag{21}$$

The equations (19) and (21) yield (5). In this way the proof of Theorem 3 is completed.

5. *Proof of Theorem* 4. Similarly, as in the proof of Theorem 1, we apply to P_m^* a lemma stating that a polynomial P_n of the form (1) belongs to $\mathscr{P}_m$ $(m\geqslant n)$ if and only if all the roots of P_n' lie outside or on the boundary of μ_{P_n}. We shall prove that in our case all the roots of $P_m^{*\prime}$ satisfy this condition because they satisfy the relations

$$|P_m^*(w_\lambda)|=1 \quad (\lambda=1,\ldots,m-1). \tag{22}$$

In fact, the relations (6) are equivalent to (19). Hence the system of (5) and (6) is equivalent to the system of (19) and (21). Now, since also

$$\delta_\lambda=w_\lambda/w_{m-1} \quad (\lambda=1,\ldots,m-2),$$

we see that (21) is equivalent to (20), where in both cases

$$|\sigma_1|=|\sigma_2|=\ldots=|\sigma_{m-2}|=\sigma_{m-1}=1.$$

But, in view of (12), the relations (20) yield (16) and, consequently, also (22). Hence Theorem 4 is proved.

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6. *Proof of Theorem* 5. It can be easily seen that $P_m^*(w) = w - \{(m-1)^{m-1}/m^m\}w^m$ is of the form (4) and that the derived polynomial has $m-1$ distinct zeros, $w_1, \ldots, w_{m-1}$, say, where, e.g. $P_m^*(w_{m-1}) = 1$. Hence we have to verify the relations (5) and $\mu_1 = \mu_2 = \ldots = \mu_{m-2} = 1$.

Let $W_\lambda(\delta_1, \ldots, \delta_{m-2})$ denote the left-hand side of the λth of the equations (5). In the case of the polynomial considered we have $\delta_1 = \eta_1, \ldots, \delta_{m-2} = \eta_{m-2}$, where $\eta_1, \ldots, \eta_{m-2}, 1$ are the distinct solutions of the equation $\eta^{m-1} - 1 = 0$. Substituting the corresponding values for $\tau_{m-\nu}(\delta_1, \ldots, \delta_{m-2}, 1)$, where $\nu = 1, \ldots, m$, we have

$$(-1)^{m-1}W_\lambda(\eta_1, \ldots, \eta_{m-2}) = \left\{\eta_\lambda^{-1}\prod_{\substack{\kappa=1\\ \kappa\neq\lambda}}^{m-2}(1-\eta_\kappa) + \mu_\lambda\prod_{\substack{\kappa=1\\ \kappa\neq\lambda}}^{m-2}(\eta_\lambda - \eta_\kappa)\right\}$$

$$-\frac{1}{m}\left\{\eta_\lambda^{m-2}\prod_{\substack{\kappa=1\\ \kappa\neq\lambda}}^{m-2}(1-\eta_\kappa) + \mu_\lambda\prod_{\substack{\kappa=1\\ \kappa\neq\lambda}}^{m-2}(\eta_\lambda - \eta_\kappa)\right\} \quad (\lambda = 1, \ldots, m-2).$$

Clearly,
$$\prod_{\substack{\kappa=1\\ \kappa\neq\lambda}}^{m-2}(\eta_\lambda - \eta_\kappa) = \frac{\eta_\lambda^{m-3}}{1-\eta_{m-\lambda-1}}\prod_{\kappa=1}^{m-2}(1-\eta_\kappa) \quad (\lambda = 1, \ldots, m-2) \tag{23}$$

and $\eta_\lambda^{m-1} = 1$ for $\lambda = 1, \ldots, m-2$, so

$$(-1)^{m-1}W_\lambda(\eta_1, \ldots, \eta_{m-2}) = \frac{m-1}{m}\eta_\lambda^{-2}\left(\frac{\eta_\lambda}{1-\eta_\lambda} + \frac{\mu_\lambda}{1-\eta_{m-\lambda-1}}\right)\prod_{\kappa=1}^{m-2}(1-\eta_\lambda)$$

$$(\lambda = 1, \ldots, m-2).$$

Since $\eta_\lambda\eta_{m-\lambda-1} = 1$ for $\lambda = 1, \ldots, m-2$, the system of numbers $\delta_1 = \eta_1, \ldots, \delta_{m-2} = \eta_{m-2}$ satisfies all the equations (5) provided $\mu_1 = \mu_2 = \ldots = \mu_{m-2} = 1$.

