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Partitioning Matroids with Only Small Cocircuits

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We show that, for every positive integer c^*, there is an integer n such that, if M is a matroid whose largest cocircuit has size c^*, then $E(M)$ can be partitioned into two sets E_1 and E_2 such that every connected component of each of $M|E_1$ and $M|E_2$ has at most n elements.

1. Introduction

The common theme of graph colourings is to partition the set of vertices or the set of edges of a graph into the smallest possible number of classes (colours) so that every subgraph induced by such a class, that is, every monochromatic subgraph, has only trivial (one-vertex or one-edge) connected components. One possible way to generalize such colourings is to allow the monochromatic subgraphs to have nontrivial connected components that are of bounded size. Indeed, several such results have recently appeared in the literature. Thomassen [7] proved the following.

Theorem 1.1. *The edges of every graph whose maximum vertex degree is at most three can be partitioned into two classes, each of which consists of a disjoint union of paths of length at most five.* □

In [2], the author and others independently obtained a weaker version of Theorem 1.1 in which the length of the paths is bounded by seven. The same paper proved that no such decomposition into two graphs with bounded-size components can be guaranteed for graphs whose maximum vertex degree is six, even if the graphs are additionally assumed to be planar. More precisely, for a positive integer n, let T_n be the graph whose vertices are the triples of nonnegative integers summing to n, with an edge connecting two triples

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if they agree in one coordinate and differ by one in each of the other two coordinates. The graph T_n may be viewed as being embedded in the plane $x + y + z = n$ in $\mathbb{R}^3$, where the name of each vertex forms its coordinates, and the edges are straight line segments. Clearly, each T_n is planar and has maximum degree at most six. Paper [2] contains the following.

Theorem 1.2. *If $\{E_1, E_2\}$ is a partition of $E(T_n)$, then at least one of E_1 and E_2 induces a subgraph of T_n with a connected component on at least n edges.* □

Even though the existence of an edge partition into two graphs with bounded-size components cannot be guaranteed for planar graphs with bounded maximum degree, it can always be found for graphs for which both the tree-width and maximum degree are bounded. Roughly speaking, tree-width measures how close a graph is to being a tree. We now define this more precisely. Let k be a positive integer. A *k-clique* is a complete graph or subgraph. A *k-tree* is a graph defined inductively as follows. A k-clique is a k-tree. If G is a k-tree, and K is a k-clique of G, then the graph obtained from G by adding a new vertex and joining it by new edges to all vertices of K is a k-tree. Any subgraph of a k-tree is a *partial k-tree*. The *tree-width* of a graph G is zero if G is edgeless; otherwise it is the smallest integer k such that G is a partial k-tree. The result from [2] mentioned above can now be stated as follows.

Theorem 1.3. *Let k and Δ be positive integers, and let G be a graph whose tree-width is at most k and whose maximum vertex degree is at most Δ. Then the edges of G can be partitioned into two sets E_1 and E_2 such that every connected component of the subgraph of G induced by one of E_1 and E_2 has at most $24k\Delta(\Delta + 1)$ vertices.* □

In this paper, we investigate an analogue of Theorem 1.3 for matroids. Since most terms used in this theorem apply to graphs, their matroid analogues must be discussed before a matroid version of Theorem 1.3 can be stated. This paper uses only basic matroid terminology. For reference, the reader may consult [3].

First, we note that a 'connected component' (or *component*) of a matroid is analogous to a 'block' of a graph, rather than to a 'connected component of a graph'. The last concept has no matroid equivalent. A matroid analogue of a vertex in a graph is a *non-separating cocircuit*, that is, a cocircuit whose deletion does not increase the number of components of the matroid. A matroid analogue of tree-width, which is called branch-width, is based on the concept of branch-width introduced for hypergraphs in [5] and can be defined as follows.

