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Nonlinear Stability and B-convergence of Additive Runge-Kutta Methods for Nonlinear Stiff Problems

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Abstract. In this paper, we are devoted to nonlinear stability and B-convergence of additive Runge-Kutta (ARK) methods for nonlinear stiff problems with multiple stiffness. The concept of $(\theta,\bar{p},\bar{q})$-algebraic stability of ARK methods for a class of stiff problems $K_{\sigma,\tau}$ is introduced, and it is proven that this stability implies some contractive properties of the ARK methods. Some results on optimal B-convergence of ARK methods for $K_{\sigma,0}$ are given. These new results extend the existing ones of RK methods and ARK methods in the references. Numerical examples test the correctness of our theoretical analysis.

AMS subject classifications: 65L08, 65L20

Key words: Stiff problem, additive Runge-Kutta method, implicit-explicit method, B-convergence, algebraic stability.

1 Introduction

Consider the initial value problems of stiff ordinary differential equations

$$\begin{cases} y'(t)=f(t,y)=f^{[1]}(t,y(t))+\cdots+f^{[N]}(t,y(t)), \quad t\in[0,T], \\ y(0)=y_0, \end{cases} \tag{1.1}$$

where $y(t),y_0\in R^m$, f and $f^{[i]}:[0,T]\times R^m\to R^m$ $(i=1,2,\cdots,N)$ are sufficiently smooth vector functions with multiple stiffness. Assume that the problems (1.1) satisfy

$$\begin{aligned} &2\langle f^{[i]}(t,y)-f^{[i]}(t,\tilde{y}),y-\tilde{y}\rangle \\ &\leq\sigma_i\|y-\tilde{y}\|^2+\tau_i\|f^{[i]}(t,y)-f^{[i]}(t,\tilde{y})\|^2, \quad i=1,2,\cdots,N, \quad \forall y,\tilde{y}\in R^m, \end{aligned} \tag{1.2}$$

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where σ_i, τ_i $(i=1,2,\cdots,N)$ are real numbers, the norm $\|\cdot\|$ is induced by the standard inner product $\langle\cdot,\cdot\rangle$ on R^m. Let $\sigma=[\sigma_1,\sigma_2,\cdots,\sigma_N]$, $\tau=[\tau_1,\tau_2,\cdots,\tau_N]$. The class of all problems (1.1) satisfying the condition (1.2) is called the class $K_{\sigma,\tau}$. We assume that the true solution $y(t)$ of (1.1) is unique and sufficiently smooth.

In this paper, the symbol $G\geq 0$ ($G>0$) means that the matrix G is non-negative definite (positive definite), the symbol $x\geq\widetilde{x}$ ($x>\widetilde{x}$) means that the vectors $x=[x_1,x_2,\cdots,x_k]^T$, $\widetilde{x}=[\widetilde{x}_1,\widetilde{x}_2,\cdots,\widetilde{x}_k]^T\in R^k$ satisfy $x_i\geq\widetilde{x}_i$ ($x_i>\widetilde{x}_i$), $i=1,2,\cdots,k$. $\|G\|$ denotes the norm of the matrix G, which is subject to the vector norm $\|\cdot\|$; $\mu(G)$ denotes the logarithmic matrix norm of G.

For the class $K_{\sigma,\tau}$ being non-empty, it is easy to prove that $\sigma_j\tau_j\leq 1$ when $\tau_j\leq 0$ for $j\in\{1,2,\cdots,N\}$. In fact, if $K_{\sigma,\tau}$ is not empty, then there exists an initial value problem belonging to $K_{\sigma,\tau}$ with the vector function $f^{[i]}(t,y)$ satisfying the condition (1.2), $i=1,2,\cdots,N$. Obviously, $\sigma_j\tau_j\leq 0$ when $\tau_j=0$ or $\tau_j<0$, $\sigma_j\geq 0$ for $j\in\{1,2,\cdots,N\}$. When $\tau_j<0$, $\sigma_j<0$, we have

$$\begin{aligned}
&\sigma_j\|y-\widetilde{y}\|^2+\tau_j\|f^{[j]}(t,y)-f^{[j]}(t,\widetilde{y})\|^2\\
\geq&2\langle f^{[j]}(t,y)-f^{[j]}(t,\widetilde{y}),y-\widetilde{y}\rangle\\
\geq&-2\|y-\widetilde{y}\|\cdot\|f^{[j]}(t,y)-f^{[j]}(t,\widetilde{y})\|\\
\geq&\frac{1}{\tau_j}\|y-\widetilde{y}\|^2+\tau_j\|f^{[j]}(t,y)-f^{[j]}(t,\widetilde{y})\|^2,
\end{aligned}$$

