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Effective mixing and counting in Bruhat–Tits trees

SANGHOON KWON

Department of Mathematical Science, Seoul National University, 1 Gwanak-ro, Gwanak-gu, 08826 Seoul, Korea
(e-mail: sanghoon.kwon@snu.ac.kr)

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Abstract. Let $\mathcal{T}$ be a locally finite tree, Γ be a discrete subgroup of $\mathrm{Aut}(\mathcal{T})$ and $\widetilde{F}$ be a Γ-invariant potential. Suppose that the length spectrum of Γ is not arithmetic. In this case, we prove the exponential mixing property of the geodesic translation map $\phi\colon \Gamma\backslash S\mathcal{T} \to \Gamma\backslash S\mathcal{T}$ with respect to the measure $m_{\Gamma,\widetilde{F}}^{\nu^-,\nu^+}$ under the assumption that Γ is *full* and $(\Gamma, \widetilde{F})$ has a *weighted spectral gap*. We also obtain the effective formula for the number of Γ-orbits with weights in a Bruhat–Tits tree $\mathcal{T}$ of an algebraic group.

1. *Introduction*

Let X be a geodesic metric space, let $G < \mathrm{Aut}(X)$ be a closed subgroup of the isometry group X and let $\{B_R : R > 1\}$ be a family of compact subsets of X whose volume goes to infinity as $R \to \infty$. We are interested in the following fundamental problem: given a discrete subgroup Γ of G, what are the asymptotics and error rates of $|x\Gamma \cap B_R|$ for a point $x \in X$?

If Γ is a lattice subgroup of G, or, equivalently, if $\Gamma\backslash G$ admits a finite G-invariant measure, then this problem is well understood for various G. In this case, $|x\Gamma \cap B_R|$ is asymptotically proportional to the volume of B_R with the suitable error term assuming that the boundaries of B_R are sufficiently regular [**8**–**10**]. However, if Γ is not a lattice subgroup, then this type of counting problem is little understood, in general.

When G is the identity component of $SO(n, 1)$, $(n \geq 2)$ and $H < G$ is either a horospherical subgroup or a symmetric subgroup, Oh and Shah computed an asymptotic formula of $|x\Gamma \cap B_R|$ for a family of compact subsets $\{B_R \subset G/H\}$, provided that the Bowen–Margulis measure on $T^1(\Gamma\backslash\mathbb{H}^n)$ and the Γ-skinning size of x are finite [**13**]. These conditions are satisfied when $\Gamma < G$ is a *geometrically finite* subgroup, that is, when the unit neighborhood of the convex core $\mathcal{C}_\Gamma$ has finite volume. In addition, Mohammadi and Oh computed an asymptotic formula with an error term via the effective asymptotic expansion of matrix coefficients of the complementary series [**11**].

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Our main focus in this paper is the case when G is the $\mathbb{F}$-points of a semi-simple $\mathbb{F}$-rank-one algebraic group over a *non-Archimedean* local field $\mathbb{F}$. In this case, we can view G as a closed subgroup of the automorphism group of a biregular tree $\mathcal{T}$ and hence G acts on the space of geodesics $S\mathcal{T}$ in $\mathcal{T}$. We are interested in the formula which counts the number of points of a Γ-orbit in $\mathcal{T}$ when Γ is a discrete subgroup of G for which certain measure $m_{\Gamma,F}^{\nu^-,\nu^+}$ on the space of geodesics $\Gamma\backslash S\mathcal{T}$ is finite.

Let $\phi\colon \Gamma\backslash S\mathcal{T} \to \Gamma\backslash S\mathcal{T}$ be the geodesic translation map and $m_{\Gamma,F}^{\nu^-,\nu^+}$ be the measure on $\Gamma\backslash S\mathcal{T}$ associated to the potentials $\widetilde{F}^{\pm}\colon E\mathcal{T} \to \mathbb{R}$ and the normalized Patterson densities $\nu^{\pm}$. We will consider a Markov chain Z_n with the countable state space $\mathcal{S}$, the transition probabilities p_{ij} and the stationary distribution π_j induced from the dynamical system $(\Gamma\backslash S\mathcal{T}, \phi, m_{\Gamma,F}^{\nu^-,\nu^+})$ (see §3). To count the number of points of discrete orbits with weights in $\mathcal{T}$ effectively, the main ingredient we show is the exponential mixing property of geodesic 2-translation map $\phi^{\circ 2}$ with respect to the measure $m_{\Gamma,F}^{\nu^-,\nu^+}$.

When the *length spectrum* L_Γ of the group Γ (see §2) is equal to $k\mathbb{Z}$, the geodesic k-translation map $\phi^{\circ k}$ is mixing on $\Gamma\backslash S_o^k\mathcal{T}$ [**4**, **17**]. If we fix a reference state $s_0 \in \mathcal{S}$, then we have a partition $(A_0, \ldots, A_{k-1})$ of $\mathcal{S}$ given by $A_m = \{s \in \mathcal{S}\colon p_{s_0 s}^{(kn+m)} > 0 \text{ for some } n \in \mathbb{N}\}$. The exponential mixing property of $\phi^{\circ k}$ follows directly if the irreducible aperiodic k-step Markov chain Z_{kn} on each subset A_m $(m = 0, \ldots, k-1)$ converges exponentially to the stationary distribution. To require such a good convergence property, we introduce the following notion, called WSG (*weighted spectral gap*), for a pair of a full discrete group Γ and a potential $\widetilde{F}$.

Definition 1.1. For a given discrete subgroup $\Gamma < \mathrm{Aut}(\mathcal{T})$, let $\pi : \mathcal{T} \to \Gamma\backslash\mathcal{T}$ be the natural projection. We call $\Gamma_f = \{g \in \mathrm{Aut}(\mathcal{T}) \mid \pi \circ g = \pi\}$ the *associated full subgroup* of Γ. A discrete subgroup $\Gamma < \mathrm{Aut}(\mathcal{T})$ is called *full* if $\Gamma = \Gamma_f$.

Definition 1.2. Let $\Gamma < \mathrm{Aut}(\mathcal{T})$ be a full discrete subgroup and $\widetilde{F}\colon E\mathcal{T} \to \mathbb{R}$ be a Γ-invariant real-valued function. Let Z_n be the Markov chain with the data $(\mathcal{S}, p_{ij}, \pi_j)$ associated with $(\Gamma\backslash S\mathcal{T}, \phi, m_{\Gamma,F}^{\nu^-,\nu^+})$ (see §3). Suppose that there is a function $t\colon \mathcal{S} \to \mathbb{R}_{\geq 0}$ given by $t(s_i) = t_i$, a finite subset $B \subset \mathcal{S}$ and a constant $0 < \rho < 1$ such that, for every $s_i \in \mathcal{S} - B$,

$$\sum_{s_j} p_{ij} t_j t_i^{-1} \leq \rho. \tag{1}$$

Then we say that $(\Gamma, \widetilde{F})$ has a WSG with (t, B, ρ). If $\widetilde{F}$ is a constant, then we say that Γ has a WSG with (t, B, ρ).

Now we can state our main theorem.

THEOREM 1.1. *Let $\mathcal{T}$ be a locally finite uniform tree and let Γ be a non-elementary full discrete subgroup of* $\mathrm{Aut}(\mathcal{T})$. *Let $\widetilde{F}$ be a potential for Γ such that $(\Gamma, \widetilde{F})$ has a WSG. If $|m_{\Gamma,F}^{\nu^-,\nu^+}| < \infty$ and $L_\Gamma = k\mathbb{Z}$, then, for any $f, g \in C_c(\Gamma\backslash S_o^k\mathcal{T})$, as $n \to \infty$,*

$$\left| \int_{\Gamma\backslash S_o^k\mathcal{T}} (f \circ \phi^{\circ kn}) \cdot g \, dm_{\Gamma,F}^{\nu^-,\nu^+} - \int_{\Gamma\backslash S_o^k\mathcal{T}} f \, dm_{\Gamma,F}^{\nu^-,\nu^+} \int_{\Gamma\backslash S_o^k\mathcal{T}} g \, dm_{\Gamma,F}^{\nu^-,\nu^+} \right| = O(\theta^n),$$

for some constant $0 < \theta < 1$. For given $(\Gamma, \widetilde{F})$, the implied constant depends only on f and g.

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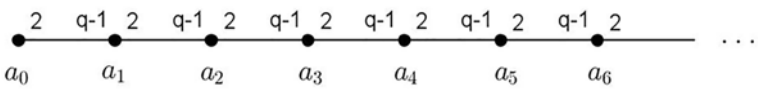


FIGURE 1. Ray of type $(2, q-1)$.

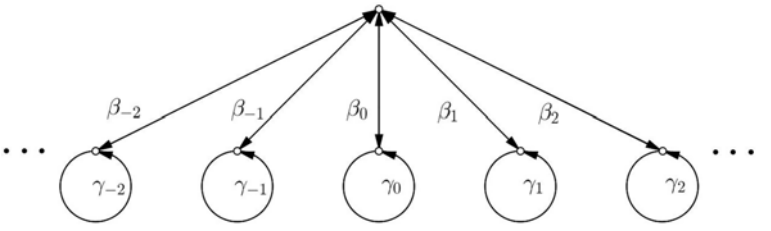


FIGURE 2. Example of a Markov chain which does not satisfy the equation (1).

We remark here that if $(\Gamma, \widetilde{F})$ has a WSG with (t, B, ρ), then $p_{ij}^{(n),B} \leq t_i t_j^{-1} \rho^n$ for each $s_i, s_j \in \mathcal{S}$ and this inequality is the key ingredient in the proof of Theorem 1.1.

Since every geometrically finite discrete subgroup $\Gamma < \mathrm{Aut}(\mathcal{T})$ of a biregular tree $\mathcal{T}_{r+1,s+1}$ has a WSG (see §4), the above theorem implies that the geodesic k-translation map $\phi^{\circ k}\colon \Gamma\backslash S_o^k\mathcal{T} \to \Gamma\backslash S_o^k\mathcal{T}$ is exponentially mixing for every geometrically finite group Γ. We also give an example of a group Γ which is not geometrically finite but which has a WSG in Example 1.1 (see §4 for more examples and the detail). Although there exists a Markov chain $(\mathcal{S}, p_{ij}, \pi_j)$ which does not satisfy equation (1) (see Example 1.2), we leave as a question whether there exists a $\Gamma < \mathrm{Aut}(\mathcal{T})$ such that $|m_\Gamma^{\mathrm{BM}}| < \infty$ without a WSG.

Example 1.1. Let $q \geq 5$ be an arbitrary odd integer. Consider an edge-indexed ray of type $(2, q-1)$ (see Figure 1). If $\Gamma\backslash\backslash\mathcal{T}$ is a union of a finite graph of groups, finite such rays and finite funnels, then Γ has a WSG. Therefore, by Theorem 1.1 the geodesic 2-translation map $\phi^{\circ 2}$ is exponentially mixing with respect to m_Γ^{BM}, which corresponds to the measure $m_{\Gamma,\widetilde{F}}^{\nu^-,\nu^+}$ with the constant potential $\widetilde{F}$.

Example 1.2. This example shows the existence of a positive recurrent Markov chain which does not converge exponentially to the invariant distribution. Let Z_n be the Markov chain on the state space $\mathcal{S} = \mathbb{Z} \cup \{\infty\}$ with the transition probabilities $p_{\infty n} = \beta_n$, $p_{nn} = \gamma_n$ and $p_{n\infty} = 1 - \gamma_n$ (see Figure 2). If $\gamma_n \to 1$ as $|n| \to \infty$, then the equation (1) does not hold for any $t\colon \mathcal{S} \to \mathbb{R}_{\geq 0}$. Nevertheless, if β_n satisfies

$$\sum_{n=-\infty}^{+\infty} \sum_{k=0}^{+\infty} \beta_n \gamma_n^k (1-\gamma_n)(k+2) < \infty,$$

then the invariant distribution $\pi_j = \lim_{k\to\infty} p_{ij}^{(k)}$ exists (see §3.2).

Now we consider an $\mathbb{F}$-rank-one algebraic group $\underline{G}$ and its $\mathbb{F}$-rational points $G = \underline{G}(\mathbb{F})$. We will define a measure $\mathcal{M}_G^{(\Gamma)}$ on G which depends on the group Γ in §5. Due to the result of Roblin [**17**], for a family of compact subset $\{B_R \subset G\}$ which is *wellrounded* (see §5 for the definition), the number of points in the intersection of B_R with a Γ-orbit is

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asymptotically $\mathcal{M}_G^{(\Gamma)}(B_R)$. From Theorem 1.1, we have the following corollary about the error rate of $|\Gamma g \cap B_R|$.

COROLLARY 1.2. *Let $\underline{G}$ be an $\mathbb{F}$-rank-one semi-simple algebraic group over a local field $\mathbb{F}$, $G = \underline{G}(\mathbb{F})$ and let $\mathcal{T}$ be the Bruhat–Tits tree of G. Let $\Gamma < G$ be a discrete subgroup with $|m_\Gamma^{\mathrm{BM}}| < \infty$ and having a WSG. Then, for any well-rounded family $\{B_R : R > 1\}$ of compact subsets of G, there is a constant $\eta > 0$ for which*

$$|\Gamma g \cap B_R| = \mathcal{M}_G^{(\Gamma)}(B_R) + O(\mathcal{M}_G^{(\Gamma)}(B_R)^{1-\eta})$$

as $R \to \infty$.

More generally, we can count the number of Γ-orbits with weights corresponding to the given potential $\widetilde{F} : E\mathcal{T} \to \mathbb{R}$. Let

$$\mathcal{N}_x(n) = \sum_{\gamma :\, d_{\mathcal{T}}(\gamma x, x) \le n} e^{\int_x^{\gamma x} \widetilde{F}}$$

be the weighted counting function of paths of length at most n in $\mathcal{T}$ which start at x and end at γx. Then, using the equidistribution of the skinning measure, as in [**16**], we have the following corollary.

COROLLARY 1.3. *Let x be a degree $q^d + 1$ vertex of the Bruhat–Tits tree $\mathcal{T}$ of G and suppose that $(\Gamma, \widetilde{F})$ is a pair of discrete subgroup $\Gamma < G$ and a potential $\widetilde{F}$ for Γ having a WSG. Then, as $n \to \infty$,*

$$\mathcal{N}_x(2n) = \frac{e^{2\delta_{\Gamma,F}} \|\nu_x^-\| \|\nu_x^+\| |\Gamma_x|}{(e^{2\delta_{\Gamma,F}} - 1)\|m_{\Gamma,F}^{\nu^-,\nu^+}\|} e^{2n\delta_{\Gamma,F}} + O(e^{(2\delta_{\Gamma,F}-\kappa)n})$$

for some $\kappa > 0$.

This paper is organized as follows. In §2, we recall preliminaries including the notion of edge-indexed graphs, graphs of groups, limit points and conformal densities on the space $\Gamma \backslash \mathrm{Aut}(\mathcal{T})$ and the Bruhat–Tits tree of rank-one algebraic groups. In §3, we translate the dynamical system $(\Gamma \backslash S\mathcal{T}, \phi, m_{\Gamma,F}^{\nu^-,\nu^+})$ into a countable Markov chain $(\mathcal{S}, p_{ij}, \pi_j)$ following [**6**]. In §4, we define the notion WSG and prove our main theorem. We also give some examples of discrete groups having a WSG. Finally, in §5, we prove an effective formula of the number of Γ orbits in the Bruhat–Tits tree $\mathcal{T}$ of G.