Let M be a matroid with rank function r. If M has fewer than two elements, then the *branch-width* of M is zero. If M has two or more elements, we proceed as follows. A tree T is *ternary* if every vertex of T has degree three or degree one. In the latter case, the vertex is a *leaf* of T. A *branch decomposition* of M is a pair (T, φ), where T is a ternary tree and φ is a bijection from $E(M)$ to the set of leaves of T. For every edge e of T, let $\{X_e, Y_e\}$ denote the partition of $E(M)$ determined by the components of $T \backslash e$, that is, two elements a and b of M belong to the same part of the partition if and only if $\varphi(a)$ and

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$\varphi(b)$ are in the same component of $T\backslash e$. The *width* of the branch decomposition is the maximum taken over all edges e of T of $r(X_e) + r(X_e) - r(M) + 1$. The *branch-width* of M is the minimum width of a branch decomposition of M. The result of [5] that relates tree-width of hypergraphs to their branch-width can easily be translated to the following.

Theorem 1.4. *Let w denote the tree-width of a graph G and let k denote the branch-width of the cycle matroid of G. Then $k \leqslant w + 1 \leqslant \max(\lfloor (3/2)k \rfloor, 2)$.* □

The following question is motivated by Theorem 1.3.

Question 1.5. *Is there a function $f(k)$ such that, if M is a matroid whose branch-width is at most k and all of whose non-separating cocircuits have size at most k, then $E(M)$ can be partitioned into two sets E_1 and E_2 so that every component of each of $M|E_1$ and $M|E_2$ has at most $f(k)$ elements?*

In general, the answer to Question 1.5 is negative: it is easy to see that no such function can be found for uniform matroids of the form $U_{3,n}$. For graphic matroids, it follows from Theorems 1.3 and 1.4 that Question 1.5 has an affirmative answer, but for many other classes of matroids, such as the class of binary matroids, the question remains open.

In this paper, we investigate a modified version of Question 1.5, in which all cocircuits, not just the non-separating ones, have bounded size. The following is the main result of this paper.

Theorem 1.6. *For every positive integer c^*, there is an integer n such that, if M is a matroid whose largest cocircuit has size c^*, then $E(M)$ can be partitioned into two sets E_1 and E_2 such that every connected component of each of $M|E_1$ and $M|E_2$ has at most n elements.*

An anonymous referee pointed out that bounding the maximum size of a cocircuit in a matroid results in bounding the matroid's branch-width. More precisely, the following proposition can easily be obtained using Lemma 2.1 and the fact that the branch-width of a matroid equals the maximum branch-width of its components.

Proposition 1.7. *If M is a matroid with positive rank and largest cocircuit size c^*, then the branch-width of M is at most $\binom{c^*+1}{2}$.* □

The next section contains some auxiliary lemmas, while the section that follows contains a proof of Theorem 1.6.

2. Auxiliary lemmas

The proof of Theorem 1.6 has been motivated by the following result of Seymour [6] (for a proof of the dual result, see [1]).

Lemma 2.1. *If C^* is a largest cocircuit of a connected matroid M, then all cocircuits of $M\backslash C^*$ have at most $|C^*| - 1$ elements.* □

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The second lemma needed for our proof is the following bound on the number of elements in a matroid. It was communicated to the author by Oxley [4], although Oxley suspects it is unlikely to be new.

Lemma 2.2. *Let M be a loopless matroid of rank r and largest cocircuit size c^*. Then M has at most rc^* elements.*

Proof. Take a basis B of M. For each e in B, consider the fundamental cocircuit $C^*(e, B^*)$ where $B^* = E(M) - B$. Now take an element f in B^*. Since M is loopless, the fundamental circuit $C(f, B)$ contains some element g of B. Since no circuit can intersect a cocircuit in exactly one element, $|C(f,B) \cap C^*(g, B^*)| \neq 1$, and, consequently, $f \in C^*(g, B^*)$. It follows that $E(M) = \bigcup_{e \in B} C^*(e, B^*)$. Therefore $|E(M)| \leqslant \sum_{e \in B} |C^*(e, B^*)| \leqslant rc^*$, as required. □

Since the rank of a set X of elements of a matroid M is the same as the rank of its closure $\mathrm{cl}_M(X)$, the following corollary follows immediately from the preceding lemma.

Corollary 2.3. *If M is a loopless matroid all of whose cocircuits have size at most c^*, and X is an m-element subset of $E(M)$, then $\mathrm{cl}_M(X)$ has at most mc^* elements.* □

The following is the last, and the most technical, of the auxiliary lemmas.