Finally, we verify that $\mu_1 = \mu_2 = \ldots = \mu_{m-2} = 1$ which is a simple consequence of (6) and $\delta_1 = \eta_1, \ldots, \delta_{m-2} = \eta_{m-2}$. Hence the proof of Theorem 5 is completed.

Remark 3. For any $m > 1$ the polynomial $P_m^*(w) = w - \{(m-1)^{m-1}/m^m\}w^m$ satisfies the necessary condition for the maximum of $|C_m|$ given in Theorem C. Moreover,

$$\rho_1 = \rho_2 = \ldots = \rho_{m-1} = 1.$$

As the μ_λ $(\lambda = 1, \ldots, m-2)$ which appear in Theorem 3 are equal to

$$\rho_\lambda/\rho_{m-1} \quad (\lambda = 1, \ldots, m-2),$$

respectively, this is an extension of Theorem 5.

In order to prove $\rho_1 = \rho_2 = \ldots = \rho_{m-1} = 1$ we apply (18) where $w_\lambda = \eta_\lambda m/(m-1)$, $\sigma_\lambda = \eta_\lambda$ for $\lambda = 1, \ldots, m-2$, and $w_{m-1} = m/(m-1)$, $\sigma_{m-1} = 1$. Thus we have

$$\rho_\lambda = (m-1)/\eta_\lambda(\eta_\lambda - 1)\prod_{\substack{\kappa=1\\ \kappa\neq\lambda}}^{m-2}(\eta_\lambda - \eta_\kappa) \quad (\lambda = 1, \ldots, m-2) \tag{24}$$

and
$$\rho_{m-1} = (m-1)\Big/\prod_{\kappa=1}^{m-2}(1-\eta_\kappa). \tag{25}$$

Formulae (24) and identities (23) imply

$$\rho_\lambda = \left\{(m-1)\Big/\prod_{\kappa=1}^{m-2}(1-\eta_\kappa)\right\}\frac{1-\eta_{m-\lambda-1}}{\eta_\lambda^{m-2}(\eta_\lambda - 1)} \quad (\lambda = 1, \ldots, m-2).$$

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Clearly, $(1-\eta_{m-\lambda-1})/\eta_\lambda^{m-2}(\eta_\lambda-1)=1$ for $\lambda=1,\ldots,m-2$, so

$$\rho_1=\rho_2=\ldots=\rho_{m-2}=(m-1)\Big/\prod_{\kappa=1}^{m-2}(1-\eta_\kappa). \tag{26}$$

Finally, in view of the well-known identity

$$\prod_{\kappa=1}^{m-2}(\eta-\eta_\kappa)=\sum_{\kappa=0}^{m-2}\eta^\kappa,$$

formulae (26) and (25) give $\rho_1=\rho_2=\ldots=\rho_{m-1}=1$, as desired.

Part II

7. In this part we prove Theorem 6.

We adopt the notation of Theorem C. By this theorem there exists in $\mathscr{P}_4$ an extremal polynomial

$$P_4^*(w)=w+C_2^*w^2+C_3^*w^3+C_4^*w^4 \quad (C_4^*\neq 0) \tag{27}$$

for which the functional $\Re E(C_2,C_3,C_4)=\Re\log C_4$ attains the maximal value, and, consequently, the estimate $|C_4|\leqslant|C_4^*|$ is best possible in $\mathscr{P}_4$. Moreover, C_4^* can be obtained by solving the system of algebraic equations described in Theorem C, and, in view of Remark B, we may, without loss of generality, assume $\sigma_l=1$.

We consider, separately, the cases $l=1,2,3$.

First, let $l=1$. Hence $P_4^{*\prime}(w)=(1-w/w_1)^3$. Since E is defined as in our lemma, we may apply formula (13) with $n=k=4$, whence

$$\phi_4(w)=-4\{3(w_1/w)-3(w_1/w)^2+(w_1/w)^3\}.$$

Consequently, since $P_4^*(w_1)=\sigma_1=1$, we obtain

$$\operatorname{Res}_{w_1}\{P_4^*(w)\phi_4(w)/w^2P_4^{*\prime}(w)\}=12/w_1,$$

whence, by Theorem C,

$$12/w_1=\rho_1>0. \tag{28}$$

On the other hand, we may apply for $\lambda=1$ formulae (20) with $m=4$ and $w_3=w_2=w_1$, since they are a consequence of (16), and of (12) with $n=m$. Hence, since $\sigma_1=1$, we get $w_1=4$, and, consequently, $C_4^*=-1/4w_1^3=-\frac{1}{256}$. Comparing this result with (28) we see that there is no contradiction in this case, but as, according to Theorem 2, $M_{44}\geqslant\frac{27}{256}$, the case considered does not give the absolute maximum of $|C_4|$ in $\mathscr{P}_4$.