for $\forall t\geq 0$, $\forall y,\widetilde{y}\in R^m$, $y\neq\widetilde{y}$. Thus $\sigma_j\geq 1/\tau_j$, i.e., $\sigma_j\tau_j\leq 1$. This fact for the class $K_{\sigma,\tau}$ with $N=1$ was shown in [25]. Therefore, in this paper, we further assume that the class $K_{\sigma,\tau}$ satisfies the conditions $\sigma_i\tau_i\leq 1$, $i=1,2,\cdots,N$.

In [25], some properties of the class $K_{\sigma,\tau}$ with $N=1$ are given. Now, we extend them to the case $N>1$.

Lemma 1.1. *If $\sigma<0$, then*

$$K_{\sigma,\tau}\subset K_{\sigma-\varepsilon,\tau+\widetilde{M}(\varepsilon,\sigma,\tau)}\quad \textit{for}\ \ \forall\varepsilon=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_N)^T\geq 0,$$

where

$$\widetilde{M}(\varepsilon,\sigma,\tau)=(\varepsilon_1\overline{M}(\sigma_1,\tau_1),\varepsilon_2\overline{M}(\sigma_2,\tau_2),\cdots,\varepsilon_N\overline{M}(\sigma_N,\tau_N))^T,\quad \overline{M}(\sigma_i,\tau_i)=\Big(\frac{1+\sqrt{1-\sigma_i\tau_i}}{\sigma_i}\Big)^2.$$

Proof. For any problems belonging to the class $K_{\sigma,\tau}$, Eqs. (1.1)-(1.2) yield

$$\sigma_i\|y-\widetilde{y}\|^2+2\|y-\widetilde{y}\|\|f^{[i]}(t,y)-f^{[i]}(t,\widetilde{y})\|+\tau_i\|f^{[i]}(t,y)-f^{[i]}(t,\widetilde{y})\|^2\geq 0,$$

and

$$|\sigma_i\|y-\widetilde{y}\|+\|f^{[i]}(t,y)-f^{[i]}(t,\widetilde{y})\||\leq\sqrt{1-\sigma_i\tau_i}\|f^{[i]}(t,y)-f^{[i]}(t,\widetilde{y})\|,$$

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thus, $\|y-\widetilde{y}\|^2 \le \overline{M}(\sigma_i,\tau_i)\|f^{[i]}(t,y)-f^{[i]}(t,\widetilde{y})\|^2$, for $\forall y,\widetilde{y}\in R^m$, $t\ge 0$. It follows from the above formula and (1.2) that

$$\begin{aligned}&2\langle f^{[i]}(t,y)-f^{[i]}(t,\widetilde{y}),y-\widetilde{y}\rangle\\ \le&(\sigma_i-\varepsilon_i)\|y-\widetilde{y}\|^2+(\tau_i+\varepsilon_i\overline{M}(\sigma_i,\tau_i))\|f^{[i]}(t,y)-f^{[i]}(t,\widetilde{y})\|^2,\quad \forall y,\widetilde{y}\in R^m,\quad t\ge 0.\end{aligned}$$

This means that the conclusion holds. □

Lemma 1.2. *Assume that $y(t)$ and $z(t)$ are any two solutions of a problem belonging to the class $K_{\sigma,0}$. Then*

$$\|y(t_2)-z(t_2)\|\le \exp(\widetilde{\sigma}(t_2-t_1))\|y(t_1)-z(t_1)\|,\quad \forall t_1,t_2:\ t_2\ge t_1\ge 0, \tag{1.3}$$

where

$$\widetilde{\sigma}=\sum_{i=1}^{N}\frac{\sigma_i}{2}.$$

And the class $K_{\sigma,0}$ has contractive solutions if $\sigma\le 0$.