Lemma 2.4. *Suppose M is a matroid and let c^* denote the size of a largest cocircuit of M. For every subset X of $E(M)$, there is a subset $Y(M, X)$ of $E(M)$ such that:*

(i) $X \cap Y(M, X) = \emptyset$;

(ii) *for every component N of M, the number of elements in $Y(M, X) \cap E(N)$ is at most $|X \cap E(N)|c^*$; and*

(iii) *if N is a component of $M \backslash Y(M, X)$ that meets X, then $E(N)$ is contained in $\mathrm{cl}_M(X \cap E(N))$.*

Proof. Let B be a basis of M such that $X \cap B$ spans X. For every element $x \in X \cap B$, let C_x^* denote the cocircuit of M disjoint from $B - x$. Put

$$Y(M, X) = \left(\bigcup_{x \in X \cap B} C_x^* \right) - X.$$

It is clear that (i) holds.

To see (ii), suppose that y_1 and y_2 are two elements of $Y(M, X)$. For each $i \in \{1, 2\}$, let x_i be an element of $X \cap B$ for which $C_{x_i}^*$ contains y_i. If x_1 and x_2 are in the same component, say, N, of M, then each of $C_{x_1}^*$ and $C_{x_2}^*$ is also contained in $E(N)$, and, consequently, both y_1 and y_2 are in $E(N)$. Therefore, $Y(M, X) \cap E(N) \subseteq \bigcup_{x \in X \cap E(N)} C_x^*$, and (ii) follows.

Suppose now that (iii) fails. Then $M \backslash Y(M, X)$ has a circuit that meets X but is not contained in $\mathrm{cl}_M(X)$. From among all such circuits, pick one, say, C, for which

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$|C \cap (X - B)|$ is the minimum. Suppose x is an element of $C \cap (X - B)$. Then let C_x be the circuit contained in $(X \cap B) \cup x$. Let y be an element of C that is not in $\mathrm{cl}_M(X)$. By the strong circuit elimination axiom, M has a circuit C' that includes y and is contained in $(C_x \cup C) - x$. Clearly, C' meets X, avoids $Y(M,X)$, and is not contained in $\mathrm{cl}_M(X)$. But $|C' \cap (X - B)| < |C \cap (X - B)|$, which contradicts the choice of C. Consequently, $C \cap (X - B) = \emptyset$.

By the choice of C, the set $X \cap C$ contains an element b, which is also a member of B. We shall examine the set $C \cap C_b^*$. Clearly, this set contains b. Suppose a is an element of $C \cap C_b^*$ other than b. Then $a \in X$, since $C_b^* - X \subseteq Y(M,X)$ and $Y(M,X)$ avoids C. But then $a \in B$ and $a \notin C_b^*$, as C_b^* is disjoint from $B - b$; a contradiction. Hence $C \cap C_b^* = \{b\}$, which is impossible since no circuit can intersect a cocircuit in exactly one element. This contradiction proves (iii). □

3. Proof of Theorem 1.6

Suppose M is a loopless matroid and X is a subset of $E(M)$. Let d denote the size of a largest cocircuit of $M\backslash X$ and let $N_1, N_2, \ldots, N_{n(d)}$ be the list of components of $M\backslash X$ that contain a cocircuit of size d. For each $i \in \{1,2,\ldots,n(d)\}$, let C_i^* denote a size d cocircuit of N_i and let $Z_d(M,X) = Y(M,X) \cup C_1^* \cup C_2^* \cup \ldots \cup C_{n(d)}^*$, where $Y(M,X)$ is a set from Lemma 2.4. It is worth noting that, in general, the set $Z_d(M,X)$ is not uniquely determined by M and X.

Since each loop in a matroid forms a one-element component, it suffices to prove the theorem for loopless matroids. Let M be a loopless matroid and let c^* denote the size of a largest cocircuit of M. We define sequences of matroids and sets as follows. Let $M_{c^*} = M$ and let $X_{c^*} = Z_{c^*}(M_{c^*}, \emptyset)$. Suppose that M_i and X_i have been defined for some $i \in \{c^*, c^* - 1, \ldots, 2\}$. Let $M_{i-1} = M_i\backslash X_i$ and let $X_{i-1} = Z_{i-1}(M_i, X_i)$. The following observation follows immediately from Lemma 2.1.

$$\text{For each } i \in \{1,2,\ldots,c^*\}, \text{ no cocircuit of } M_i \text{ has more than } i \text{ elements.} \tag{3.1}$$