8. *The case* $l=2$. We may write $P_4^{*\prime}(w)=(1-w/w_1)(1-w/w_2)^2$ where $w_1\neq w_2$. Applying for w_1 our lemma and Theorem C we get

$$4\sigma_1w_2^2/w_1(w_1-w_2)^2=\rho_1>0 \quad (|\sigma_1|=1). \tag{29}$$

Next, applying again formula (13) and Theorem C for w_2, we obtain, since $P_4^*(w_2)=\sigma_2=1$,

$$4w_1(2w_1-3w_2)/w_2(w_1-w_2)^2=\rho_2>0. \tag{30}$$

Comparing (29) with (30) we get

$$\mu_1\delta_1^2(2\delta_1-3)=\sigma_1 \quad (\mu_1>0,\ |\sigma_1|=1), \tag{31}$$

where $\delta_1=w_1/w_2$ and $\mu_1=\rho_1/\rho_2$.

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On the other hand, we may apply for $\lambda = 1$ formulae (21) with $m = 4$ and $\delta_2 = 1$, since they are a consequence of (16), and of (12) with $n = m$. Hence we get

$$\delta_1^2(\delta_1^2 - 4\delta_1 + 6) - \sigma_1(4\delta_1 - 1) = 0. \tag{32}$$

Eliminating σ_1 from (31) and (32), we obtain

$$(8\mu_1 - 1)\,\delta_1^2 - 2(7\mu_1 - 2)\,\delta_1 + 3(\mu_1 - 2) = 0 \quad (\mu_1 > 0). \tag{33}$$

Suppose first $\Im\delta_1 \neq 0$, or, equivalently, that $0 < \mu_1 < \frac{2}{25}$. Since, by (33),

$$\delta_1 = \frac{1}{8\mu_1 - 1}\{7\mu_1 - 2 + \gamma\sqrt{[(\mu_1 + 1)\,(25\mu_1 - 2)]}\} \quad (\gamma = 1 \text{ or } -1),$$

then $|\delta_1|^2 = 3(2-\mu_1)/(1-8\mu_1)$ and $|4\delta_1 - 1|^2 = 81/(1-8\mu_1)$. Applying now for $\lambda = 2$ formulae (20) with $m = 4$, $w_3 = w_2$ and $\sigma_2 = 1$, we get $w_2 = 12/(4 - \delta_1^{-1})$, whence

$$C_4^* = -1/4w_1w_2^2 = -3^{-3}.4^{-4}\delta_1^{-1}(4 - \delta_1^{-1})^3, \tag{34}$$

and, consequently, $|C_4^*| = 3.4^{-4}\sqrt{(1-8\mu_1)}/(2-\mu_1)^2 < 3.4^{-4}(\frac{2}{4}\frac{5}{8})^2 < M_{44}$.

Suppose now that δ_1 is real. Hence, in view of (31), we have $\sigma_1 = 1$ or -1. If $\sigma_1 = 1$, then (32) yields $\delta_1 = 1$ which contradicts $w_1 \neq w_2$. If $\sigma_1 = -1$, then (32) takes the form $\delta_1^4 - 4\delta_1^3 + 6\delta_1^2 + 4\delta_1 - 1 = 0$. It can be easily seen that this equation has only two real roots: they are situated in the intervals $-0{\cdot}7 < \delta_1 < -0{\cdot}6$ and $0{\cdot}19 < \delta_1 < 0{\cdot}2$, respectively. Now we apply (34), as before. If $-0{\cdot}7 < \delta_1 < -0{\cdot}6$, then

$$|C_4^*| < 3^{-3}.4^{-4}.\tfrac{5}{3}(\tfrac{17}{3})^3 < M_{44}.$$

If $0{\cdot}19 < \delta_1 < 0{\cdot}2$, then $|C_4^*| < 3^{-3}.4^{-4}.\frac{100}{19}(\frac{24}{19})^3 < M_{44}$ again.

