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On invariant knots

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It was first proved by R. Lashof in [4], using the work of S. Cappell and J. Shaneson on four-dimensional surgery (see [1]), that there exist locally flat topological knots $S^3 \subset S^5$ which are not smoothable. In [2] (compare also [6]) S. Cappell and J. Shaneson have constructed infinitely many non-smoothable locally flat topological knots as the fixed points of locally nice (= locally smoothable) Z_p actions on S^5, therefore giving non-trivial examples of locally smoothable but equivariantly non-smoothable actions of Z_p on S^5.

They also suggested that the methods of [2] can be used to construct non-smoothable locally flat topological knots invariant for certain smooth free actions on S^5.

In this note (which can be thought of as a complement to [2]) we show how to construct such knots. In fact we show that (up to twisting) the knots constructed in [2] are invariant with respect to some smooth free actions of Z_p on S^5 for all odd p.

To make this note a little bit more self-contained we recall briefly the construction from [2]. The crucial role in this construction is played by the following:

PROPOSITION (see [2]). *Let* $pZ \subset Z$, $p \neq 0$, $\gamma \in L_5(Z)$, $X = S^3 \times S^1$. *Then there is a topological normal map* (f, b):

$$f\colon (W; \partial_0 W, \partial_1 W) \to (X \times I; X \times \{0\}, X \times \{1\}),$$

with

(*a*) $f|_{\partial_i W}\colon \partial_i W \to X \times \{i\}$, $i = 0, 1$ *is a homeomorphism,*

(*b*) $\Theta(f, b) = \gamma$,

(*c*) *f induces an isomorphism of fundamental groups,*

(*d*) *if* $\hat{}$ *denotes the cover associated to* $pZ \subset Z$ *then* $\hat{f}\colon \hat{W} \to (X \times I)^{\wedge}$ *induces an isomorphism of integral homology groups.*

Now let γ be an odd multiple of a generator of $L_5(Z) \approx Z$. We form (using the Proposition) the manifold Q as $Q = W \cup_{S^3 \times S^1} D^4 \times S^1$. Then $\partial \hat{Q} = S^3 \times S^1$ and $\hat{Q} \cup_{S^3 \times S^1} S^3 \times D^2$ is homotopy equivalent to S^5 (using (*c*) and (*d*) of the Proposition) hence homeomorphic to S^5.

Clearly Z_p acts on $S^5 = \hat{Q} \cup_{S^3 \times S^1} S^3 \times D^2$, just by extending the covering translation of $\hat{Q}$, with fixed point set $S^3 \times \{0\} \subset S^3 \times D^2 \subset S^5 = \hat{Q} \cup_{S^3 \times S^1} S^3 \times D^2$. It turns out (see [2]) that this embedding of S^3 in S^5 gives the non-smoothable locally flat topological knot.

Now we prove the following:

THEOREM. *Let* $p > 1$ *be an odd number. There exist infinitely many smooth free* Z_p *actions on* S^5 *which preserve a non-smoothable locally flat topological knot* $S^3 \subset S^5$.

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Proof. Let us consider the manifold $Q = W \cup_{S^3\times S^1} D^4 \times S^1$ as described above. Then $\hat{Q}$ is a free Z_p-space and $\partial\hat{Q} = S^3 \times S^1$. Now we form the space V as

$$D^2 V = \hat{Q} \underset{\sim}{\sqcup}_{S^3\times S^1} S^3 \times$$

where the identification of $\partial\hat{Q} = S^3 \times S^1$ with $\partial(S^3 \times D^2)$ is given by twisting using the standard action of S^1 on S^3. Namely, if $S^3 = \{(z_1, z_2) \in C^2 | \, |z_1|^2 + |z_2|^2 = 1\}$ and $S^1 = \{z \in C | \, |z|^2 = 1\}$ then the twisting is given by

$$\partial\hat{Q} = S^3 \times S^1 \ni ((z_1, z_2), z) \overset{T}{\to} ((z_1 . z, z_2 . z), z) \in S^3 \times S^1 = \partial(S^3 \times D^2).$$

Observe that $$T_*: \Pi_1(S^3 \times S^1) \to \Pi_1(S^3 \times S^1)$$

is the identity homomorphism, analogously

$$T_*: H_*(S^3 \times S^1; Z) \to H_*(S^3 \times S^1; Z)$$

is the identity map. This observation has the following two consequences. First, the manifold V has the homotopy type of S^5 (as follows from (c) and (d) of the Proposition) and consequently it is homeomorphic to S^5. Second, the manifold V is a free Z_p-manifold with

$$S^3 \times \{0\} \subset \hat{Q} \underset{S_3\times S^1}{\sqcup} S^3 \times D^2 = V = S^5$$

invariant.

Therefore we have a locally flat topological knot $S^3 \subset S^5$ invariant with respect to this free topological action of Z_p on S^5. In fact this action is smooth. To see it, let us consider the quotient space $L = S^5/Z_p$. From the Universal Coefficient Theorem we infer $H^4(L; Z_2) \approx \mathrm{Ext}(Z_p, Z_2) = 0$ as p is an odd number. Therefore the Kirby–Siebenmann obstruction $k \in H^4(L; Z_2)$ is trivial, so (see [**3**]) the manifold L has a PL structure and in fact a smooth structure because $\Pi_n(PL/O) = 0$, $n \leqslant 6$ (see [**3**]). This implies the action of Z_p on S^5 is smooth.

Now we show that the knot $S^3 \subset S^5$ constructed above is non-smoothable. Let us, as before, take γ to be an odd multiple of the generator of $L_5(Z) \approx Z$. Consider the knot $S^3 \subset S^5$ constructed in [**2**]. Using topological transversality (see [**5**]) we can construct a topological Seifert surface S_1 for this knot. The signature $\sigma(S_1)$ of this surface is detected by γ. Namely,

$$\sigma(S_1) = \mathrm{tr}\,(\gamma) \in 8Z$$

where $\mathrm{tr}: L_5(Z) \to L_5(Z)$ is the transfer homomorphism and the identification $L_5(Z) \approx L_4(e) \approx 8Z$ is made.

This implies the non-smoothability of this knot, as smoothability would contradict Rohlin's Theorem.

The boundary $\partial(S_1)$ of S_1 is equal to $S^3 \times \{a\} \subset S^3 \times S^1 = \partial Q$ for some $a \in S^1$. Now we form $S_2 = S_1 \cup S^3 \times \widehat{(a, 1)}$, where $\widehat{(a, 1)} \subset S^1$ is the arc. As $T|_{S^3\times\{1\}} = \mathrm{id}$, then we conclude that S_2 is the topological Seifert surface for our knot $S^3 \subset S^5$, so this knot is non-smoothable.

Remark 1. As in [**2**], different choices of γ give knots which are not even cobordant.

Remark 2. The non-smoothability of the knots constructed above can be proved without appealing to 4-dimensional topological transversality; namely, one can use a suspended Seifert surface as in [**2**].

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As a consequence of the naturality of Kirby–Siebenmann obstruction with respect to odd-fold coverings we obtain the following:

COROLLARY. *There exist infinitely many non-smoothable locally flat topological embeddings of a standard 3-dimensional lens space into 5-dimensional smooth fake lens space which preserve the preferred generators of fundamental groups.*

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