2. *Notation and preliminaries*

2.1. *Graphs of groups and edge-indexed graphs.* For a graph A, we denote by VA the set of vertices of A and by EA the set of *oriented* edges of A. For $e \in EA$, let $\overline{e} \in EA$ be the opposite edge of e and let $\partial_0 e$ and $\partial_1 e$ be the initial vertex and the terminal vertex of e, respectively.

By a *graph of groups* $\mathbf{A} = (A, \mathcal{A})$ we mean a connected graph A together with groups $\mathcal{A}_a$ $(a \in VA)$, $\mathcal{A}_e = \mathcal{A}_{\overline{e}}$ $(e \in EA)$ and monomorphisms $\alpha_e : \mathcal{A}_e \to \mathcal{A}_{\partial_1 e}$ $(e \in EA)$. An *isomorphism* between two graphs of groups $\mathbf{A} = (A, \mathcal{A})$ and $\mathbf{A}' = (A', \mathcal{A}')$ is an isomomorphism $\phi : A \to A'$ between two underlying graphs together with the set

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of isomomorphisms $\phi_a\colon \mathcal{A}_a \to \mathcal{A}'_{\phi(a)}$ and $\phi_e\colon \mathcal{A}_e \to \mathcal{A}'_{\phi(e)}$ satisfying the following property: for each $e \in EA$, there is an element $h_e \in \mathcal{A}'_{\phi(\partial_1 e))}$ such that

$$\phi_{\partial_1 e}(\alpha_e(g)) = h_e \cdot (\alpha'_{\phi(e)}(\phi_e(g))) \cdot h_e^{-1}$$

for all $g \in \mathcal{A}_e$ [**1**].

For each $e \in EA$, we put $i(e) = [\mathcal{A}_a\colon \alpha_e\mathcal{A}_e]$. We then call $I(\mathbf{A}) = (A, i)$ the *edge-indexed graph* of $\mathbf{A}$. We will assume that all indices $i(e)$ are finite and define $q(e) = i(\overline{e})/i(e) \in \mathbb{Q}^{\times}_{>0}$ for each $e \in EA$. Let $\pi(A)$ be the path group of A. Since $q(\overline{e}) = q(e)^{-1}$, it follows that $q\colon \pi(A) \to \mathbb{Q}^{\times}_{>0}$ defines a homomorphism.

Given a locally finite tree $\mathcal{T}$, let Γ be a closed subgroup of $\mathrm{Aut}(\mathcal{T})$, which is the group of all isometries of $\mathcal{T}$ acting without inversions. Then the quotient graph $\Gamma\backslash\mathcal{T}$ has a natural structure of a graph of groups, which we will denote by $\Gamma\backslash\backslash\mathcal{T}$, as follows. For each $v \in V(\Gamma\backslash\mathcal{T})$ and $e \in E(\Gamma\backslash\mathcal{T})$, choose any corresponding vertex $\widetilde{v} \in V\mathcal{T}$ and edge $\widetilde{e} \in E\mathcal{T}$. Let $\overline{\widetilde{e}} = \widetilde{\overline{e}}$ and fix an element $\gamma_e \in \Gamma$ which satisfies $\gamma_e \widetilde{\partial_1(e)} = \partial_1(\widetilde{e})$. Define $\mathcal{A}_v$ and $\mathcal{A}_e$ to be the stabilizers of $\widetilde{v}$ and $\widetilde{e}$ in Γ, respectively, and let $\alpha_e\colon \mathcal{A}_e \to \mathcal{A}_{\partial_1 e}$ be the monomorphism $h \mapsto \gamma_e^{-1} h \gamma_e$. Then the *quotient graph of groups* $\Gamma\backslash\backslash\mathcal{T} = (\Gamma\backslash\mathcal{T}, \mathcal{A})$ does not depend on the choice of $\widetilde{v}$, $\widetilde{e}$ and γ_e, up to isomorphism of a graph of groups. Let $I(\Gamma\backslash\backslash\mathcal{T}) = (\Gamma\backslash\mathcal{T}, i)$ be the edge-indexed graph of $\Gamma\backslash\backslash\mathcal{T}$. Conversely, if $(A, \mathcal{A})$ is a graph of groups and (A, i) is the corresponding edge-indexed graph, then, fixing a basepoint $a_0 \in VA$, the universal covering tree $\mathcal{T} = \widetilde{(A, a_0)}$ and the natural projection $\pi\colon \mathcal{T} \to A$ depend only on the edge-indexed graph (A, i) [**18**].

2.2. *Geometrically finite discrete groups acting on locally finite trees.* For a given locally finite simplicial tree $\mathcal{T}$, we define a metric d on $V\mathcal{T}$ so that $d(u, v)$ is the number of edges of the segment between u and v. Let us denote by $S\mathcal{T}$ the space of all bi-infinite geodesics in $\mathcal{T}$, that is, the set of isometries $\xi\colon \mathbb{Z} \to V\mathcal{T}$. We call an isometry $r\colon \mathbb{N} \cup \{0\} \to V\mathcal{T}$ a *ray*. Let $\widetilde{\phi}\colon S\mathcal{T} \to S\mathcal{T}$ be the *forward geodesic translation* given by $(\widetilde{\phi}\xi)(t) = \xi(t+1)$ and let $\widetilde{\iota}\colon S\mathcal{T} \to S\mathcal{T}$ be the *geodesic inversion* given by $(\widetilde{\iota}\xi)(t) = \xi(-t+1)$. Then $\widetilde{\phi}$ and $\widetilde{\iota}$ commute with the action of $\psi \in \mathrm{Aut}(\mathcal{T})$. The group $\mathbb{Z}$ acts on $S\mathcal{T}$ by $n \mapsto \widetilde{\phi}^{\circ n}$.

Let $\widetilde{\pi}\colon S\mathcal{T} \to \Gamma\backslash S\mathcal{T}$ be the natural projection map, let $\phi\colon \Gamma\backslash S\mathcal{T} \to \Gamma\backslash S\mathcal{T}$ be the induced forward geodesic translation given by $\phi \circ \widetilde{\pi} = \widetilde{\pi} \circ \widetilde{\phi}$ and let $\iota\colon \Gamma\backslash S\mathcal{T} \to \Gamma\backslash S\mathcal{T}$ be the induced geodesic inversion, defined similarly.

The *Busemann cocycle* is the map $\beta\colon \partial_\infty\mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathbb{Z}$, given by

$$(\omega, x, y) \mapsto \beta_\omega(x, y) = d(x, v) - d(y, v)$$

for some (and hence any) $v \in r_x \cap r_y$, where r_x and r_y are the rays starting from x and y, respectively, which converge to ω. The Busemann cocycle satisfies the following properties: $\beta_\omega(x, y) + \beta_\omega(y, z) = \beta(x, z)$, $\beta_\omega(x, y) = -\beta_\omega(y, x)$ and $\beta_{g\cdot\omega}(g \cdot x, g \cdot y) = \beta_\omega(x, y)$ for all $g \in G$, $x, y, z \in \mathcal{T}$ and $\omega \in \partial_\infty\mathcal{T}$.

Given $\xi \in S\mathcal{T}$, let $\xi^+ \in \partial_\infty\mathcal{T}$ be the positive end and $\xi^- \in \partial_\infty\mathcal{T}$ be the negative end. A *contracting horosphere* based at ξ^+ is the subset $\mathcal{H}^+_\xi = \{\eta \in S\mathcal{T} \mid \eta^+ = \xi^+ \text{ and } \beta_{\xi^+}(\xi(0), \eta(0)) = 0\}$. An *expanding horosphere* based at ξ^- is the subset $\mathcal{H}^-_\xi = \{\eta \in S\mathcal{T} \mid \eta^- = \xi^- \text{ and } \beta_{\xi^-}(\xi(0), \eta(0)) = 0\}$.

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Let Γ be a discrete subgroup of $\mathrm{Aut}(\mathcal{T})$. The *limit set* Λ_Γ of Γ is the set of accumulation points of a Γ-orbit in $\mathcal{T}$. By the discreteness of Γ, $\Lambda_\Gamma \subseteq \partial_\infty \mathcal{T}$. The *convex hull* $C\Lambda_\Gamma$ of Γ is the smallest convex subset of $\overline{\mathcal{T}} = \mathcal{T} \cup \partial_\infty \mathcal{T}$ containing Λ_Γ.

A point $\omega \in \Lambda_\Gamma$ is called a *conical limit point* if there is a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of Γ such that, for any point $x \in \mathcal{T}$, any ray c which converges to ω and any $n \in \mathbb{N}$, $d(\gamma_n x, c) \leq C$ for some $C > 0$. A point $\omega \in \Lambda_\Gamma$ is called a *horocyclic limit point* if there is a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of Γ such that, for any point $x \in \mathcal{T}$, $\lim_{n\to\infty} \beta_\omega(\gamma_n x, x) = \infty$.

Definition 2.1. A point $\omega \in \partial_\infty \mathcal{T}$ is called a Γ-*parabolic point* if the stabilizer Γ_ω fixes no point in $\partial_\infty \mathcal{T}$ other than ω and fixes no vertex of $\mathcal{T}$. It is called Γ-*cuspidal* if, further, for any ray with vertex sequence $x_0, x_1, x_2, x_3, \ldots$ towards ω, $\Gamma_{x_n} \leq \Gamma_\omega$ for any sufficiently large $n > 0$.

A point $\omega \in \Lambda_\Gamma$ is called a *bounded parabolic point* if the stabilizer Γ_ω of ω acts properly discontinuously and cocompactly on $\Lambda_\Gamma \backslash \{\omega\}$.

PROPOSITION 2.1. [**15**, Theorem 1.1] *Let Γ be a discrete subgroup of* $\mathrm{Aut}(\mathcal{T})$ *and let us denote the minimal Γ-invariant subtree of $\mathcal{T}$ by $\mathcal{T}_{min}$. Then the following are equivalent.*

(a) *Every limit point of Γ is either a conical limit point or a bounded parabolic point.*
(b) *Every limit point of Γ is either a horocyclic limit point or a bounded parabolic point.*
(c) *The quotient graph of groups $\Gamma \backslash\backslash \mathcal{T}_{min}$ is a union of a finite graph of finite groups and a finite number of Γ-cuspidal rays of groups.*

If the above equivalent conditions hold, then we say the subgroup Γ is *geometrically finite*. If, furthermore, $\Gamma \backslash\backslash \mathcal{T}_{\min}$ is a finite graph of finite groups, then we say that Γ is *convex cocompact*. Note that there is a lattice $\Gamma < \mathrm{Aut}(\mathcal{T})$ which is *not* geometrically finite. We also remark here that a discrete subgroup Γ of $\mathrm{Aut}(\mathbb{H}^n)$ is geometrically finite if one of the equivalent conditions (a) and (b) in the above proposition holds, or, equivalently, if the unit neighborhood of the convex core $\mathcal{C}_\Gamma = \Gamma \backslash C\Lambda_\Gamma$ has finite volume.

A *funnel* F is a subtree of $\mathcal{T}$ with exactly one vertex of degree one such that $\mathcal{T} - F$ is connected. Imitating the proof for hyperbolic plane case in [**3**] and using (c) of the above theorem, we have the following proposition about the structure of quotient graph of groups $\Gamma \backslash\backslash \mathcal{T}$ when Γ is geometrically finite.

PROPOSITION 2.2. *If Γ is geometrically finite, then there are a finite graph of groups D, finite Γ-cuspidal rays $C_1, \ldots, C_k$ and finite funnels $F_1, \ldots, F_l$ so that:*

(1) $VA = VD \cup VC_1 \cup \cdots \cup VC_k \cup VF_1 \cup \cdots \cup VF_l$;
(2) $|VD \cap VC_j| = |VD \cap VF_m| = 1$ *and* $VC_j \cap VF_m = \phi$; *and*
(3) *if* $a_{j,0} \in VD \cap VC_j$ *and* $e_{j,1} \in EC_j$ *with* $\partial_1 e_{j,1} = a_{j,0}$, *then* $i(e_{j,1}) = 1$.

Let L_Γ be the subgroup of $\mathbb{Z}$ generated by the translation length $l(\gamma) = \min_{v\in V\mathcal{T}}\{d(v, \gamma \cdot v)\}$ of every element $\gamma \in \Gamma$. We call L_Γ by the *length spectrum* of Γ. If $\mathcal{T}$ has no proper non-empty Γ-invariant subtree and has no vertices of degree two, then either $L_\Gamma = \mathbb{Z}$ or $L_\Gamma = 2\mathbb{Z}$ [**4**]. If $L_\Gamma = \mathbb{Z}$, then we say that the length spectrum of Γ is *not arithmetic*. This property is required to prove that the geodesic translation map is mixing (see Theorem 2.3).

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2.3. *Patterson–Sullivan theory with potential $\widetilde{F}$ for groups acting on trees.* The theory of *Patterson–Sullivan densities* was developed by Patterson (for $\mathbb{H}^2$) and Sullivan (for $\mathbb{H}^n$, $n \geq 3$) [**19**]. It was generalized to groups acting on CAT(-1) spaces by Burger and Mozes [**6**]. See also [**7**] for the case of universal covers of finite simplicial graphs. In this subsection, we review the Patterson–Sullivan theory together with a non-zero potential $\widetilde{F}$, following [**5**, **16**].

A *potential* $\widetilde{F}$ for a discrete group $\Gamma < \mathrm{Aut}(\mathcal{T})$ is a Γ-invariant continuous function $\widetilde{F}\colon E\mathcal{T} \to \mathbb{R}$. We denote by F the induced function $F\colon \Gamma\backslash E\mathcal{T} \to \mathbb{R}$ on the quotient. For all $x, y \in V\mathcal{T}$ with $d(x, y) = n$, we define

$$\int_x^y \widetilde{F} = \sum_{i=0}^{n-1} \widetilde{F}(e_i),$$

where $\xi \in S\mathcal{T}$ is any bi-infinite geodesic with $\xi(0) = x$ and $\xi(n) = y$, and e_i is the edge satisfying $\partial_0 e_i = \xi(i)$ and $\partial_1 e_i = \xi(i+1)$. This definition does not depend on the choice of $\xi \in S\mathcal{T}$.

Remark 2.1. In general, a potential is a function defined on the unit tangent bundle $T^1\widetilde{M}$ of a negatively curved manifold $\widetilde{M}$ or the space of germs T^1X of a CAT(-1) space X. For simplicial trees $\mathcal{T}$, the unit tangent vector bundle (or the space of germs) is defined as the quotient space $S\mathcal{T}/\sim$, where $\xi \sim \xi'$ if and only if $\xi(0) = \xi'(0)$ and $\xi(1) = \xi'(1)$. Thus, we can canonically identify $S\mathcal{T}/\sim$ with $E\mathcal{T}$.