9. *The case* $l = 3$. We have $P_4^{*\prime}(w) = (1 - w/w_1)\,(1 - w/w_2)\,(1 - w/w_3)$ where w_1, w_2, w_3 are distinct. Applying Theorem 3 we get

$$\begin{aligned} &-\delta_\lambda\delta_\kappa\{\delta_\lambda^{-1}(1-\delta_\kappa) + \mu_\lambda(\delta_\lambda - \delta_\kappa)\} + \tfrac{1}{2}(\delta_\lambda\delta_\kappa + \delta_\lambda + \delta_\kappa)\{(1-\delta_\kappa) + \mu_\lambda(\delta_\lambda - \delta_\kappa)\} \\ &\quad - \tfrac{1}{3}(\delta_\lambda + \delta_\kappa + 1)\{\delta_\lambda(1-\delta_\kappa) + \mu_\lambda(\delta_\lambda - \delta_\kappa)\} \\ &\quad + \tfrac{1}{4}\{\delta_\lambda^2(1-\delta_\kappa) + \mu_\lambda(\delta_\lambda - \delta_\kappa)\} = 0 \quad (\lambda = 1, 2;\ \kappa = 3 - \lambda), \end{aligned} \tag{35}$$

where

$$\mu_\lambda = |(1-\delta_\kappa)/\delta_\lambda^2(\delta_\lambda - \delta_\kappa)| > 0 \quad (\lambda = 1, 2;\ \kappa = 3 - \lambda). \tag{36}$$

Equations (35) may be written in the simpler form

$$5\delta_\lambda^2\delta_\kappa - 4\delta_\lambda\delta_\kappa^2 - \delta_\lambda^2 - 4\delta_\kappa^2 - \delta_\lambda + 5\delta_\kappa - \epsilon_\lambda(\delta_\lambda - \delta_\kappa)\,(6\delta_\lambda\delta_\kappa - 2\delta_\lambda - 2\delta_\kappa + 1) = 0$$
$$(\lambda = 1, 2;\ \kappa = 3 - \lambda),$$

where $\epsilon_\lambda = 1 - \mu_\lambda$ $(\lambda = 1, 2)$. By adding and by subtracting together the first and the second of them we get

$$\begin{aligned} &(\delta_1 + \delta_2 + 10)\,\delta_1\delta_2 - 5(\delta_1 + \delta_2)^2 + 4(\delta_1 + \delta_2) \\ &\qquad - (\epsilon_1 - \epsilon_2)\,(\delta_1 - \delta_2)\,(6\delta_1\delta_2 - 2\delta_1 - 2\delta_2 + 1) = 0 \end{aligned} \tag{37}$$

and

$$(\delta_1 - \delta_2)\{3[3 - 2(\epsilon_1 + \epsilon_2)]\,\delta_1\delta_2 + [3 + 2(\epsilon_1 + \epsilon_2)]\,(\delta_1 + \delta_2) - (6 + \epsilon_1 + \epsilon_2)\} = 0, \tag{38}$$

respectively

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Suppose first that $\Im\delta_1 \neq 0$ or $\Im\delta_2 \neq 0$. If $\epsilon_1+\epsilon_2 = \frac{3}{2}$, then, as $\delta_1 = \delta_2$ contradicts $w_1 \neq w_2$, equation (38) yields $\delta_1+\delta_2 = \frac{5}{4}$, and from (37) we obtain $\delta_1\delta_2 = \frac{1}{4}$ or $\epsilon_1-\epsilon_2 = \frac{15}{8}(\delta_1-\delta_2)^{-1}$. In both cases δ_1 and δ_2 are real, which contradicts our assumption. Thus $\delta_1 \neq \delta_2$ and $\epsilon_1+\epsilon_2 \neq \frac{3}{2}$. Consequently, equation (38) implies

$$\delta_1\delta_2 = -\tfrac{1}{3}(1+16\alpha)(\delta_1+\delta_2)+\tfrac{2}{3}(1+10\alpha), \tag{39}$$

where

$$\alpha = \tfrac{1}{12}(\epsilon_1+\epsilon_2)/\{1-\tfrac{2}{3}(\epsilon_1+\epsilon_2)\}. \tag{40}$$