It is easy to show using (3.1) that the sets $X_{c^*}, X_{c^*-1}, \ldots, X_1$ partition $E(M)$. We will now establish an upper bound on the number of elements that may be common to X_k and a fixed component of M_k. More precisely, for each k in $\{1,2,\ldots,c^*\}$, we show the following:

$$\text{If } N \text{ is a component of } M_k, \text{ then } |X_k \cap E(N)| \leqslant f(k,c^*), \text{ where} \tag{3.2}$$

$$f(k,c^*) = \sum_{i=k}^{c^*} i \prod_{j=k+1}^{i} j.$$

We prove this inductively, starting with $k = c^*$ and proceeding down to $k = 1$. Note that each component of X_{c^*} is a c^*-element cocircuit, and that $f(c^*,c^*) = c^*$. Consequently, (3.2) holds for $k = c^*$.

For carrying out the inductive step, the following recurrence relation for the function f

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will be helpful:

$$\begin{aligned} f(m,c^*) &= \sum_{i=m}^{c^*} i \prod_{j=m+1}^{i} j \qquad (3.3)\\ &= \left(\sum_{i=m+1}^{c^*} i \prod_{j=m+1}^{i} j\right) + m \\ &= \left(\sum_{i=m+1}^{c^*} i \prod_{j=m+2}^{i} j\right)(m+1) + m \\ &= f(m+1,c^*)(m+1) + m. \end{aligned}$$

Suppose now that m is an integer such that $1 \leqslant m < c^*$ and (3.2) holds for all k in $\{m+1, m+2, \ldots, c^*\}$. Let N be a component of M_m, and let N' be the component of M_{m+1} that contains N. Then

$$\begin{aligned} |Y(M_{m+1}, X_{m+1}) \cap E(N)| &\leqslant |Y(M_{m+1}, X_{m+1}) \cap E(N')| \qquad (3.4)\\ &\leqslant |X_{m+1} \cap E(N')|(m+1) \\ &\leqslant f(m+1,c^*)(m+1), \end{aligned}$$

where the last inequality follows from the inductive hypothesis, and the preceding one follows from Lemma 2.4(ii) and (3.1) Observe now that $Z_m(M_{m+1}, X_{m+1})$ is obtained from $Y(M_{m+1}, X_{m+1})$ by adjoining some m-element cocircuits, each of which is in a different component of M_m. Since N is a component of M_m, it can meet at most one of these cocircuits. Upon combining the above observations with (3.1) and (3.4), we conclude that $|X_m \cap E(N)| = |Z_m(M_{m+1}, X_{m+1}) \cap E(N)| \leqslant f(m+1,c^*)(m+1) + m$, and the desired bound on $|X_m \cap E(N)|$ follows from (3.3).

Let E_1 be the union of the sets X_i for the odd indices i, and let E_2 be the union of these sets for the even indices. Let N be a component of $M|E_1$ or of $M|E_2$, and let $n = f(1,c^*)$. We shall show that $|E(N)| \leqslant n$.

Let k be the largest integer such that $E(N)$ meets X_k. Then $E(N) \subseteq E(M_k)$. If $k = 1$, then $|E(N)| = 1$ and the theorem follows, since each component of M_1 is a coloop. We may therefore assume that $k > 1$. Then $E(N)$ avoids X_{k-1}, which contains $Y(M_k, X_k)$. Let N' denote the component of $M_k \backslash Y(M_k, X_k)$ that contains N. From Lemma 2.4(iii), we conclude that $E(N') \subseteq \mathrm{cl}_{M_k}(X_k \cap E(N'))$. Note that, by (3.2), the set $X_k \cap E(N')$ has at most $f(k,c^*)$ elements. Then Corollary 2.3 asserts that $|\mathrm{cl}_{M_k}(X_k \cap E(N'))| \leqslant f(k,c^*)k$.

Note that the recurrence relation (3.3) implies that $f(1,c^*) \geqslant f(k,c^*)k!$ for each positive integer k. Consequently, $|E(N)| \leqslant |E(N')| \leqslant |\mathrm{cl}_{M_k}(X_k \cap E(N'))| \leqslant f(k,c^*)k \leqslant f(1,c^*) = n$, as required. □

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