The *critical exponent* of (Γ, F) is the element $\delta_{\Gamma,F} \in [-\infty, +\infty]$ defined by

$$\delta_{\Gamma,F} = \limsup_{n\to+\infty} \frac{1}{n} \log \sum_{\gamma\in\Gamma, d(x,\gamma y)=n} e^{\int_x^{\gamma y} \widetilde{F}}$$

and the *Poincaré series* of (Γ, F) is the map $Q = Q_{\Gamma,F,x,y}\colon \mathbb{R} \to [0, \infty]$ given by

$$Q\colon s \mapsto \sum_{\gamma\in\Gamma} e^{\int_x^{\gamma y}(\widetilde{F}-s)}.$$

If $F = 0$, then $\delta_{\Gamma,0}$ and $Q_{\Gamma,0}$ are the usual critical exponent δ_Γ and Poincaré series $\sum_{\gamma\in\Gamma} e^{-sd(x,\gamma y)}$ of Γ, respectively.

Let $\widetilde{F}^+ = \widetilde{F}$ and $\widetilde{F}^- = \widetilde{F} \circ \widetilde{\iota}$. We denote by $F^\pm\colon \Gamma\backslash E\mathcal{T} \to \mathbb{R}$ their induced maps. In the rest of paper, we assume that $\delta = \delta_{\Gamma,F^+} = \delta_{\Gamma,F^-}$ is finite. Generalizing the Busemann cocycle, the authors in [**16**] defined the *Gibbs cocycle* associated with the group Γ and the potential $\widetilde{F}$ as the map $C_F\colon \partial_\infty\mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathbb{R}$ given by

$$(\omega, x, y) \mapsto C_{F,\omega}(x, y) = \int_y^v \widetilde{F} - \int_x^v \widetilde{F}$$

for some (and hence any) $v \in r_x \cap r_y$. We call $C_{F-\delta}$ a *normalized Gibbs cocycle* associated with Γ and $\widetilde{F}$.

Definition 2.2. (Patterson density for $(\Gamma, \widetilde{F})$ [**16**]) A *Patterson density* of dimension α for the pair $(\Gamma, \widetilde{F})$ is a family of finite non-zero positive Borel measures $\{\mu_x\}_{x\in V\mathcal{T}}$ on $\partial_\infty\mathcal{T}$ such that, for every $\gamma \in \Gamma$, for all $x, y \in \mathcal{T}$ and for every $\omega \in \partial_\infty\mathcal{T}$,

$$\gamma_*\mu_x = \mu_{\gamma\cdot x} \quad \text{and} \quad \frac{d\mu_x}{d\mu_y}(\omega) = e^{-C_{F-\alpha,\omega}(x,y)}.$$

In particular, a $\delta_{\Gamma,F}$-dimensional Patterson density for $(\Gamma, \widetilde{F})$ is called *normalized*.

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There is at least one normalized Patterson density for the pair $(\Gamma, \widetilde{F})$. Moreover, if $Q_{\Gamma,F}(\delta) = \infty$, then the normalized Patterson density is unique up to a multiplicative constant [**5**, **16**].

Now let $\{\nu_x^{\pm}\}$ be the *normalized* Patterson densities for the pairs $(\Gamma, \widetilde{F}^{\pm})$, respectively, and let $C^{\pm}$ be the *normalized* Gibbs cocycle for $(\Gamma, \widetilde{F}^{\pm})$. Fixing $o \in \mathcal{T}$, the map $\xi \mapsto (\xi^+, \xi^-, s)$ gives a homeomorphism between $S\mathcal{T}$ and $((\partial_\infty\mathcal{T} \times \partial_\infty\mathcal{T})\backslash\Delta) \times \mathbb{Z}$, where $\xi(s)$ is the closest point to o on the geodesic line ξ. Let us define the measure $\widetilde{m}_F^{\nu^-,\nu^+}$ on $S\mathcal{T}$ associated to $\{\nu_x^-\}$ and $\{\nu_x^+\}$ by

$$d\widetilde{m}_F^{\nu^-,\nu^+}(\xi) = e^{C^-_{\xi^-}(o,\xi_0)} e^{C^+_{\xi^+}(o,\xi_0)}\, d\nu_o^-(\xi^-)\, d\nu_o^+(\xi^+)\, ds,$$

where ds is the counting measure on $\mathbb{Z}$. It follows, from the Γ-invariant conformal property of $\{\nu_x^{\pm}\}$, that the definition of $\widetilde{m}_F$ is independent of the choice of $o \in \mathcal{T}$ and that $\widetilde{m}_F^{\nu^-,\nu^+}$ is left Γ-invariant. Hence it induces a Radon measure $m_{\Gamma,F}^{\nu^-,\nu^+}$ of the quotient space $\Gamma\backslash S\mathcal{T}$. This definition is also invariant under the geodesic translation map ϕ_Γ on $\Gamma\backslash S\mathcal{T}$. Now we may lift the measure $m_{\Gamma,F}^{\nu^-,\nu^+}$ to $\Gamma\backslash G$ for arbitrary closed subgroup G of $\mathrm{Aut}(\mathcal{T})$ via $M = \mathrm{Stab}_G(\xi)$-invariant extension: for $\widetilde{f} \in C_c(\Gamma\backslash G)$, define

$$m_{\Gamma,F}^{\nu^-,\nu^+}(\widetilde{f}) = \int_{\eta\in\Gamma\backslash S\mathcal{T}} \int_{g\in M} \widetilde{f}(\eta g)\, dg\, dm_{\Gamma,F}^{\nu^-,\nu^+}.$$

When F is constant, the normalized Patterson density is equal to the usual δ_Γ-dimensional *Patterson–Sullivan* density and the measure $m_{\Gamma,F}^{\nu^-,\nu^+}$ coincides with the *Bowen–Margulis* measure m_Γ^{BM} given by

$$dm_\Gamma^{\mathrm{BM}} = e^{\delta_\Gamma \beta_{\xi^+}(o,\xi_0)} e^{\delta_\Gamma \beta_{\xi^-}(o,\xi_0)}\, d\nu_o(\xi^+)\, d\nu_o(\xi^-)\, ds.$$

Recall that the length spectrum of Γ is said to be not arithmetic when $L_\Gamma = \mathbb{Z}$ for $\Gamma < \mathrm{Aut}(\mathcal{T})$. The following mixing property of the geodesic flow is due to [**5**, **17**].

THEOREM 2.3. *Let $\Gamma < \mathrm{Aut}(\mathcal{T})$ be a discrete subgroup whose length spectrum is not arithmetic. Let $\widetilde{F}$ be a given potential for Γ and let $\nu^{\pm}$ be the normalized Patterson densities for $(\Gamma, \widetilde{F}^{\pm})$. Suppose that $|m_{\Gamma,F}^{\nu^-,\nu^+}| < \infty$. Then, for all compactly supported continuous functions $f, g\colon \Gamma\backslash S\mathcal{T} \to \mathbb{R}$,*

$$\int_{\Gamma\backslash S\mathcal{T}} f \cdot (g \circ \phi_\Gamma^{\circ n})\, dm_{\Gamma,F}^{\nu^-,\nu^+} \longrightarrow \frac{1}{|m_{\Gamma,F}^{\nu^-,\nu^+}|} \int_{\Gamma\backslash S\mathcal{T}} f\, dm_{\Gamma,F}^{\nu^-,\nu^+} \int_{\Gamma\backslash S\mathcal{T}} g\, dm_{\Gamma,F}^{\nu^-,\nu^+}.$$

If $L_\Gamma = k\mathbb{Z}$, let $S_o^k\mathcal{T}$ be the subset $\{\xi \in S\mathcal{T} \mid d(\xi(0), o) \in k\mathbb{Z}\}$ of $S\mathcal{T}$. This is invariant under Γ and $\phi^{\circ k}$, so we may restrict $m_{\Gamma,F}^{\nu^-,\nu^+}$ to $\Gamma\backslash S_o^k\mathcal{T}$. In this case, the dynamical system $(\Gamma\backslash S_o^k\mathcal{T}, \phi^{\circ k}, m_{\Gamma,F}^{\nu^-,\nu^+})$ is mixing, for any choice of $o \in V\mathcal{T}$ (cf. [**4**]).

2.4. *Bruhat–Tits tree of rank-one algebraic groups.* Let $\mathbb{F}$ be a non-Archimedean local field with the valuation $v\colon \mathbb{F} \to \mathbb{Z}$. If $G = \underline{G}(\mathbb{F})$ consists of $\mathbb{F}$-rational points of an arbitrary $\mathbb{F}$-rank-one algebraic group over $\mathbb{F}$ whose residue field is $\mathbb{F}_q$, then the Bruhat–Tits tree $\mathcal{T}$ of G is a $(q^d + 1, q^{d'} + 1)$-biregular tree, where d and d' are integers attached to the two vertices of the relative local Dynkin diagram for H. Indeed, if B is the minimal parabolic

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subgroup of G, and P_1 and P_2 are the two maximal proper parabolic subgroups of G, then $q^d + 1 = [P_1 : B]$ and $q^{d'} + 1 = [P_2 : B]$. By the classical result of the classification, up to strict isogeny we have nine absolutely simple rank-one algebraic groups. If G is not simply connected, then G may act with inversions. In this case, we can replace G by a subgroup of index two which acts without inversions (see [**20**]). From now on, we assume that G acts on $\mathcal{T}$ without inversions; hence, $d_{\mathcal{T}}(x, g \cdot x)$ is always even for every $g \in G$ and every vertex x of $\mathcal{T}$.

We now introduce some subgroups of G. Let N^+ and N^- be the set of elements of G which stabilize the contracting horosphere $\mathcal{H}^+_\xi(0)$ and expanding horosphere $\mathcal{H}^-_\xi(0)$, respectively. We can also choose a maximal $\mathbb{F}$-split torus S of $\underline{G}$ which fixes ξ^+ and ξ^- in $\partial_\infty \mathcal{T}$ pointwise. Let $Z = Z_{\underline{G}}(S)_{\mathbb{F}}$ be the centralizer of S in $\underline{G}$. For each $j \in \mathbb{Z}$, there is a unique element z in Z such that $z \cdot H^+_\xi(0) = H^+_\xi(2j)$; we denote such z by a_{2j} (see [**20**]). It follows that $Z = \{a_{2j} : j \in \mathbb{Z}\}$. Define $Z^+ = \{a_{2j} : j \geq 0\}$. Finally, let $P^+ = ZN^+$ and $P^- = ZN^-$.

3. *Changing the dynamical system of geodesic translation map into a Markov chain*

In this section, we introduce the Markov chain $(\mathcal{S}, p_{ij}, \pi_j)$ with countable states which is associated to the dynamical system $(\Gamma\backslash S\mathcal{T}, \phi, m_{\Gamma,F}^{\nu^-,\nu^+})$. We adopt a similar approach to that taken in [**6**] and use some of the ideas, but with slight modifications. The work in this section is novel because we interpret the exponential mixing properties of the dynamical system $(\Gamma\backslash S\mathcal{T}, \phi, m_{\Gamma,F}^{\nu^-,\nu^+})$ in terms of a countable Markov chain.

3.1. *Countable Markov shift $X_{(A,i)}$.*

Let $\mathcal{T}$ be a locally finite tree and $S\mathcal{T}$ be the set of all bi-infinite geodesics in $\mathcal{T}$. Let $\widetilde{\phi} : S\mathcal{T} \to S\mathcal{T}$ be the *forward geodesic translation* given by $(\widetilde{\phi}\xi)(t) = \xi(t+1)$ and let $\widetilde{\iota} : S\mathcal{T} \to S\mathcal{T}$ be the *geodesic inversion* given by $(\widetilde{\iota}\xi)(t) = \xi(-t+1)$. The group $\mathbb{Z}$ acts on $S\mathcal{T}$ by $n \mapsto \widetilde{\phi}^{\circ n}$. Then $\widetilde{\phi}$ and $\widetilde{\iota}$ commute with the action of $\psi \in \mathrm{Aut}(\mathcal{T})$.

Let $\widetilde{\pi} : S\mathcal{T} \to \Gamma\backslash S\mathcal{T}$ be the natural projection map, let $\phi : \Gamma\backslash S\mathcal{T} \to \Gamma\backslash S\mathcal{T}$ be the induced forward geodesic translation given by $\phi \circ \widetilde{\pi} = \widetilde{\pi} \circ \widetilde{\phi}$ and let $\iota : \Gamma\backslash S\mathcal{T} \to \Gamma\backslash S\mathcal{T}$ be the induced geodesic inversion, defined similarly. We will assume that Γ is a full discrete subgroup of $\mathrm{Aut}(\mathcal{T})$, that is, a discrete subgroup of $\mathrm{Aut}(\mathcal{T})$ which is equal to its associated full subgroup $\Gamma_f = \{g \in \mathrm{Aut}(\mathcal{T}) \mid \pi \circ g = \pi\}$, where $\pi : \mathcal{T} \to \Gamma\backslash\mathcal{T}$ is the canonical projection (see Definition 1.1).

For a given edge-indexed graph (A, i), consider the subset

$$X_{(A,i)} = \{x = (e_j)_{j\in\mathbb{Z}} \mid \partial_0 e_{j+1} = \partial_1 e_j \text{ and if } e_{j+1} = \overline{e_j}, \text{ then } i^A(e_j) > 1\}$$

of $(EA)^{\mathbb{Z}}$. The family of *cylinders*

$$[e_0, \ldots, e_{n-1}] := \{x \in X_{(A,i)} : x_i = e_i,\ i = 0, \ldots, n-1\}$$

is a basis of open sets for a topology on X. Let $\sigma : X_{(A,i)} \to X_{(A,i)}$ be given by $\sigma(x)_i := x_{i+1}$. Then $(S\mathcal{T}, \widetilde{\phi})$ is conjugate to $(X_{\mathcal{T}}, \sigma)$ (consider $\mathcal{T}$ as $(\mathcal{T}, i^0)$ with $i^0(e) = 1$, for all $e \in E\mathcal{T}$). If we denote by (A, i) the edge-indexed graph associated with $\Gamma\backslash\backslash\mathcal{T}$, then

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we also have a bijection Φ between two spaces $(\Gamma\backslash S\mathcal{T}, \phi)$ and $(X_{(A,i)}, \sigma)$ so that the following diagram commutes.

$$\begin{array}{ccc} \Gamma\backslash S\mathcal{T} & \xrightarrow{\Phi} & X_{(A,i)} \\ \phi \downarrow & & \downarrow \sigma \\ \Gamma\backslash S\mathcal{T} & \xrightarrow{\Phi} & X_{(A,i)} \end{array} \tag{2}$$

Indeed, let $\pi\colon \mathcal{T} \to \Gamma\backslash\mathcal{T}$ be the canonical projection and let $\xi\colon \mathbb{Z} \to \mathcal{T}$ be an element of $S\mathcal{T}$. Let ξ_j be the edge satisfying $\partial_0\xi_j = \xi(j)$ and $\partial_1\xi_j = \xi(j+1)$. Define $\widetilde{\Phi}(\xi) = (\pi(\xi_j))_{j\in\mathbb{Z}}$. Then $\widetilde{\Phi}\colon S\mathcal{T} \to X_{(A,i)}$ is Γ-invariant, so it induces the map $\Phi\colon \Gamma\backslash S\mathcal{T} \to X_{(A,i)}$. Given $e_1, e_2 \in E\mathcal{T}$ satisfying $\partial_0 e_1 = \partial_0 e_2$ and $\pi(e_1) = \pi(e_2)$, there is $\gamma \in \mathrm{Aut}(\mathcal{T})$ such that $\gamma e_1 = e_2$, $\pi \circ \gamma = \pi$ and γ acts trivially on the connected component of $\mathcal{T} - \{e_1, e_2\}$ containing $\partial_0 e_1$. Therefore, Φ is a ϕ-equivariant homeomorphism (cf. [**6**]).