Substituting the expression obtained for $\delta_1\delta_2$ in (37) we get

$$(\delta_1+\delta_2-\tfrac{5}{4})\{(1+\alpha)(\delta_1+\delta_2)-\beta(\delta_1-\delta_2)+1+10\alpha\} = 0, \tag{41}$$

where

$$\beta = \tfrac{3}{4}(1+8\alpha)(\epsilon_1-\epsilon_2). \tag{42}$$

If $\delta_1+\delta_2 = \frac{5}{4}$, we obtain again that δ_1 and δ_2 should be real. Thus from (41) we have

$$(1+\alpha-\beta)\delta_1+(1+\alpha+\beta)\delta_2+1+10\alpha = 0. \tag{43}$$

Since $\Im\delta_1 \neq 0$ or $\Im\delta_2 \neq 0$, then, according to (43) and (39), $\beta = 1+\alpha$ implies $\beta = -1-\alpha$ and vice versa. As $\beta = 1+\alpha = -1-\alpha$ implies $\alpha = -1$ against (43), the system of equations (39) and (43) has the solution

$$\delta_\lambda = -\tfrac{1}{2}\{1+10\alpha+\tfrac{2}{3}(-1)^\lambda(1+16\alpha)\beta+\tfrac{1}{3}(-1)^\lambda\gamma\sqrt{\Delta}\}/\{1+\alpha+(-1)^\lambda\beta\} \quad (\lambda = 1,2), \tag{44}$$

where

$$\Delta = -27(1+4\alpha)\{(1+2\alpha)(1+10\alpha)-\tfrac{4}{27}(7+64\alpha)\beta^2\} < 0 \tag{45}$$

and $\gamma = 1$ or -1.

From (45) and (42) we infer at once that either $-\frac{1}{2} < \alpha < -\frac{7}{64}$ and $\alpha \neq -\frac{1}{8}, \alpha \neq -\frac{1}{4}$, or $\alpha > -\frac{1}{10}$. On the other hand, according to (36) we have $\epsilon_1 = 1-\mu_1 < 1$ and $\epsilon_2 = 1-\mu_2 < 1$. Since $\alpha \neq -\frac{1}{8}$, then (40) and (42) imply

$$\epsilon_1+\epsilon_2 = 12\alpha/(1+8\alpha) \quad \text{and} \quad \epsilon_1-\epsilon_2 = \tfrac{4}{3}\beta/(1+8\alpha), \tag{46}$$

and, consequently,

$$6\alpha(1+8\alpha)^{-1}-\tfrac{2}{3}(-1)^\lambda\beta(1+8\alpha)^{-1} = \epsilon_\lambda < 1 \quad (\lambda = 1,2).$$

Hence either $\alpha < -\frac{1}{2}$ and $|\beta| < -\frac{3}{2}(1+2\alpha)$, or $\alpha > -\frac{1}{8}$ and $|\beta| < \frac{3}{2}(1+2\alpha)$. Comparing this result with the previous one, derived from (45) and (42), we see that either $-\frac{1}{8} < \alpha < -\frac{7}{64}$ or $\alpha > -\frac{1}{10}$.

The first case is impossible. In fact, comparing in this case the inequality $\beta^2 < \frac{4}{9}(1+2\alpha)^2$ with (45) we have $1+2\alpha > 3(1+10\alpha)/(7+64\alpha)$, whence $-\frac{1}{4} < \alpha < -\frac{1}{8}$, which contradicts $-\frac{1}{8} < \alpha < -\frac{7}{64}$. Since, in view of (42), we may assume $\beta \geqslant 0$ without loss of generality, then we have finally

$$\alpha > -\tfrac{1}{10} \quad \text{and} \quad 0 \leqslant \beta < \tfrac{3}{2}(1+2\alpha). \tag{47}$$

Next, let us notice that if $\alpha > -\frac{1}{10}$, the inequality (45) is stronger than $\beta^2 < \frac{9}{4}(1+2\alpha)^2$. As the last inequality is not strong enough for our further purpose, and (45) is too complicated to be convenient, we calculate $|\delta_1-\delta_2|^2$ which must be positive since $w_1 \neq w_2$. The result is:

$$|\delta_1-\delta_2|^2 = 3(1+2\alpha)(1+4\alpha)(1+10\alpha)/(1+\alpha-\beta)(1+\alpha+\beta). \tag{48}$$

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Hence the inequalities (47) may be replaced by

$$\alpha > -\tfrac{1}{10} \quad \text{and} \quad 0 \leqslant \beta < 1+\alpha. \tag{49}$$

Now we turn to the problem of solving the system of equations (36), where δ_1 and δ_2 are given by (44). We have

$$|\delta_\lambda|^2 = (1+10\alpha)\{(1+\alpha)(1+6\alpha)+\tfrac{1}{3}(-1)^\lambda(1+16\alpha)\beta-\tfrac{2}{3}\beta^2\}\{1+\alpha+(-1)^\lambda\beta\}^{-2} \quad (\lambda = 1,2) \tag{50}$$

and

$$|1-\delta_\lambda|^2 = 3(1+4\alpha)\{(1+\alpha)(1+5\alpha)+\tfrac{4}{3}(-1)^\lambda(1+4\alpha)\beta+\tfrac{1}{3}\beta^2\} \times\{1+\alpha+(-1)^\lambda\beta\}^{-2} \quad (\lambda = 1,2). \tag{51}$$

Moreover, the relations (46) imply

$$\epsilon_\lambda = 1-\{1+2\alpha+\tfrac{2}{3}(-1)^\lambda\beta\}/(1+8\alpha) \quad (\lambda = 1,2). \tag{52}$$

Applying the relations (48), (50), (51), (52) and $\mu_\lambda = 1-\epsilon_\lambda$ $(\lambda = 1,2)$ for the equations (36) we get

$$\frac{\{1+2\alpha+\tfrac{2}{3}(-1)^\lambda\beta\}^2}{(1+8\alpha)^2} = \frac{\{1+\alpha+(-1)^\lambda\beta\}^5\{(1+\alpha)(1+5\alpha)-\tfrac{4}{3}(-1)^\lambda(1+4\alpha)\beta+\tfrac{1}{3}\beta^2\}}{(1+2\alpha)(1+10\alpha)^3\{1+\alpha-(-1)^\lambda\beta\}\{(1+\alpha)(1+6\alpha)+\tfrac{1}{3}(-1)^\lambda(1+16\alpha)\beta-\tfrac{2}{3}\beta^2\}^2} \quad (\lambda = 1,2). \tag{53}$$

Dividing the first of the equations obtained by the second one we have

$$\left(\frac{1+\alpha+\beta}{1+\alpha-\beta}\right)^6\left(\frac{1+2\alpha-\tfrac{2}{3}\beta}{1+2\alpha+\tfrac{2}{3}\beta}\right)^2\left(\frac{(1+\alpha)(1+6\alpha)-\tfrac{1}{3}(1+16\alpha)\beta-\tfrac{2}{3}\beta^2}{(1+\alpha)(1+6\alpha)+\tfrac{1}{3}(1+16\alpha)\beta-\tfrac{2}{3}\beta^2}\right)^2 \times\frac{(1+\alpha)(1+5\alpha)-\tfrac{4}{3}(1+4\alpha)\beta+\tfrac{1}{3}\beta^2}{(1+\alpha)(1+5\alpha)+\tfrac{4}{3}(1+4\alpha)\beta+\tfrac{1}{3}\beta^2} = 1. \tag{54}$$

This is justified because, in view of the previous considerations, especially the inequalities (49), $1+2\alpha \neq 0$, $1+4\alpha \neq 0$, $1+10\alpha \neq 0$, and all factors appearing in the numerator and in the denominator of (54) also are different from zero.

Now, in view of (49), it seems convenient to consider separately two possibilities: $\beta > 0$ and $\beta = 0$. Because we have considered a lot of cases, we recall, in view of possible ambiguities, that we still suppose $l = 3$, and $\Im\delta_1$ or $\Im\delta_2$ to be different from zero.

First, let $\beta > 0$. Since, by (49), we have $\alpha > -\frac{1}{4}$ and $\beta < 1+\alpha$, then

$$\frac{1+\alpha+\beta}{1+\alpha-\beta}\,\frac{1+2\alpha-\tfrac{2}{3}\beta}{1+2\alpha+\tfrac{2}{3}\beta} > 1. \tag{55}$$

Similarly, applying $\beta > 0$ and (49) we verify that

$$\frac{1+\alpha+\beta}{1+\alpha-\beta}\,\frac{(1+\alpha)(1+6\alpha)-\tfrac{1}{3}(1+16\alpha)\beta-\tfrac{2}{3}\beta^2}{(1+\alpha)(1+6\alpha)+\tfrac{1}{3}(1+16\alpha)\beta-\tfrac{2}{3}\beta^2} > 1 \tag{56}$$

and

$$\left(\frac{1+\alpha+\beta}{1+\alpha-\beta}\right)^2\frac{(1+\alpha)(1+5\alpha)-\tfrac{4}{3}(1+4\alpha)\beta+\tfrac{1}{3}\beta^2}{(1+\alpha)(1+5\alpha)+\tfrac{4}{3}(1+4\alpha)\beta+\tfrac{1}{3}\beta^2} > 1. \tag{57}$$

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Inequalities (55), (56) and (57) imply that the left-hand side of (54) is greater than 1, and then we have a contradiction. Consequently, $\beta = 0$.