3.2. *Invariant measures on* $(X_{(A,i)}, \sigma)$. Let $\mathbf{A} = (A, \mathcal{A})$ be a graph of groups and (A, i) be the corresponding edge-indexed graph. In this subsection, we will get a *Markov measure* λ_F on the dynamical system $(X_{(A,i)}, \sigma)$ corresponding to $m_{\Gamma,F}^{\nu^-,\nu^+}$. In other words, we require

$$\begin{aligned} \sum_{e_j\colon \partial_0 e_k = \partial_1 e_j} \lambda_F([e_j, e_k]) &= \lambda_F([e_k]), \\ \sum_{e_k\colon \partial_0 e_k = \partial_1 ej} \lambda_F([e_j, e_k]) &= \lambda_F([e_j]) \end{aligned} \tag{3}$$

and

$$\int_{\Gamma\backslash S\mathcal{T}} f\, dm_{\Gamma,F}^{\nu^-,\nu^+} = \int_{X_{(A,i)}} \Phi^*(f)\, d\lambda_F, \tag{4}$$

where $\Phi^*(f)[(x_i)_{i\in\mathbb{Z}}] = f[\Phi^{-1}((x_i)_{i\in\mathbb{Z}})]$.

Fixing a basepoint $a_0 \in VA$, the universal covering $\mathcal{T} \simeq \widetilde{(\mathbf{A}, a_0)}$ is a locally finite tree. Let $\pi\colon \mathcal{T} \to \Gamma\backslash\mathcal{T}$ be the natural projection. We define the invariant measure λ_F on $(X_{(A,i)}, \sigma)$ as follows. For $e \in E\mathcal{T}$, we denote the *shadow* of an edge e by

$$\mathcal{O}(e) = \{\omega \in \partial_\infty\mathcal{T} \mid \exists \xi \in S\mathcal{T} \text{ such that } \xi(0) = \partial_0 e,\ \xi(1) = \partial_1 e \text{ and } \xi^+ = \omega\}.$$

For each cylinder $[e_0, \ldots, e_{n-1}]$, let $\xi_0, \ldots, \xi_{n-1} \in E\mathcal{T}$ be the edges of a path in $\mathcal{T}$ satisfying $\pi(\xi_j) = e_j$ and let

$$\Gamma_{\xi_0,\ldots,\xi_{n-1}} := \mathrm{Stab}_\Gamma(\xi_0) \cap \cdots \cap \mathrm{Stab}_\Gamma(\xi_{n-1}).$$

Let $\nu^\pm$ be the normalized Patterson density for $(\Gamma, \widetilde{F}^\pm)$ constructed in §2.3. Now we define

$$\lambda_F([e_0, \ldots, e_{n-1}]) = \frac{\nu^-_{\partial_0\xi_0}(\mathcal{O}(\overline{\xi_0}))\nu^+_{\partial_1\xi_{n-1}}(\mathcal{O}(\xi_{n-1}))}{|\Gamma_{\xi_0,\ldots,\xi_{n-1}}|} e^{\int_{\partial_0\xi_0}^{\partial_1\xi_{n-1}} \widetilde{F}-\delta}.$$

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Since $|\Gamma_{\xi_0,\dots,\xi_{n-1}}|$ depends only on $e_0, \dots, e_{n-1}$ and $\nu^{\pm}$ are Γ-invariant, the measure λ_F is well defined. Moreover, equation (4) holds since

$$\begin{aligned}
\lambda_F([e_0, \dots, e_{n-1}]) &= \frac{\nu^-_{\partial_0\xi_0}(\mathcal{O}(\overline{\xi_0}))\nu^+_{\partial_1\xi_{n-1}}(\mathcal{O}(\xi_{n-1}))}{|\Gamma_{\xi_0,\dots,\xi_{n-1}}|} e^{\int_{\partial_0\xi_0}^{\partial_1\xi_{n-1}} \widetilde{F}-\delta} \\
&= \frac{1}{|\Gamma_{\xi_0,\dots,\xi_{n-1}}|} \int_{\xi^+\in\partial_\infty\mathcal{T}} \mathbb{1}_{\mathcal{O}(\xi_{n-1})} e^{-C_{\xi^+}(\partial_1\xi_{n-1},\partial_0\xi_0)} \\
&\quad \times \int_{\xi^-\in\partial_\infty\mathcal{T}} \mathbb{1}_{\mathcal{O}(\overline{\xi_0})}\, d\nu^-_{\partial_0\xi_0}(\xi^-)\, d\nu^+_{\partial_1\xi_{n-1}}(\xi^+) \\
&= \int_{\Gamma\backslash S\mathcal{T}} \mathbb{1}_{\Phi^{-1}[e_0,\dots,e_{n-1}]}\, d\nu^-_{\partial_0\xi_0} d\nu^+_{\partial_0\xi_0} \\
&= m_{\Gamma,F}^{\nu^-,\nu^+}(\Phi^{-1}[e_0, \dots, e_{n-1}]).
\end{aligned}$$

It remains for us to prove the Markov property (3) of λ_F.

PROPOSITION 3.1. (Markov property) *Let us define*

$$p_{e_j,e_k} := \frac{\lambda_F([e_j, e_k])}{\lambda_F([e_j])}.$$

Then the measures λ_F and p_{e_j,e_k} satisfy

$$\sum_{e_j:\, \partial_0 e_k=\partial_1 e_j} \lambda_F([e_j])\, p_{e_j,e_k} = \lambda_F([e_k]) \quad \textit{and} \quad \sum_{e_k:\, \partial_0 e_k=\partial_1 e_j} p_{e_j,e_k} = 1.$$

Proof. Let $\Gamma\backslash\backslash\mathcal{T} = (A, \mathcal{A})$ and $e_k \in EA$ with $\partial_0 e_k = \partial_1 e_j$. If $e_k = \overline{e_j}$, then $|\Gamma_{\xi_k}| = (i^A(e_j) - 1)|\Gamma_{\xi_j,\xi_k}|$ and, if $e_k \neq \overline{e_j}$, then $|\Gamma_{\xi_k}| = i^A(\overline{e_k})|\Gamma_{\xi_j,\xi_k}|$. In each case, let

$$\rho_{e_j,e_k} = \frac{|\Gamma_{\xi_j}|}{|\Gamma_{\xi_j,\xi_k}|}.$$

Since $\nu^+_{\partial_1\xi_j}(\mathcal{O}_{\xi_k}) = \nu^+_{\partial_1\xi_k}(\mathcal{O}_{\xi_k})e^{\widetilde{F}(\xi_k)-\delta}$ and $\nu^-_{\partial_0\xi_k}(\mathcal{O}_{\overline{\xi_j}}) = \nu^-_{\partial_0\xi_j}(\mathcal{O}_{\overline{\xi_j}})e^{\widetilde{F}(\xi_j)-\delta}$,

$$p_{e_j,e_k} = \frac{|\Gamma_{\xi_j}|\nu^+_{\partial_1\xi_j}(\mathcal{O}(\xi_k))}{|\Gamma_{\xi_j,\xi_k}|\nu^+_{\partial_1\xi_j}(\mathcal{O}(\xi_j))} \quad \text{and} \quad \frac{\lambda_F([e_j, e_k])}{\lambda_F([e_k])} = \frac{|\Gamma_{\xi_k}|\nu^-_{\partial_0\xi_k}(\mathcal{O}(\overline{\xi_j}))}{|\Gamma_{\xi_j,\xi_k}|\nu^-_{\partial_0\xi_k}(\mathcal{O}(\overline{\xi_k}))}.$$

By the definition of an edge-indexed graph (A, i) (see §2.1) and the finite (even countable) additivity of the normalized Patterson density ν,

$$\nu^+_{\partial_1\xi_j}(\mathcal{O}(\xi_j)) = \sum_{e_k:\, \partial_0 e_k=\partial_1 e_j} \rho_{e_j,e_k}\nu^+_{\partial_1\xi_j}(\mathcal{O}(\xi_k))$$

and

$$\nu^-_{\partial_0\xi_k}(\mathcal{O}(\xi_k)) = \sum_{e_j:\, \partial_1 e_j=\partial_0 e_k} \rho_{e_k,e_j}\nu^-_{\partial_0\xi_k}(\mathcal{O}(\xi_j)),$$

which completes the proof. □

The equations in (3) follow from the above proposition. Therefore, it follows that the measure λ_F is a Markov measure and the two dynamical systems

$$(\Gamma\backslash S\mathcal{T}, \phi, m_{\Gamma,F}^{\nu^-,\nu^+}) \quad \text{and} \quad (X_{(A,i)}, \sigma, \lambda_F)$$

are isomorphic, by (2) and (4).

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3.3. *Decay of correlation in terms of Markov chain* $(\mathcal{S}, p_{ij}, \pi_j)$. We consider a Markov chain Z_n with phase space $\mathcal{S} = \{s_1, s_2, \ldots\}$ and transition probabilities

$$p_{ij} = p_{s_i s_j} = P\{Z_{n+1} = s_j \mid Z_n = s_i\}.$$

For a subset $B \subset \mathcal{S}$ of alphabets, let

$$\begin{aligned} f_{ij}^{(n),B} &= P\{Z_1 \notin B \cup s_j, \ldots, Z_{n-1} \notin B \cup s_j, Z_n = s_j \mid Z_0 = s_i\}, \\ p_{ij}^{(n),B} &= P\{Z_1 \notin B, \ldots, Z_{n-1} \notin B, Z_n = s_j \mid Z_0 = s_i\}. \end{aligned} \tag{5}$$

We denote $f_{ij}^{(n)} = f_{ij}^{(n),\phi}$ and $p_{ij}^{(n)} = p_{ij}^{(n),\phi}$. By convention, $f_{ij}^{(0)} = 0$ and $p_{ij}^{(0)} = \delta_{ij}$. Meanwhile, we have the convolution relation

$$p_{s_i s_j}^{(n),B} = \sum_{r=1}^{n} f_{s_i s_i}^{(r),B} p_{s_i s_j}^{(n-r),B}. \tag{6}$$

Suppose that the Markov chain Z_n is irreducible, that is, for any $s_i, s_j \in \mathcal{S}$, there exists $n > 0$ such that $p_{ij}^{(n)} > 0$. We say that π_j is the *stationary* (or *invariant*) distribution if it satisfies $\pi_j = \sum_{i \in \mathcal{S}} \pi_i p_{ij}$. If it exists, then we say that the Markov chain $(\mathcal{S}, p_{ij})$ is *recurrent* (otherwise we say *transient*). When the Markov chain $(\mathcal{S}, p_{ij}, \pi_j)$ is recurrent, it is called *positive recurrent* if $\sum_{n=1}^{\infty} n f_{jj}^{(n)} < \infty$ (otherwise it is called *null recurrent*). When $(\mathcal{S}, p_{ij}, \pi_j)$ is positive recurrent,

$$\pi_j = \frac{1}{\sum_{n=1}^{\infty} n f_{jj}^{(n)}}.$$

A *period* of a state $s_i \in \mathcal{S}$ is defined by $k = \gcd\{n\colon p_{ii}^{(n)} > 0\}$. An irreducible Markov chain is called *aperiodic* if for some (and hence every) state $s_i \in \mathcal{S}$ the period is one. If the positive recurrent Markov chain Z_n is irreducible and aperiodic, then $\pi_j = \lim_{n\to\infty} p_{ij}^{(n)}$ and π_j does not depend on the choice of $i \in \mathcal{S}$ [**12**].

We are interested in the Markov chain $(\mathcal{S}, p_{ij}, \pi_j)$ for

$$\mathcal{S} = E(\Gamma \backslash\backslash \mathcal{T}), \quad p_{ij} = p_{s_i s_j} = \frac{\lambda_F([s_i, s_j])}{\lambda_F([s_i])} \quad \text{and} \quad \pi_j = \lambda_F([s_j]). \tag{7}$$

If the length spectrum of Γ is not arithmetic, then $(\Gamma \backslash S\mathcal{T}, \phi, m_{\Gamma,F}^{\nu^-,\nu^+})$ is mixing and hence $(\mathcal{S}, p_{ij}, \pi_j)$ is an irreducible aperiodic Markov chain.

Now we can deduce the exponential decay of correlation from the information of the speed of convergence to stationary distribution.

PROPOSITION 3.2. *Suppose there are constants* $C_{s_i s_j} > 0$ *and* $0 < \theta < 1$ *such that* $|p_{s_i s_j}^{(n)} - \pi_{s_j}| \le C_{s_i s_j} \theta^n$. *Then*

$$|\mathrm{Cov}(f, g)| = \left| \int (f \circ \sigma^{\circ n}) \cdot g \, d\lambda_F - \int f \, d\lambda_F \int g \, d\lambda_F \right| \le C(f, g)\theta^n.$$

Proof. Since every measurable set can be approximated by finite disjoint unions of cylinder sets, we only need to show that $|\lambda_F([\underline{a}] \cap \sigma^{-n}[\underline{b}]) - \lambda_F([\underline{a}])\lambda_F([\underline{b}])|$ converges

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to zero exponentially, for all cylinder sets $[\underline{a}] = [a_0, \dots, a_{k-1}]$ and $[\underline{b}] = [b_0, \dots, b_{l-1}]$. Now

$$\begin{aligned}|\mathrm{Cov}(\mathbb{1}_{[\underline{a}]}, \mathbb{1}_{[\underline{b}]})| &= |\lambda_F([\underline{a}] \cap \sigma^{-n}[\underline{b}]) - \lambda_F([\underline{a}])\lambda_F([\underline{b}])| \\ &= \left|\lambda_F\left(\bigcup_{\underline{c}\in W_{n-k}} [\underline{a}, \underline{c}, \underline{b}]\right) - \lambda_F([\underline{a}])\lambda_F([\underline{b}])\right| \\ &= \left|\lambda_F([\underline{a}])\left(\sum_{\underline{c}\in W_{n-k}} p_{a_{k-1}c_0}\cdots p_{c_{n-k-1}b_0}\cdot\frac{\lambda_F([\underline{b}])}{\pi_{b_0}}\right) - \lambda_F([\underline{a}])\lambda_F([\underline{b}])\right| \\ &= \lambda_F([\underline{a}])\lambda_F([\underline{b}])\left|\frac{p^{(n-k)}_{a_{k-1}b_0} - \pi_{b_0}}{\pi_{b_0}}\right| \le \|\mathbb{1}_{[\underline{a}]}\|\|\mathbb{1}_{[\underline{b}]}\|\frac{C_{a_{k-1}b_0}}{\pi_{b_0}\theta^k}\theta^n.\end{aligned}$$

This completes the proof. □

4. *Exponential decay of the correlation function*

Let $\mathcal{T}$ be a locally finite tree. Let $(\Gamma, \widetilde{F})$ be a pair of full discrete subgroup $\Gamma < \mathrm{Aut}(\mathcal{T})$ and a potential $\widetilde{F}$ for Γ with $|m^{\nu^-,\nu^+}_{\Gamma,F}| < \infty$. Assume that $L_\Gamma = \mathbb{Z}/k\mathbb{Z}$ and let $S^k_o\mathcal{T}$ be a subset $\{\xi \in S\mathcal{T} \mid d(\xi_0, o) \in k\mathbb{Z}\}$ of $S\mathcal{T}$ for $o \in V\mathcal{T}$. Our goal in this section is to prove the exponential decay of the correlation function

$$|\mathrm{Cov}(f, g)| = \left|\int_{\Gamma\backslash S^k_o\mathcal{T}} (f\circ\sigma^{\circ n})\cdot g\, dm^{\nu^-,\nu^+}_{\Gamma,F} - \int_{\Gamma\backslash S^k_o\mathcal{T}} f\, dm^{\nu^-,\nu^+}_{\Gamma,F}\int_{\Gamma\backslash S^k_o\mathcal{T}} g\, dm^{\nu^-,\nu^+}_{\Gamma,F}\right|$$

for $f, g \in C_c(\Gamma\backslash S^k_o\mathcal{T})$ when $\Gamma < \mathrm{Aut}(\mathcal{T})$ has a WSG (see Definition 8). By Proposition 3.2, it suffices to show that $|p^{(n)}_{s_is_j} - \pi_{s_j}| \le C_{s_is_j}\theta^n$ for some $C_{s_is_j} > 0$ and $0 < \theta < 1$. Inspired by Young's observation about the relation between the speed of convergence to equilibrium states and the recurrent time to a compact set [**21**], we investigate the speed of decay of the probability $p^{(n),B}_{s_is_j}$ for a finite set $B \subset E(\Gamma\backslash\mathcal{T})$ (see (5)).