Since $\beta = 0$, then each of the equations (53) takes the form

$$(1+2\alpha)^3(1+6\alpha)^2\ (1+10\alpha)^3-(1+\alpha)^3(1+5\alpha)(1+8\alpha)^2 = 0. \tag{58}$$

Clearly, the left-hand side of (58) is positive for $\alpha > 0$, and negative for $-\frac{1}{10} < \alpha < 0$. Hence, in view of (47), we have $\alpha = 0$. Consequently, formulae (44) imply

$$\delta_1 = -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{3}\gamma i \quad \text{and} \quad \delta_2 = -\tfrac{1}{2}-\tfrac{1}{2}\sqrt{3}\gamma i \quad (\gamma = 1 \text{ or } -1). \tag{59}$$

Applying now $w_1 = \delta_1 w_3$, $w_2 = \delta_2 w_3$, and formulae (20) with $m = 4$, $\sigma_3 = 1$ for $\lambda = 3$, we get

$$C_4^* = -\frac{1}{4w_1w_2w_3} = -\frac{1}{3^3\,4^4\delta_1\delta_2}\left(6-\frac{2}{\delta_1}-\frac{2}{\delta_2}+\frac{1}{\delta_1\delta_2}\right)^3 = -\frac{3^3}{4^4}. \tag{60}$$

On the other hand, since $\alpha = 0$ and $\beta = 0$, then, according to (46), we have $\epsilon_1 = 0$ and $\epsilon_2 = 0$. Hence $\mu_1 = 1-\epsilon_1 = 1$ and $\mu_2 = 1-\epsilon_2 = 1$.

The above system of values (59) and $\mu_1 = 1$, $\mu_2 = 1$ satisfies the system of equations (35) and (36).

10. Suppose now that δ_1 and δ_2 are both real. Since equations (36) are a consequence of (19), we conclude that σ_1 and σ_2 must also be real. As $|\sigma_1| = |\sigma_2| = 1$ and $\sigma_3 = 1$, there exist at least two indices λ and κ such that $\sigma_\lambda = \sigma_\kappa$. Since $\sigma_\lambda = P_4^*(w_\lambda)$, $\sigma_\kappa = P_4^*(w_\kappa)$ and $P_4^{*\prime}(w_\lambda) = P_4^{*\prime}(w_\kappa) = 0$, then $P_4^*(w)-\sigma_\lambda = C_4^*(w-w_\lambda)^2(w-w_\kappa)^2$. Differentiating the relation obtained we see that $w_\lambda+w_\kappa = 2w_\nu$, where w_ν denotes the third zero of $P_4^{*\prime}$. Thus we have three possibilities: $w_1+w_2 = 2w_3$, $w_2+w_3 = 2w_1$ and $w_3+w_1 = 2w_2$. Clearly, we may confine ourselves to the first and second possibilities without loss of generality. Morover, it can be easily seen, that the second possibility is equivalent to the first one if the polynomial $\hat{P}_4^*(w) = -P_4^*(-w)$ instead of P_4^* is considered, and if we set $\hat{w}_1 = -w_2$, $\hat{w}_2 = -w_3$ and $\hat{w}_3 = -w_1$ (cf. Remark B).

Suppose then, that δ_1 and δ_2 are real, and $\delta_1+\delta_2 = 2$. Since $w_1 \neq w_2$ we have $\delta_1 \neq \delta_2$. Moreover, if $\epsilon_1+\epsilon_2 = \frac{3}{2}$, then (38) implies $\delta_1+\delta_2 = \frac{5}{4}$ which contradicts $\delta_1+\delta_2 = 2$. Hence $\epsilon_1+\epsilon_2 \neq \frac{3}{2}$, and we derive the equation (43) with α and β defined by (40) and (42), respectively, as in the case where $\Im\delta_1 \neq 0$ or $\Im\delta_2 \neq 0$. Consequently, as $\beta = 0$ implies either $\alpha = -\frac{1}{8}$ and $\alpha = -\frac{1}{4}$ simultaneously, or $\epsilon_1 = \epsilon_2 = \frac{3}{2} > 1$, we get $\beta \neq 0$, and

$$\delta_\lambda = -(-1)^\lambda(3/2\beta)\{1+4\alpha-\tfrac{2}{3}(-1)^\lambda\beta\} \quad (\lambda = 1, 2). \tag{61}$$

Substituting the expression obtained for δ_1 and δ_2 in (36), and applying the relations $\mu_\lambda = 1-\epsilon_\lambda$ $(\lambda = 1, 2)$, we get

$$1-\epsilon_\lambda = \tfrac{2}{9}\beta^2\gamma_\lambda/\{1+4\alpha-\tfrac{2}{3}(-1)^\lambda\beta\}^2 \quad (\lambda = 1, 2), \tag{62}$$

where $\gamma_1 = 1$ or -1, and $\gamma_2 = 1$ or -1.

Now, applying again the relations (19) with $m = 4$, we see that $\gamma_1 = -\sigma_1$ and $\gamma_2 = -\sigma_2$. Since $\sigma_1 = \sigma_2$, then we have

$$\gamma_1 = \gamma_2 = \gamma \text{ (say)}. \tag{63}$$

If $\alpha = -\frac{1}{8}$ we obtain, in view of (42), a contradiction with $\beta \neq 0$. Hence we may apply formulae (52). From (62), (63) and (52) we infer that

$$\{1+2\alpha+\tfrac{2}{3}(-1)^\lambda\beta\}\{1+4\alpha-\tfrac{2}{3}(-1)^\lambda\beta\}^2 = \tfrac{2}{9}\gamma(1+8\alpha)\beta^2 \quad (\lambda = 1, 2).$$

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By adding and by subtracting together the first and the second of the equations obtained, we get

$$2\{(1+2\alpha)(1+4\alpha)^2-\tfrac{4}{9}[1+6\alpha+\tfrac{1}{2}\gamma(1+8\alpha)]\beta^2\}=0 \tag{64}$$

and

$$\tfrac{4}{3}\beta(1+4\alpha-\tfrac{4}{9}\beta^2)=0. \tag{65}$$

Since, as shown before, $\beta \neq 0$, then (65) implies $\beta^2=\frac{9}{4}(1+4\alpha)$, and from (64) we obtain $16\alpha^2-8\gamma\alpha-\gamma=0$. If $\gamma=-1$, then $\alpha=-\frac{1}{4}$, whence we get a contradiction with $\beta \neq 0$. Thus $\gamma=1$ and, consequently, $\alpha=\frac{1}{4}(1+\gamma'\sqrt{2})$ and $\beta=\frac{3}{2}\gamma''\sqrt{(2+\gamma'\sqrt{2})}$, where $\gamma'=1$ or -1, and $\gamma''=1$ or -1. In view of (42) we may assume $\beta \geqslant 0$ without loss of generality, and then we set $\gamma''=1$. Since formulae (61) imply $\delta_1=1+\sqrt{(2+\gamma'\sqrt{2})}$ and $\delta_2=1-\sqrt{(2+\gamma'\sqrt{2})}$, we have finally (cf. formula (60))

$$C_4^*=(3^3/4^4)(1+\tfrac{2}{3}\gamma'\sqrt{2})^3/(1+\gamma'\sqrt{2})^4, \quad \text{and} \quad |C_4^*|<M_{44}.$$

Hence we have proved that $|C_4| \leqslant |C_4^*|$ in $\mathscr{P}_4$ where C_4^* is given by (60). Using the equations $w_1=\delta_1 w_3$, $w_2=\delta_2 w_3$, $w_3=12/(6-2\delta_1^{-1}-2\delta_2^{-1}+\delta_1^{-1}\delta_2^{-1})$, the relations (59), and Remark B, we see that equality is attained in $|C_4| \leqslant |C_4^*|=3^3/4^4$ if and only if $P_4(w)=w-(3^3/4^4)e^{i\theta}w^4$ (θ real). The proof of Theorem 6 is completed.

Added in proof

1. From the proof of our lemma it follows that if P_n^* is a polynomial of the form (8), w_λ ($\lambda=1, \ldots, n-1$), denote all zeros of the derived polynomial, not necessarily distinct, ϕ_k is defined by (10), and $E(z_2, \ldots, z_k)=\log z_k$, then

$$\phi_k(w_\lambda)=-k\zeta_{n-k}(w_1, \ldots, w_{\lambda-1}, w_{\lambda+1}, \ldots, w_{n-1})/\tau_{n-k}(w_1, \ldots, w_{n-1}) \quad (\lambda=1, \ldots, n-1).$$

We do not assume here that P_n^* fulfils the conditions of Theorem C.

2. The case considered in section 10 can also be excluded in a simpler way by Theorem 1 of (1) because the function A described there can be continuously extended onto the closed unit disc.

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