4.1. *Exponential convergence to the stationary distribution.*

Definition 4.1. (WSG) Let $\Gamma < \mathrm{Aut}(\mathcal{T})$ be a full discrete subgroup and let $\widetilde{F}\colon E\mathcal{T} \to \mathbb{R}$ be a potential for Γ such that $|m^{\nu^-,\nu^+}_{\Gamma,F}| < \infty$. Let Z_n be the Markov chain with the data $(\mathcal{S}, p_{ij}, \pi_j)$ associated with $(\Gamma\backslash S\mathcal{T}, \phi, m^{\nu^-,\nu^+}_{\Gamma,F})$ (see (7)). Suppose that there is a function $t\colon \mathcal{S} \to \mathbb{R}_{\ge 0}$ given by $t(s_i) = t_i$, a finite subset $B \subset \mathcal{S}$ and a constant $0 < \rho < 1$ such that, for $s_i \in \mathcal{S} - B$,

$$\sum_{s_j} p_{ij}t_jt_i^{-1} \le \rho. \tag{8}$$

Then we say that $(\Gamma, \widetilde{F})$ has a WSG with (t, B, ρ).

LEMMA 4.1. *Suppose* $(\Gamma, \widetilde{F})$ *has a WSG with* $t\colon \mathcal{S} \to \mathbb{R}_{\ge 0}$, $B \subset \mathcal{S}$ *and* $0 < \rho < 1$. *Then, we obtain* $p^{(n),B}_{ij} \le t_it_j^{-1}\rho^n$. *In particular,*

$$p^{(n),B}_{i,B} = P\{Z_1 \notin B, \dots, Z_{n-1} \notin B, Z_n \in B \mid Z_0 = s_i\} \le Mt_i\rho^n$$

for $M = \max\{t_j^{-1} \mid s_j \in B\}$.

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Proof. Let $t_i = t(s_i)$ and $t_j = t(s_j)$. Then

$$\begin{aligned} p_{ij}^{(n),B} &= \sum_{r_k \in \mathcal{S}-B} p_{s_i r_1} p_{r_1 r_2} \cdots p_{r_{n-1} s_j} = \frac{t_i}{t_j} \sum_{r_k \in \mathcal{S}-B} \left(p_{s_i r_1} \frac{t(r_1)}{t_i} \right) \cdots \left(p_{r_{n-1} s_j} \frac{t_j}{t(r_{n-1})} \right) \\ &\le \frac{t_i}{t_j} \left(\sum_{r_1} p_{s_i r_1} \frac{t(r_1)}{t_i} \right) \left(\max_{r_1} \sum_{r_2} p_{r_1 r_2} \frac{t(r_2)}{t(r_1)} \right) \cdots \left(\max_{r_{n-1}} p_{r_{n-1} j} \frac{t_j}{t(r_{n-1})} \right) \\ &\le t_i t_j^{-1} \rho^n. \end{aligned}$$

The second statement follows directly from this inequality. □

The following two propositions are the main ingredients of our proof of the exponential mixing property.

PROPOSITION 4.2. *Let $(\mathcal{S}, p_{ij})$ be an irreducible aperiodic countable Markov chain. If there exists a state $b_0 \in \mathcal{S}$ which satisfies $p_{s_i s_j}^{(n),\{b_0\}} \le C_{s_i,s_j} \theta^n$ for some $0 < \theta < 1$, then it converges to the stationary distribution exponentially, that is, $|p_{s_i s_j}^{(n)} - \pi_{s_j}| < \widehat{C}_{s_i,s_j} \widehat{\theta}^n$ for some $0 < \widehat{\theta} < 1$, with the constants $\widehat{C}_{s_i,s_j}$ depending only on s_i and s_j.*

Proof. Let $F_{s_i s_j}(z) = \sum_{n=0}^{\infty} f_{ij}^{(n)} z^n$ and $P_{s_i s_j}(z) = \sum_{n=0}^{\infty} p_{ij}^{(n)} z^n$ be the generating functions. By the assumption, $f_{s_i b_0}^{(n)} = p_{s_i b_0}^{(n),\{b_0\}} \le C_{s_i} \theta^n$. Thus, power series $F_{s_i b_0}(z)$ is analytic in the disk $C_{\theta^{-1}} = \{z \in \mathbb{C} \mid |z| < \theta^{-1}\}$. Moreover, from the convolution relation (6), the equality

$$P_{b_0 b_0}(z) = \frac{1}{1 - F_{b_0 b_0}(z)}$$

holds and $P_{b_0 b_0}(z)$ has a unique simple pole at the point $z = 1$ in the disk C_R for some $1 < R \le \theta^{-1}$. Since

$$\lim_{z \to 1} \frac{z-1}{1 - F_{b_0 b_0}(z)} = \frac{1}{-\sum_{n=1}^{\infty} n f_{b_0 b_0}^{(n)}} = -\pi_{b_0},$$

it follows that $P_{b_0 b_0}(z) - \pi_{b_0}/(1-z) = \sum_{n=0}^{\infty} (p_{b_0 b_0}^{(n)} - \pi_{b_0}) z^n$ is analytic for $|z| < R$, and hence there is a constant $c > 0$ such that

$$|p_{b_0 b_0}^{(n)} - \pi_{b_0}| < c \theta_1^n$$

for some $\theta_1 \le R^{-1} < 1$.

Now using the equality

$$p_{s_i b_0}^{(n)} = f_{s_i b_0}^{(n)} + \sum_{r=1}^{n-1} f_{s_i, b_0}^{(r)} p_{b_0 b_0}^{(n-r)}$$

and the triangle inequality,

$$|p_{s_i b_0}^{(n)} - \pi_0| \le \left| \sum_{r=1}^{n} f_{s_i b_0}^{(r)} (p_{b_0 b_0}^{(n-r)} - \pi_{b_0}) \right| + \pi_{b_0} \sum_{r=n+1}^{\infty} f_{s_i b_0}^{(r)}.$$

This implies the existence of constant $C'_{s_i} > 0$ for which

$$|p_{s_i b_0}^{(n)} - \pi_{b_0}| < C'_{s_i} \theta_2^n$$

with some $R^{-1} < \theta_2 < 1$.

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Finally, since the Markov chain $(\mathcal{S}, p_{ij}, \pi_j)$ is irreducible and aperiodic, $\pi_{s_j} = \pi_{b_0} \sum_{n=0}^{\infty} p_{b_0,s_j}^{(n),\{b_0\}}$ and

$$p_{s_i s_j}^{(n)} = p_{s_i,s_j}^{(n),\{b_0\}} + \sum_{r=0}^{n-1} p_{s_i b_0}^{(r)} p_{b_0 s_j}^{(n-r),\{b_0\}}.$$

Hence, we obtain

$$|p_{s_i s_j}^{(n)} - \pi_{s_j}| \le p_{s_i s_j}^{(n),\{b_0\}} + \left| \sum_{r=1}^{n-1} (p_{s_i b_0}^{(r)} - \pi_{b_0}) p_{b_0 s_j}^{(n-r),\{b_0\}} \right| + \pi_{b_0} \sum_{r=n}^{\infty} p_{b_0,s_j}^{(r),\{b_0\}},$$

which implies that there is a constant $C(s_i, s_j)$ such that

$$|p_{s_i s_j}^{(n)} - \pi_{s_j}| < \widehat{C}_{s_i s_j} \widehat{\theta}^n$$

for some $\theta_2 < \widehat{\theta} < 1$. This completes the proof. □

PROPOSITION 4.3. *Let $C_{s_i s_j}$ be a positive constant depending only on s_i and s_j. If there is a finite subset $B_N = \{b_0, b_1, \ldots, b_N\} \subset \mathcal{S}$ of alphabets such that $p_{s_i s_j}^{(n),B_N} \le C_{s_i s_j} \tau^n$ for some $0 < \tau < 1$, then the Markov chain $(\mathcal{S}, p_{ij})$ converges to the stationary distribution exponentially.*

Proof. Since

$$p_{s_i s_j}^{(n),B_{N-1}} = p_{s_i s_j}^{(n),B_N} + \sum_{r=1}^{n-1} f_{s_i b_N}^{(r),B_{N-1}} p_{b_N s_j}^{(n-r),B_{N-1}}, \tag{9}$$

setting $s_i = b_N$, it follows that

$$P_{b_N s_j}^{B_{N-1}}(z) = \frac{P_{b_N s_j}^{B_N}(z)}{1 - F_{b_N b_N}^{B_{N-1}}(z)}.$$

Thus, $P_{b_N s_j}^{B_{N-1}}(z)$ is analytic in the disk C_R for some $R > 1$ and we obtain the estimate $p_{b_N s_j}^{(n),B_{N-1}} < c_{s_j} \widetilde{\tau}^n$ for some $\widetilde{\tau} < 1$. Moreover, $f_{s_i b_N}^{(n),B_{N-1}} = p_{s_i b_N}^{(n),B_N} \le C_{s_i b_N} \tau^n$. Now from (9), we get

$$p_{s_i s_j}^{(n),B_{N-1}} \le \widehat{C}_{s_i s_j} \widehat{\tau}^n$$

for some $\widehat{C}_{s_i s_j} > 0$ and $\widetilde{\tau} < \widehat{\tau} < 1$. Using induction for the sets $B_{N-1}, B_{N-2}, \ldots$, it finally gives us

$$p_{s_i s_j}^{(n),\{b_0\}} \le C'_{s_i s_j} \theta^n$$

for some $\tau < \theta < 1$. Now we can apply the previous Lemma. □

Now we state and prove the main theorem of this section.

THEOREM 4.4. *Let $\mathcal{T}$ be a locally finite tree. Suppose that $(\Gamma, \widetilde{F})$ is a pair of a full discrete subgroup $\Gamma < \mathrm{Aut}(\mathcal{T})$ and a potential $\widetilde{F}$ for Γ with $|m_{\Gamma,F}^{\nu^-,\nu^+}| < \infty$. Let us fix a vertex $o \in V\mathcal{T}$. If $L_\Gamma = k\mathbb{Z}$ and if $(\Gamma, \widetilde{F})$ has a WSG, then, for any $f, g \in C_c(\Gamma \backslash S_o^k \mathcal{T})$,*

$$\left| \int_{\Gamma \backslash S_o^k \mathcal{T}} (f \circ \phi^{\circ kn}) \cdot g \, dm_{\Gamma,F}^{\nu^-,\nu^+} - \int_{\Gamma \backslash S_o^k \mathcal{T}} f \, dm_{\Gamma,F}^{\nu^-,\nu^+} \int_{\Gamma \backslash S_o^k \mathcal{T}} g \, dm_{\Gamma,F}^{\nu^-,\nu^+} \right| = O(\theta^n)$$

for some constant $0 < \theta < 1$. The implied constant depends only on f and g.

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Proof. The dynamical system $(\Gamma\backslash S\mathcal{T}, \phi, m_{\Gamma,F}^{\nu^-,\nu^+})$ is isomorphic to $(X_{(A,i)}, \sigma, \lambda_F)$. Let Z_n be the Markov chain with the data $(\mathcal{S}, p_{ij}, \pi_j)$, where $\mathcal{S} = E(\Gamma\backslash S\mathcal{T})$, $p_{ij} = \lambda_F([s_i, s_j])/\lambda_F([s_i])$ and $\pi_j = \lambda_F([s_j])$. If $L_\Gamma = k\mathbb{Z}$, then $(\Gamma\backslash S_o^k\mathcal{T}, \phi^{\circ k}, m_{\Gamma,F}^{\nu^-,\nu^+})$ is mixing. Therefore, if we let Z_{kn} be the k-step Markov chain on the equivalence of periodic classes in $(\mathcal{S}, p_{ij}, \pi_j)$, then it is irreducible and aperiodic. Since $(\Gamma, \widetilde{F})$ has a WSG, by Lemma 4.1, there is a finite subset B of $\mathcal{S}$ such that $p_{ij}^{(kn),B} \le t_i t_j^{-1}\rho^{kn}$ for some $0 < \rho < 1$. It follows that there exists $C_{ij}, \theta > 0$ such that $|p_{ij}^{(kn)} - \pi_j| < C_{ij}\theta^{kn}$, by Proposition 4.3. Now Proposition 3.2 completes the proof. □

4.2. *Examples of* $\Gamma < \mathrm{Aut}(\mathcal{T})$ *with a WSG.* First, we state and prove the fact about the largeness of the critical exponent δ_Γ of Γ when $\Gamma\backslash\backslash\mathcal{T}$ has a cuspidal ray. This is due to the finiteness of the normalized Patterson density.

PROPOSITION 4.5. *Let $\mathcal{T}$ be a locally finite tree. Let $(\Gamma, \widetilde{F})$ be a pair consisting of non-elementary full discrete subgroup $\Gamma < \mathrm{Aut}(\mathcal{T})$ and a potential $\widetilde{F}$ for Γ, with $|m_{\Gamma,F}^{\nu^-,\nu^+}| < \infty$. If $\Gamma\backslash\backslash\mathcal{T}$ has at least one cuspidal ray (see Definition 2.1) with edges $e_1, e_2, e_3, \ldots$ and indices $r_1, r_2, r_3, \ldots$ (see Figure 3), then*

$$\delta_{\Gamma,F} \ge \tfrac{1}{2}\log \varlimsup_{n\to\infty}(c_n^{1/n}),$$

where $c_n = (r_n - 1)r_{n-1}r_{n-2}\cdots r_1 e^{F(e_1)+F(\overline{e_1})+\cdots F(e_n)+F(\overline{e_n})}$. Furthermore, if the tree $\mathcal{T}$ is regular or biregular and F is constant, then the inequality is strict.

Proof. Let e_i be the edge with $\partial_0 e_i = a_{i-1}$ and $\partial_1 e_i = a_i$. Let $\nu^\pm$ be the Patterson density for $(\Gamma, \widetilde{F}^\pm)$ defined in §2.3 and let $x = \nu_{a_0}^+(\mathcal{O}(\overline{e_1}))$. Since Γ is non-elementary, ν^+ has no atoms (cf. [**3**]) and it follows that $x \ne 0$. Let $G\colon E\mathcal{T} \to \mathbb{R}$ be a Γ-invariant function given by $G(e) = F(e) + F(\overline{e})$. Because of the countable additivity and conformal property of ν^+,

$$\begin{aligned}\nu_{a_0}^+(\mathcal{O}(e_1)) &= (r_1 - 1)\nu_{a_1}^+(\mathcal{O}(\overline{e_1}))e^{F(e_1)-\delta} + (r_2 - 1)r_1\nu_{a_2}^+(\mathcal{O}(\overline{e_1}))e^{F(e_1)+F(e_2)-2\delta} + \cdots\\ &= (r_1 - 1)xe^{G(e_1)-2\delta} + (r_2 - 1)r_1 xe^{G(e_1)+G(e_2)-4\delta} + \cdots\\ &= \sum_{n=1}^{\infty} c_n x e^{-2n\delta} < \infty,\end{aligned}$$

which implies that $\delta_{\Gamma,F} \ge \frac{1}{2}\log\varlimsup_{n\to\infty}(c_n^{1/n})$. If we assume, further, that $\mathcal{T}$ is a $(r+1, s+1)$ biregular tree $\mathcal{T}_{r+1,s+1}$ and that F is constant, then

$$\nu_{a_0}^+(\mathcal{O}(e_1)) = \left[(r-1)xe^{2(F-\delta)} + (s-1)rxe^{4(F-\delta)}\right]\sum_{n=1}^{\infty}(rs)^n e^{4n(F-\delta)},$$

so we obtain $\delta_{\Gamma,F} > F + \frac{1}{2}\log\sqrt{rs}$. □

In the rest of this subsection, we assume that the given potential $\widetilde{F}\colon E\mathcal{T} \to \mathbb{R}$ for $\Gamma < \mathrm{Aut}(\mathcal{T})$ is constant. We give some examples of groups Γ which have a WSG with these potentials.

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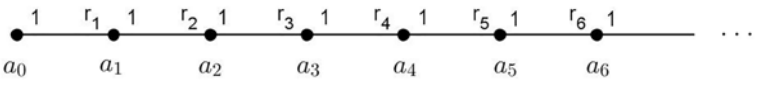


FIGURE 3. Cuspidal ray with indices $r_1, r_2, r_3, \ldots$.

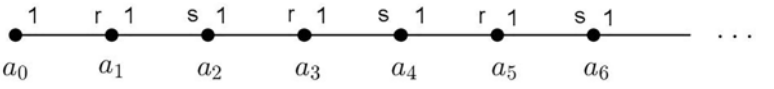


FIGURE 4. Cuspidal ray of $\mathcal{T}_{r+1,s+1}$.

Example 4.1. Let $\mathcal{T}_{r+1,s+1}$ be a $(r+1, s+1)$ biregular tree and let $\Gamma < \mathrm{Aut}(\mathcal{T})$ be a geometrically finite full discrete subgroup with critical exponent δ_Γ for the zero potential. Then we have a decomposition $V(\Gamma\backslash\mathcal{T}) = VD \cup VC_1 \cup \cdots \cup VC_k \cup VF_1 \cup \cdots \cup VF_l$ of the set of vertices, as in Proposition (2.2), and there is a finite set $B \subseteq E(\Gamma\backslash\mathcal{T})$ such that $E(\Gamma\backslash\mathcal{T}) = EB \cup EC_1 \cup \cdots \cup EC_k \cup EF_1 \cup \cdots EF_l$.

Let $\widetilde{F}$ be a given constant potential. Note that $F - \delta_{\Gamma,F} = -\delta_\Gamma$. For a cuspidal ray with the sequence of vertices $a_0, a_1, a_2, \ldots$ (see Figure 4), if we let e_i be the oriented edge with $\partial_0 e_i = a_{i-1}$ and $\partial_1 e_i = a_i$, then

$$p_{e_{2i-1}e_{2i}} = \frac{(s-1)re^{-2\delta_\Gamma} + (r-1)rse^{-4\delta_\Gamma}}{(r-1) + (s-1)re^{-2\delta_\Gamma}} < 1,$$

$$p_{e_{2i}e_{2i+1}} = \frac{(r-1)se^{-2\delta_\Gamma} + (s-1)rse^{-4\delta_\Gamma}}{(s-1) + (r-1)se^{-2\delta_\Gamma}} < 1,$$

$$p_{e_i\overline{e_i}} = 1 - p_{e_ie_{i+1}}$$

and

$$p_{\overline{e_{i+1}}\,\overline{e_i}} = 1.$$

Let $p = p_{e_{2i-1}e_{2i}}$ and $q = p_{e_{2i}e_{2i+1}}$. Define $t\colon E\mathcal{T} \to \mathbb{R}_{\geq 0}$ as follows. Choose $1 < R < \sqrt[4]{pq}$ and t_1 sufficiently large. Let $t(\overline{e}_i) = R^i$ and $t(e_i) = t_i$ with

$$t_{2i} = \frac{1}{p^i q^{i-1} R^{2i-1}}\left(t_1 - (1-p)R^2\sum_{k=1}^{i}(pqR^4)^k - (1-q)pR^4\sum_{k=1}^{i-1}(pqR^4)^k\right)$$

$$t_{2i+1} = \frac{1}{p^i q^i R^{2i}}\left(t_1 - (1-p)R^2\sum_{k=1}^{i}(pqR^4)^k - (1-q)pR^4\sum_{k=1}^{i}(pqR^4)^k\right)$$

for $i \geq 1$. Then,

$$pt(e_{2i}) + (1-p)t(\overline{e_{2i-1}}) = \frac{t(e_{2i-1})}{R},$$

$$qt(e_{2i+1}) + (1-q)t(\overline{e_{2i}}) = \frac{t(e_{2i})}{R} \quad \text{and} \quad t(\overline{e_{i-1}}) = \frac{t(\overline{e_i})}{R},$$

so Γ has a WSG with (t, B, R^{-1}).

Generalizing geometrically finite groups, we consider $\Gamma < \mathrm{Aut}(\mathcal{T})$ for which a quotient graph of groups $\Gamma\backslash\backslash\mathcal{T}$ is a union of a finite graph of groups, finite rays and finite funnels.

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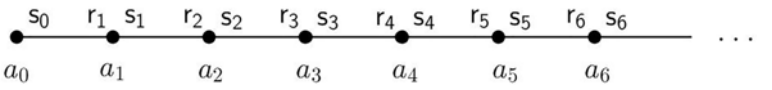


FIGURE 5. General ray.

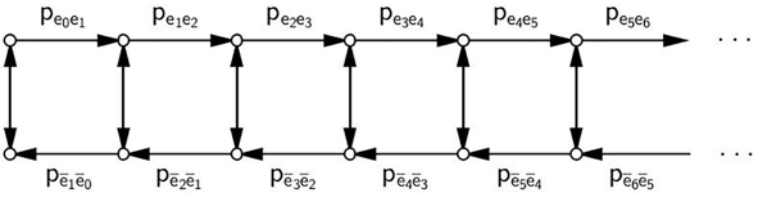


FIGURE 6. Markov chain associated to a ray.

Consider the following ray given in Figure 5. Let e_i be the edge with $\partial_0 e_i = a_{i-1}$ and $\partial_1 e_i = a_i$. Then

$$p_{e_i e_{i+1}} = \frac{|\Gamma_{\xi_i}|\nu_{a_i}^+(\mathcal{O}_{e_{i+1}})}{|\Gamma_{\xi_i,\xi_{i+1}}|\nu_{a_i}^+(\mathcal{O}_{e_i})} = \frac{s_i \cdot \nu_{a_i}^+(\mathcal{O}_{e_{i+1}})}{\nu_{a_i}^+(\mathcal{O}_{e_i})}.$$

Example 4.2. First, consider the case $s_i = 2$ and $r_i = q - 1$. Let $\Gamma < \mathrm{Aut}(\mathcal{T})$ be a full discrete subgroup such that the decomposition of quotient graph of groups $\Gamma\backslash\backslash\mathcal{T}$ consists of a finite graph of groups D, finite funnels and finite rays of type $(2, q-1)$. This is a quotient of a $(q+1)$-regular tree, but is not geometrically finite. There is $\alpha > 0$ such that

$$\alpha < p_{e_i,e_{i+1}} < 1 - \alpha \quad \text{and} \quad \alpha < p_{\overline{e_{i+1}},\overline{e_i}} < 1 - \alpha.$$

For a Markov chain given in Figure 6, we can choose a sequence $t_i \geq 1$ satisfying the following conditions. There exists $0 < \rho < 1$ such that, for any $i = 0, 1, 2, \ldots,$

$$\begin{aligned} &(1)\ \ p_{e_i,e_{i+1}} t_{i+1} + (1 - p_{e_i,e_{i+1}}) t_{i-1} \leq t_i \rho; \quad \text{and} \\ &(2)\ \ p_{\overline{e_{i+1}},\overline{e_i}} t_{i-1} + (1 - p_{\overline{e_{i+1}},\overline{e_i}}) t_{i+1} \leq t_i \rho. \end{aligned}$$

Then Γ has a WSG with (t, ED, ρ).

Example 4.3. Now we consider the case when the ray of groups itself is not even an *expander diagram*. The most famous example appeared in [**2**]. In this example, for each i, either $r_i = 1$ or q and $s_i = q + 1 - r_i$. Although the regular representation of $\mathrm{Aut}(\mathcal{T})$ into $L^2(\Gamma\backslash \mathrm{Aut}(\mathcal{T}))$ has no spectral gap (see [**2**]), we have the exponential mixing property of the geodesic 2-translation map $\phi^{\circ 2}$. Indeed, as in the previous example, we can choose a sequence t_i for $i \geq 0$ such that

$$\begin{aligned} &(1)\ \ p_{e_i,e_{i+1}} t_{i+1} + (1 - p_{e_i,e_{i+1}}) t_{i-1} \leq t_i \rho; \quad \text{and} \\ &(2)\ \ p_{\overline{e_{i+1}},\overline{e_i}} t_{i-1} + (1 - p_{\overline{e_{i+1}},\overline{e_i}}) t_{i+1} \leq t_i \rho. \end{aligned}$$

Now we can apply Lemma 4.1 and Proposition 4.3.

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5. *Decay of matrix coefficient of algebraic groups and effective counting of discrete points*

Before we proceed further, let us summarize here the results of the previous sections. Let us assume that a pair $(\Gamma, \widetilde{F})$ of a full discrete subgroup $\Gamma < \mathrm{Aut}(\mathcal{T})$ and a potential $\widetilde{F}$ for Γ has a WSG for instance, let Γ be a geometrically finite discrete subgroup with given a constant potential. If we assume that the length spectrum of Γ is not arithmetic, then the geodesic translation map is *exponentially* mixing. This is not a strong assumption since, if we consider the restricted dynamical system $(\Gamma\backslash S_o^k\mathcal{T}, \phi^{\circ k}, m_{\Gamma,F}^{\nu^-,\nu^+})$ for $k = |\mathbb{Z}/L_\Gamma|$, then this is always true. A more precise statement is the following.

Let $\mathcal{T}$ be an arbitrary locally finite tree. If $(\Gamma, \widetilde{F})$ has a WSG and the length spectrum of Γ is not arithmetic, then, for any $f, g \in C_c(\Gamma\backslash S\mathcal{T})$, there is a constant $\kappa_1 > 0$ such that

$$\int_{\Gamma\backslash S\mathcal{T}} (f \circ \phi_\Gamma^{\circ n}) \cdot g \, dm_{\Gamma,F}^{\nu^-,\nu^+} = \frac{1}{|m_{\Gamma,F}^{\nu^-,\nu^+}|} m_{\Gamma,F}^{\nu^-,\nu^+}(f) m_{\Gamma,F}^{\nu^-,\nu^+}(g) + O(e^{-\kappa_1 n}).$$

In this section, we focus on the case when $G < \mathrm{Aut}(\mathcal{T})$ consists of $\mathbb{F}$-points of an algebraic group $\underline{G}$ over a local field $\mathbb{F}$ and Γ is a non-elementary discrete subgroup of G. Thus $\mathcal{T}$ is a $(q^d + 1, q^{d'} + 1)$-biregular tree. In this case, the geodesic 2-translation map $\phi^{\circ 2}$ is exponentially mixing on $\Gamma\backslash S_o^2\mathcal{T}$ for every $o \in V\mathcal{T}$. We want to translate the exponential mixing of $\phi_\Gamma^{\circ 2}$ with respect to m_Γ^{BM} into the effective decay of the matrix coefficient of the regular representation of G on $L^2(\Gamma\backslash G)$. Furthermore, this allows us to count discrete orbit points in the Bruhat–Tits tree $\mathcal{T}$ of G effectively.

5.1. *Equidistribution of $\phi_*^{\circ n}\nu_{\mathcal{H}}^{\mathrm{Leb}}$.* In [**13**], the authors gave the following series of propositions in the case of hyperbolic space $\mathbb{H}^n$. For effective versions of these, we refer [**11**, **14**]. Following directly their arguments, we get analogous results about the effective equidistribution of expanding horospheres in $\mathcal{T}$.

We assume, throughout this section, that $L_\Gamma = 2\mathbb{Z}$ and the given potential $\widetilde{F}$ is constant. Let

$$h_{\mathcal{T}} = \lim_{n\to\infty} \frac{\log \#\{x \in V\mathcal{T} \mid d(x, o) \le n\}}{n}$$

be the *volume entropy* (which does not depend on the choice of $o \in V\mathcal{T}$) of $\mathcal{T}$. Let $\{\nu_x\}$ and $\{m_x\}$ be the δ_Γ-dimensional Patterson–Sullivan density and the $h_{\mathcal{T}}$-dimensional Lebesgue density, respectively. We denote

$$m_\Gamma^{\mathrm{BM}} = m^{\nu,\nu}, \quad m_\Gamma^{\mathrm{BR}} = m^{m,\nu}, \quad m_{*,\Gamma}^{\mathrm{BR}} = m^{\nu,m} \quad \text{and} \quad m^{\mathrm{Haar}} = m^{m,m}. \tag{10}$$

For each unstable horosphere $\widetilde{\mathcal{H}}$ in $S\mathcal{T}$, we consider the locally finite Borel measures on $\widetilde{\mathcal{H}}$:

$$d\mu_{\widetilde{\mathcal{H}}}^{\mathrm{Leb}}(\xi) = e^{h_{\mathcal{T}}\beta_{\xi^+}(o,\xi_0)} \, dm_o(\xi^+), \quad d\mu_{\widetilde{\mathcal{H}}}^{\mathrm{PS}}(\xi) = e^{\delta_\Gamma\beta_{\xi^+}(o,\xi_0)} \, d\nu_o(\xi^+)$$

for some $o \in V\mathcal{T}$. The measures $\mu_{\widetilde{\mathcal{H}}}^{\mathrm{Leb}}$ and $\mu_{\widetilde{\mathcal{H}}}^{\mathrm{PS}}$ are invariant under $\Gamma_{\widetilde{\mathcal{H}}} = \mathrm{Stab}_\Gamma(\widetilde{\mathcal{H}})$ and hence induce measures on $\mathcal{H} = \Gamma_{\widetilde{\mathcal{H}}}\backslash\widetilde{\mathcal{H}}$.

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The following proposition gives the equidistribution of the measure $\mu_{\widetilde{\mathcal{H}}}^{\mathrm{PS}}$ towards m_Γ^{BM}.

PROPOSITION 5.1. *For any $f \in C_c(\Gamma\backslash S\mathcal{T})$ and $g \in L^1(\mathcal{H}, \nu_{\mathcal{H}}^{\mathrm{PS}})$, there is a constant $\kappa_2 > 0$ such that*

$$\int_{\mathcal{H}} (f \circ \phi_\Gamma^{\circ 2n}) \cdot g \, d\mu_{\mathcal{H}}^{\mathrm{PS}} = \frac{\mu_{\mathcal{H}}^{\mathrm{PS}}(g)}{|m_\Gamma^{\mathrm{BM}}|} m^{\mathrm{BM}}(f) + O(e^{-\kappa_2 n}),$$

where the implied constant depends only on f and g.

Remark 5.1. In general, when the potential F is not constant, this statement is still true. We can also consider the generalized *skinning measure* $d\sigma_{\widetilde{\mathcal{H}}}(\xi) = e^{C_{F,\xi^+}(o,\xi_0)} \, d\nu_o^+(\xi^+)$ and this equidistributes towards $m_{\Gamma,F}^{\nu^-,\nu^+}$ when pushed by the geodesic translation map [**5**].

THEOREM 5.2. *Let $f \in C_c(\Gamma\backslash S\mathcal{T})$ and $g \in C_c(\mathcal{H})$. Then there is a constant $\kappa_3 > 0$ such that*

$$e^{(h_{\mathcal{T}}-\delta_\Gamma)n} \int_{\mathcal{H}} (f \circ \phi_\Gamma^{\circ 2n}) \cdot g \, d\mu_{\mathcal{H}}^{\mathrm{Leb}} = \frac{\mu_{\mathcal{H}}^{\mathrm{PS}}(g)}{|m_\Gamma^{\mathrm{BM}}|} m^{\mathrm{BR}}(f) + O(e^{-\kappa_3 n}).$$

The implied constant depends only on f and g.

THEOREM 5.3. *Let $f, g \in C_c(\Gamma\backslash S\mathcal{T})$. Then there is a constant $\kappa_4 > 0$ such that*

$$e^{(h_{\mathcal{T}}-\delta_\Gamma)n} \int_{\Gamma\backslash S\mathcal{T}} f \cdot (g \circ \phi_\Gamma^{\circ 2n}) \, dm_\Gamma^{\mathrm{Haar}} = \frac{1}{|m_\Gamma^{\mathrm{BM}}|} \int f \, dm_{*,\Gamma}^{\mathrm{BR}} \int g \, dm_\Gamma^{\mathrm{BR}} + O(e^{-\kappa_4 n}),$$

where the implied constant depends only on f and g.

COROLLARY 5.4. *Let $\underline{G}$ be a semi-simple $\mathbb{F}$-rank-one algebraic group over $\mathbb{F}$ and $G = \underline{G}(\mathbb{F})$. Suppose that the Bruhat–Tits tree $\mathcal{T}$ of G is $(q^d + 1, q^{d'} + 1)$-biregular and that Γ is a discrete subgroup of G with $|m_\Gamma^{\mathrm{BM}}| < \infty$. Let $M < G$ be the stabilizer of a fixed bi-infinite geodesic line $\xi \in S\mathcal{T}$. Then there exists $\kappa > 0$ such that, for any $\Phi_1, \Phi_2 \in C_c(\Gamma\backslash G/M)$,*

$$\frac{q^{(d+d')j}}{e^{2\delta_\Gamma j}} \langle a_{2j} \cdot \Phi_1, \Phi_2 \rangle = \frac{m_\Gamma^{\mathrm{BR}}(\Phi_1) \cdot m_{*,\Gamma}^{\mathrm{BR}}(\Phi_2)}{|m_\Gamma^{\mathrm{BM}}|} + O(e^{-\kappa j}).$$

5.2. *More on Burger–Roblin measure m_Γ^{BR}.* We recall the definition of $\widetilde{m}_\Gamma^{\mathrm{BR}}$ in (10). If $\psi \in C_c(S\mathcal{T})$, then

$$\widetilde{m}_\Gamma^{BR}(\psi) = \int_{(\xi^+,\xi^-,s)\in S\mathcal{T}} \psi(g) e^{h_{\mathcal{T}}\beta_{g\cdot\xi^+}(o,g\cdot o)} e^{\delta\beta_{g\cdot\xi^-}(o,g\cdot o)} \, dm_o(g\xi^+) \, d\nu_o(g\xi^-) \, ds$$

and

$$\widetilde{m}_{*,\Gamma}^{BR}(\psi) = \int_{(\xi^+,\xi^-,s)\in S\mathcal{T}} \psi(g) e^{\delta\beta_{g\cdot\xi^+}(o,g\cdot o)} e^{h_{\mathcal{T}}\beta_{g\cdot\xi^-}(o,g\cdot o)} \, d\nu_o(g\xi^+) \, dm_o(g\xi^-) \, ds.$$

Using Iwasawa decomposition, we may write $g = k_1 p = k_2 q$ for some $k_1, k_2 \in K$, $p = a_{2j}k' \in P^+$ and $q = q_{2l}k'' \in P^-$: for instance, when $G = \mathrm{SL}_2(\mathbb{F})$, we can think of them as $p = a_{2j}k' = \left(\begin{smallmatrix} \pi^j & b \\ 0 & \pi^{-j} \end{smallmatrix}\right) \in P^+$ and $q = a_{2l}k'' = \left(\begin{smallmatrix} \pi^l & 0 \\ c & \pi^{-l} \end{smallmatrix}\right) \in P^-$ for some $b, c \in \mathbb{F}$. Thus

$$\beta_{g\cdot\xi^+}(o, g\cdot o) = \beta_{k_1 p\cdot\xi^+}(o, k_1 p\cdot o) = \beta_{\xi^+}(o, p\cdot o) = -2j$$

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and

$$\beta_{g\cdot\xi^-}(o, g\cdot o) = \beta_{k_2 q\cdot\xi^-}(o, k_2 q\cdot o) = \beta_{\xi^-}(o, q\cdot o) = 2l.$$

Note, also, that if we define the measure $d\lambda_g(n_x)$ on N^+ by $e^{h\tau\beta_{gn_x\cdot\xi^+}(o,gn_x\cdot o)}$ $dm_o(gn_x\xi^+)$, then $d\lambda_g(n_x) = e^{h\tau\beta_{n_x\cdot\xi^+}(g^{-1}\cdot o,n_x\cdot o)}e^{h\tau\beta_{n_x\cdot\xi^+}(o,g^{-1}\cdot o)}\, dm_o(g\xi^+) = d\lambda_e$ (n_x). Since $d\lambda_e(gn_x) = d\lambda_g(n_x) = d\lambda_e(n_x)$, it follows that λ_e is a Haar measure on N^+. We will denote it by $dn = d\lambda_e(n)$ as an integral on N^+ (see [**13**, Lemma 7.1]).

Therefore,

$$\widetilde{m}^{\mathrm{BR}}_{\Gamma}(\psi) = \sum_{l\in\mathbb{Z}}\int_{k_2\in K}\int_{n^-\in N^-/N^-\cap K}\psi(k_2 a_l n^-)e^{2\delta l}\, dn^- dv_o(k_2\xi^-)$$

and, similarly,

$$\widetilde{m}^{\mathrm{BR}}_{*,\Gamma}(\psi) = \sum_{j\in\mathbb{Z}}\int_{k_1\in K}\int_{n\in N^+/N^+\cap K}\psi(k_1 a_j n)e^{-2\delta j}\, dn\, dv_o(k_1\xi^+).$$

5.3. *Definition of the measure $\mathcal{M}_G^{(\Gamma)}$.* Let $G = \underline{G}(\mathbb{F})$ be the $\mathbb{F}$-rational points of a semi-simple rank-one algebraic group over $\mathbb{F}$ whose Bruhat–Tits tree $\mathcal{T}$ is a $(q^d+1, q^{d'}+1)$-biregular tree. We have the natural action of G on $\mathcal{T}$. Let K be the stabilizer of some fixed vertex o in $\mathcal{T}$. Fix a geodesic $\xi \in S\mathcal{T}$ whose base point is o, and let o' be the unique vertex in $\mathcal{H}^+_\xi(1)$ for which $d_{\mathcal{T}}(o, o') = 1$. Let K' be the stabilizer of the vertex o'. Then K and K' are maximal compact subgroups of H and they are not conjugate to each other. In this case, we can identify $V\mathcal{T} \simeq G/K \cup G/K'$.

For a given measurable subset B_R of G, we define the *counting function* F_R on $\Gamma\backslash G \times \Gamma\backslash G$ by

$$F_R(g, h) = \sum_{\gamma\in\Gamma}\chi_{B_R}(g^{-1}\gamma h).$$

Then it follows that $F_R(e, g) = |\Gamma g \cap B_R|$. For any real-valued function $\Phi \in C_c(\Gamma\backslash G)$ and $k \in K$, we set $\Phi^k(g) := \Phi(gk)$ and $\Phi_k(g) := \Phi(kg)$.

LEMMA 5.5. *Suppose that o is a vertex of degree q^d+1 in $\mathcal{T}$. If $f \in C_c(G)$, then*

$$\int_{g=k_1 a_{2j} k_2\in G} f(g)\, dg = \int_{k_1\in K}\sum_{a_{2j}\in Z^+}\int_{k_2\in K} f(k_1 a_{2j} k_2)\Delta(2j)\, dk_2\, dk_1,$$

where $\Delta(0) = 1$, $\Delta(2j) = (q^d+1)q^{(d+d')j-d}$ for $j \geq 1$, and dk_1 and dk_2 denote the probability measures on K.

Proof. Note that $\Delta(2j)$, $(j \geq 0)$ is the number of vertices of a sphere of radius $2j$ (centered at x_0) in a $(q^d+1, q^{d'}+1)$-regular tree. Let f^K be the function on G/K given by $f^K(xK) = \int_{k\in K} f(xk)\, dk$. Let W_0 be a spherical Weyl group so that $K = BW_0B$ corresponds to a vertex $o \in \mathcal{T}$ and dk is a probability measure on K. Now

$$\begin{aligned}\sum_{a_{2j}\in Z^+}\int_{k_1}\int_{k_2} f(k_1 a_{2j} k_2)\Delta(2j)\, dk_1\, dk_2 &= \sum_{a_{2j}\in Z^+}\int_{k_1} f^K(k_1 a_{2j} K)\Delta(2j)\, dk_1\\ &= \sum_{k_1 a_{2j} K\in G/K} f^K(k_1 a_{2j} K).\end{aligned}$$

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Meanwhile, if we denote by $m_{G/K}$ the G-invariant measure on G/K, then

$$\begin{aligned}\int_{g=k_1a_{2j}k_2} f(g)\,dg &= \int_{k_1a_{2j}K\in G/K}\int_{k_2\in K} f(k_1a_{2j}k_2)\,dk_2\,dm_{G/K}\\ &= \int_{k_1a_{2j}K\in G/K} f^K(k_1a_{2j}K)\,dm_{G/K}.\end{aligned}$$

This completes the proof since the counting measure m on $V\mathcal{T}\simeq G/K\cup G/K'$ given by $m(E)=\#E$ is G-invariant. □

Now for any $\Phi_1,\ \Phi_2\in C_c(\Gamma\backslash G)$, using the previous lemma and following the argument of Duke *et al* [**8**],

$$\begin{aligned}&\langle F_R,\ \Phi_1\otimes\Phi_2\rangle_{\Gamma\backslash G\times\Gamma\backslash G}\\ &\quad=\int_{\Gamma\backslash G}\int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma}\chi_{B_R}(x^{-1}\gamma y)\Phi_1(x)\Phi_2(y)\,dx\,dy\\ &\quad=\int_{\Gamma\backslash G}\Phi_1(x)\int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma}\chi_{B_R}(x^{-1}\gamma y)\Phi_2(\gamma y)\,dy\,dx\\ &\quad=\int_{\Gamma\backslash G}\Phi_1(x)\int_G\chi_{B_R}(x^{-1}g)\Phi_2(g)\,dg\,dx\\ &\quad=\int_{g\in B_R}\int_{\Gamma\backslash G}\Phi_1(x)\Phi_2(xg)\,dx\,dg\\ &\quad=\int_{g\in B_R}\langle\Phi_1,\ g\cdot\Phi_2\rangle_{L^2(\Gamma\backslash G)}\,dg\\ &\quad=\int_{k_1a_{2j}k_2\in B_R}\left[\int_{\Gamma\backslash G}\Phi_1(xk_2^{-1})\Phi_2(xk_1a_{2j})\,dx\right]\cdot\Delta(2j)\,dj\,dk_1\,dk_2\\ &\quad=\int_{k_1}\int_{k_2}\sum_{a_{2j}\in k_1^{-1}B_Rk_2^{-1}}(\Delta(2j)\cdot\langle\Phi_1^{k_2^{-1}},\ a_{2j}\cdot\Phi_2^{k_1}\rangle)\,dk_1\,dk_2.\end{aligned}\tag{11}$$

Let $\delta=\delta_\Gamma$ be the critical exponent of Γ, and let ν_o be the Patterson–Sullivan density based at the vertex $o\in\mathcal{T}$ (see §2.3 for the definition). We define the measure $\mathcal{M}_G^{(\Gamma)}$ on G by

$$\begin{aligned}\mathcal{M}_G^{(\Gamma)}(\psi)&=\frac{1}{|m_\Gamma^{\mathrm{BM}}|}\int_{k_1\in K}\int_{k_2\in K}\left(\sum_{j\in\mathbb{N}\cup\{0\}}\frac{e^{2\delta j}}{q^{(d+d')j}}\Delta(2j)\psi(k_1a_{2j}k_2)\right)d\nu_o(k_1\xi^-)\,d\nu_o(k_2\xi^+)\\ &=\frac{1}{|m_\Gamma^{\mathrm{BM}}|}\iint_{k_1,k_2\in K}\left(\psi(k_1k_2)+\frac{q^d+1}{q^d}\sum_{j\in\mathbb{N}}e^{2\delta j}\psi(k_1a_{2j}k_2)\right)d\nu_o(k_1\xi^-)\,d\nu_o(k_2\xi^+)\end{aligned}$$

for any $\psi\in C_c(G)$. For example, if we let $Z_{2R}^+=\{a_{2j}\mid 0\le j\le R\}$, then we get

$$\mathcal{M}_G^{(\Gamma)}(KZ_{2R}^+K)=\left(1+\frac{q^d+1}{q^d}\cdot\frac{e^{2\delta}(e^{2\delta R}-1)}{e^{2\delta}-1}\right)\frac{1}{|m_\Gamma^{\mathrm{BM}}|}.$$

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5.4. *About well-rounded sets.* Using the exponential mixing property of the geodesic translation map $\phi^{\circ 2}\colon S\mathcal{T} \to S\mathcal{T}$, we count the number of points of a Γ-orbit in a well-rounded family of compact sets. Here we give a definition of well roundedness following [**11**].

Definition 5.1. A family of compact sets $\{B_R\} \subseteq G$ is called *effectively well rounded* with respect to Γ if there exists $p > 0$ such that, for any small $\epsilon > 0$ and sufficiently large R, $\mathcal{M}_G^{(\Gamma)}(B_{R,\epsilon}^+ - B_{R,\epsilon}^-) = O(\epsilon^p)\mathcal{M}_G^{(\Gamma)}(B_R)$.

Definition 5.2. [**11**, Definition 6.10] We say that a Borel subset $\Omega \subseteq K$ with $\nu_o(\Omega^{-1}) > 0$ is *admissible* if there exists $0 < s \leq 1$ such that, for all small $\epsilon > 0$,

$$\nu_o\left(\Omega^{-1}K_\epsilon - \bigcap_{k\in K_\epsilon} \Omega^{-1}k\right) \ll \epsilon^s,$$

with the implied constant depending only on Ω. Here, K_ϵ is the ϵ-neighborhood of identity in K with respect to the given left-invariant metric.

Example 5.1. (Admissible bisectors) Let $\Omega_1, \Omega_2 \subset K$ be Borel subsets in K such that Ω_1^{-1} and Ω_2 are admissible. If we write $S_R(\Omega_1, \Omega_2) = \Omega_1 Z_{2R}^+ \Omega_2$, then the family of compact sets $\{S_R(\Omega_1, \Omega_2)\colon R \in \mathbb{N}\}$ is effectively well rounded with respect to Γ.

Proof. First, by the definition of the measure $\mathcal{M}_G^{(\Gamma)}$,

$$\mathcal{M}_G^{(\Gamma)}(S_R(\Omega_1, \Omega_2)) = \frac{1}{|m_\Gamma^{\mathrm{BM}}|}\nu_o(\Omega_1)\nu_o(\Omega_2^{-1}) \sum_{0\leq j\leq R} \frac{\Delta(2j)}{q^{(d+d')j}} e^{2\delta j}.$$

Also, for all $R \in \mathbb{N}$ and small $\epsilon > 0$,

$$S_T(\Omega_1, \Omega_2)G_\epsilon \subset \Omega_1 Z_{2R}^+ \Omega_2^{-1} K_\epsilon$$

because K is an open subgroup of G. (In fact, this implies the strong wavefront property of Cartan decomposition.) Hence

$$\begin{aligned}
\mathcal{M}_G^{(\Gamma)}(S_R(\Omega_1, \Omega_2)G_\epsilon) &\ll \frac{1}{|m_\Gamma^{\mathrm{BM}}|}\nu_o(\Omega_1)\nu_o(\Omega_2^{-1}K_\epsilon) \sum_{0\leq j\leq R} \frac{\Delta(2j)}{q^{(d+d')j}} e^{2\delta j} \\
&\ll \frac{1}{|m_\Gamma^{\mathrm{BM}}|}(1 + O(\epsilon^s))\nu_o(\Omega_1)\nu_o(\Omega_2^{-1}) \sum_{0\leq j\leq R} \frac{\Delta(2j)}{q^{(d+d')j}} e^{2\delta j} \\
&= (1 + O(\epsilon^s))\mathcal{M}_G^{(\Gamma)}(S_R(\Omega_1, \Omega_2))
\end{aligned}$$

for some $s > 0$. Similarly, we also have that

$$\mathcal{M}_G^{(\Gamma)}\left(\bigcap_{g\in G_\epsilon} S_R(\Omega_1, \Omega_2)g\right) = (1 - O(\epsilon^s))\mathcal{M}_G^{(\Gamma)}(S_R(\Omega_1, \Omega_2)) \quad \text{for some } s > 0.$$

Therefore, the family $\{S_R(\Omega_1, \Omega_2)\colon R \in \mathbb{N}\}$ is effectively well rounded with respect to Γ. □

Meanwhile, for a subset $E \in V\mathcal{T}$, the boundary ∂E of E is defined by the set $\{x \in V\mathcal{T} \mid x \notin E \text{ and } \exists\, y \in E\colon d(y, x) = 1\}$. We can prove that if the boundaries of a family of finite sets in $V\mathcal{T}$ are sufficiently small, then the family of finite sets itself is well rounded. This is given more precisely in the following proposition.

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PROPOSITION 5.6. *Let $\{E_R\}$ be a family of finite subsets in $V\mathcal{T} \simeq G/K \cup G/K'$ satisfying the following property: $|E_R| \to \infty$ as $R \to \infty$ and there exists a universal constant $\beta > 1$ such that $|\partial E_R|/|E_R| \le R^{-\beta}$ for every $R \ge 1$. Then $\{E_R\}$ is a well-rounded family.*

Proof. Let $D_R = E_R \cap (G \cdot o)$ and $D'_R = E_R \cap (G \cdot o')$. Taking $U = Ka_2Ka_2^{-1}$,

$$\frac{\mathcal{M}_G^{(\Gamma)}(D_RKU - \cap_{u\in U} D_RKu)}{\mathcal{M}_G^{(\Gamma)}(D_RK)} \le \frac{(q^d+1)\mathcal{M}_G^{(\Gamma)}((\partial D_R)K)}{\mathcal{M}_G^{(\Gamma)}(D_RK)}$$

and, because of the condition and the definition of measure $\mathcal{M}_G^{(\Gamma)}$, it follows that

$$\frac{(q^d+1)\mathcal{M}_G^{(\Gamma)}((\partial D_R)K)}{\mathcal{M}_G^{(\Gamma)}(D_RK)} \le (q^d+1)^2 R^{-\beta}.$$

A similar argument gives us

$$\frac{\mathcal{M}_G^{(\Gamma)}(D'_RK'V - \cap_{v\in V} D'_RK'v)}{\mathcal{M}_G^{(\Gamma)}(D'_RK')} \le (q^{d'}+1)^2 R^{-\beta}$$

for $V = K'a_2K'a_2^{-1}$. These inequalities complete the proof. □

5.5. *Effective counting of discrete points in G.*

THEOREM 5.7. *For an effectively well-rounded family $\{B_R\}$ of compact subsets of G, there is a constant $\eta > 0$ for which*

$$|\Gamma g \cap B_R| = \mathcal{M}_G^{(\Gamma)}(B_R) + O(\mathcal{M}_G^{(\Gamma)}(B_R)^{1-\eta}).$$

Proof. For an element $g = a_l n^- k_1 \in ZN^-K$ define $\kappa^-(g) = k_1$ and $H^-(g) = l$. Similarly, if $g = a_{l'} n^+ k_2 \in ZN^+K$, then we define $\kappa^+(g) = k_2$ and $H^+(g) = l'$. We have an integral formula

$$\int_{g=a_l n^- k} f(g)\, dg = \int_{k\in K} \int_{n^-\in N^-/(N^-\cap K)} \sum_{l=-\infty}^{\infty} f(a_l n^- k_1)\, dn^- dk_1.$$

Let $\Phi_\epsilon \in C_c(G)$ be a continuous function whose support is a compact subset of an ϵ-neighborhood of e in G. By equation (11),

$$\begin{aligned}\langle F^+_{B_{R,\epsilon}}, \Phi_\epsilon \otimes \Phi_\epsilon\rangle &= \int_{g\in B^+_{R,\epsilon}} \langle \Phi_\epsilon, g\cdot\Phi_\epsilon\rangle_{L^2(\Gamma\backslash G)}\, dm^{\mathrm{Haar}}(g)\\ &= \int_{k_1 a_{2j} k_2 \in B_{R,\epsilon}} \langle \Phi_\epsilon^{k_1^{-1}}, a_{2j}\cdot\Phi_\epsilon^{k_2}\rangle_{L^2(\Gamma\backslash G)}\, dm^{\mathrm{Haar}}(g).\end{aligned}$$

Now, by Corollary 5.4, this is asymptotically equivalent to

$$\begin{aligned}&\frac{1}{|m_\Gamma^{\mathrm{BM}}|}\int_{k_1 a_{2j} k_2\in B_{R,\epsilon}} \frac{e^{2\delta j}}{q^{(d+d')j}} m_\Gamma^{\mathrm{BR}}(\Phi_\epsilon^{k_1^{-1}}) m_{*,\Gamma}^{\mathrm{BR}}(\Phi_\epsilon^{k_2})\, dm^{\mathrm{Haar}}(g)\\ &= \frac{1}{|m_\Gamma^{\mathrm{BM}}|}\int_{k_1\in K}\int_{k_2\in K} \sum_{a_{2j}\in k_1^{-1}B^+_{R,\epsilon}k_2^{-1}\cap Z^+} \frac{e^{2\delta j}\Delta(2j)}{q^{(d+d')j}} m_\Gamma^{\mathrm{BR}}(\Phi_\epsilon^{k_1^{-1}}) m_{*,\Gamma}^{\mathrm{BR}}(\Phi_\epsilon^{k_2})\, dk_1\, dk_2.\end{aligned}$$

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The formula of Burger–Roblin measure given in §5.2 implies that this equation is equal to

$$\frac{1}{|m_\Gamma^{\mathrm{BM}}|}\int_{k_1\in K}\int_{k_2\in K}\sum_{a_{2j}\in k_1^{-1}B^+_{R,\epsilon}k_2^{-1}\cap Z^+}\frac{e^{2\delta j}\Delta(2j)}{q^{(d+d')j}}$$
$$\times\left[\int_{k\in K}\int_{n^-\in N^-/N^-\cap K}\sum_{l=-\infty}^{\infty}(\Phi_\epsilon(ka_ln^-k_1^{-1})e^{2\delta l})\,dn^-dv_o(k\xi^-)\right]$$
$$\times\left[\int_{k'\in K}\int_{n^+\in N^+/N^+\cap K}\sum_{l'=-\infty}^{\infty}(\Phi_\epsilon(k'a_{l'}n^+k_2)e^{-2\delta l'})\,dn^+dv_o(k'\xi^+)\right]dk_1\,dk_2.$$

Taking $g_1=a_ln^-k_1$ and $g_2=a_{l'}n^+k_2$, and by the well roundedness of $\{B_R\}$, we can conclude that

$$\langle F^+_{B_{R,\epsilon}},\Phi_\epsilon\otimes\Phi_\epsilon\rangle\sim\frac{1}{|m_\Gamma^{\mathrm{BM}}|}\int_{k\in K}\int_{k'\in K}\int_{g_1\in G}\int_{g_2\in G}\Phi_\epsilon(kg_1)e^{2\delta H^-(g_1)}\Phi_\epsilon(k'g_2)e^{-2\delta H^+(g_2)}$$
$$\times\sum_{a_{2j}\in\kappa(g_1)^{-1}B^+_{R,\epsilon}\kappa(g_2)^{-1}\cap Z^+}\frac{e^{2\delta j}\Delta(2j)}{q^{(d+d')j}}\,dg_1\,dg_2\,dv_o(k\xi^-)\,dv_o(k'\xi^+)$$
$$=\frac{1}{|m_\Gamma^{\mathrm{BM}}|}\int_k\int_k\int_{g_1}\int_{g_2}\sum_{a_{2j}\in\kappa(k^{-1}g_1)^{-1}B^+_{R,\epsilon}\kappa(k'^{-1}g_2)^{-1}\cap Z^+}\frac{e^{2\delta j}\Delta(2j)}{q^{(d+d')j}}$$
$$dg_1\,dg_2\,dv_o(k\xi^-)\,dv_o(k'\xi^+)$$
$$\le\frac{1}{|m_\Gamma^{\mathrm{BM}}|}\int_{k\in K}\int_{k'\in K}\sum_{a_{2j}\in k_1^{-1}B^+_{R,c\epsilon}k_2^{-1}\cap Z^+}\frac{e^{2\delta j}\Delta(2j)}{q^{(d+d')j}}\,dv_o(k\xi^-)\,dv_o(k'\xi^+)$$
$$=\mathcal{M}_G^{(\Gamma)}(B^+_{R,c\epsilon})=(1+O(\epsilon^p))\mathcal{M}_G^{(\Gamma)}(B_R).$$

By a similar argument, $\langle F^-_{B_{R,\epsilon}},\Phi_\epsilon\otimes\Phi_\epsilon\rangle=(1-O(\epsilon^p))\mathcal{M}_G^{(\Gamma)}(B_R)$. Now the inequalities $\langle F^-_{B_{R,\epsilon}},\Phi_\epsilon\otimes\Phi_\epsilon\rangle\le F_{B_R}(e,g)=|\Gamma g\cap B_R|\le\langle F^+_{B_{R,\epsilon}},\Phi_\epsilon\otimes\Phi_\epsilon\rangle$ give us the result. □

For a subset $\Omega\subset\partial_\infty\mathcal{T}$ and $x\in V\mathcal{T}$, we denote the set of all points lying in geodesics emanating from x towards Ω by $S_x(\Omega)\subset\mathcal{T}$, and we denote the ball of radius n centered at x by $B(x,n)\subset V\mathcal{T}$.

COROLLARY 5.8. *Let x be a degree q^d+1 vertex of the Bruhat–Tits tree $\mathcal{T}$ of G and let Γ be a discrete subgroup of G with $|m_\Gamma^{\mathrm{BM}}|<\infty$ having a WSG. Then*

$$\#\{\gamma\in\Gamma\colon\gamma\cdot x\in S_x(\Omega)\cap B(x,2n)\}=\frac{e^{2\delta_\Gamma}(q^d+1)(e^{2\delta_\Gamma n}-1)|\Gamma_x|}{q^d(e^{2\delta_\Gamma}-1)\|m_\Gamma^{\mathrm{BM}}\|}\nu_x(\Omega)+O(e^{(2\delta_\Gamma-\kappa)n})$$

for some $\kappa>0$.

Using the arguments of [**5**, **16**], one can obtain the error rate on the number of edge paths of length at most n in $\Gamma\backslash\backslash\mathcal{T}$ with weights $e^{\int_x^{\gamma y}\widetilde{F}}$ using the equidistribution of the skinning measure $d\sigma_{\widetilde{\mathcal{H}}}$ (see Remark 5.1). The following corollary gives the precise statement.

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COROLLARY 5.9. [**5**] *Let x be a degree $q^d + 1$ vertex of the Bruhat–Tits tree $\mathcal{T}$ of G and suppose that $(\Gamma, \widetilde{F})$ is a pair consisting of discrete subgroup $\Gamma < G$ and a potential $\widetilde{F}$ for Γ with a WSG. Let $\mathcal{N}_x(n)$ be the weighted counting function of the closed path in $\Gamma\backslash\backslash\mathcal{T}$ of length at most n with base point Γx. In other words, let*

$$\mathcal{N}_x(n) = \sum_{\gamma\,:\, d_{\mathcal{T}}(\gamma x, x) \le n} e^{\int_x^{\gamma x} \widetilde{F}}.$$

Then as $n \to \infty$,

$$\mathcal{N}_x(2n) = \frac{e^{2\delta_{\Gamma,F}} \|\nu_x^-\| \|\nu_x^+\| |\Gamma_x|}{(e^{2\delta_{\Gamma,F}} - 1)\|m_{\Gamma,F}^{\nu^-,\nu^+}\|} e^{2n\delta_{\Gamma,F}} + O(e^{(2\delta_{\Gamma,F}-\kappa)n})$$

for some $\kappa > 0$.

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