Proof. It follows from (1.1)-(1.2) that

$$2\langle f(t,y)-f(t,z),y-z\rangle\le\Big(\sum_{i=1}^{N}\sigma_i\Big)\|y-z\|^2.$$

This means that $f(t,y)$ satisfies the one-sided Lipschitz condition with the one-sided Lipschitz constant $\sum_{i=1}^{N}\sigma_i/2$. Thus, the conclusion of Lemma 1.2 holds. □

Lemma 1.3. *Assume that $y(t)$ and $z(t)$ are any two solutions of a problem belonging to the class $K_{\sigma,\tau}$ with $\sigma,\tau\le 0$. Then $y(t)$ and $z(t)$ satisfy the contractive inequality*

$$\|y(t_2)-z(t_2)\|\le \exp(\overline{\sigma}(t_2-t_1))\|y(t_1)-z(t_1)\|,\quad \forall t_1,t_2:\ t_2\ge t_1\ge 0, \tag{1.4}$$

where

$$\overline{\sigma}=\sum_{i=1}^{N}\widehat{\sigma}_i,$$

$$\widehat{\sigma}_i=\begin{cases}\dfrac{1-\sqrt{1-\sigma_i\tau_i}}{\tau_i} & \text{for } \tau_i<0,\\ \dfrac{\sigma_i}{2} & \text{for } \tau_i=0.\end{cases}$$

Proof. When $\sigma,\tau<0$, we take $\varepsilon_i=-\tau_i/\overline{M}(\sigma_i,\tau_i)$, thus Lemma 1.1 yields

$$K_{\sigma,\tau}\subset K_{2\widehat{\sigma},0},\quad \widehat{\sigma}=(\widehat{\sigma}_1,\widehat{\sigma}_2,\cdots,\widehat{\sigma}_N)^T.$$

It follows from Lemma 1.2 that (1.4) holds.

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When $\sigma=0$, $\tau\leq 0$, $K_{\sigma,\tau}\subset K_{0,0}$. Thus, Lemma 1.2 yields (1.4) with $\widehat{\sigma}=0$, $\overline{\sigma}=0$.

When $\sigma\leq 0$, $\tau=0$, the class $K_{\sigma,\tau}$ becomes the class $K_{\sigma,0}$. Thus, Lemma 1.2 yields (1.4) with $\widehat{\sigma}=\sigma/2$, $\overline{\sigma}=\widetilde{\sigma}=\sum_{i=1}^{N}\sigma_i/2$.

Therefore, the conclusion of Lemma 1.3 holds. □

When

$$\sigma=0,\quad \tau_i=0\quad (i=1,2,\cdots,l),\quad \tau_i=2\nu_i\quad (\nu_i<0,\ i=l+1,l+2,\cdots,N),$$

the class $K_{\sigma,\tau}$ becomes the class $K_{0,\tau}$ considered in [14] and satisfies

$$\begin{aligned}&\langle f^{[i]}(t,y)-f^{[i]}(t,\tilde{y}),y-\tilde{y}\rangle\leq 0, && i=1,2,\cdots,l,\\ &\langle f^{[i]}(t,y)-f^{[i]}(t,\tilde{y}),y-\tilde{y}\rangle\leq \nu_i\|f^{[i]}(t,y)-f^{[i]}(t,\tilde{y})\|^2, && i=l+1,l+2,\cdots,N.\end{aligned}$$

When $\tau\leq 0$, the class $K_{0,\tau}$ has contractive solutions (cf. [1]).

The problems (1.1) are defined by operators which may be decomposed into sum of two or more parts, and often appear in space discretization of some initial-boundary value problems of partial differential equations, for example, the compressive Navier-Stokes equations are convection-diffusion-reaction equations including at least three terms (i.e., $N\geq 3$) with different characteristics (cf. [3–7, 16–24, 31]). To efficiently solve this class of stiff problems, ones often use additive Runge-Kutta (ARK) methods including fractional-step Runge-Kutta (FSRK) methods and implicit-explicit (IMEX) RK methods as some particular cases, and some different applications of ARK methods can be found in [3–7, 16, 18–24, 26–28, 31]. The main advantage of ARK methods is focused on reducing the computational cost. In fact, if the partition of the function f in (1.1) is suitable, then ARK methods for (1.1) are more efficient than an individual implicit RK methods for (1.1), for example, IMEX RK methods are often used to solve nonlinear stiff problems with the function f partitioned into a stiff part and a nonstiff part.

Consider the N different s-stage RK methods for the problems (1.1)

$$\begin{cases} y_{n+1}=y_n+h\displaystyle\sum_{\nu=1}^{N}\sum_{i=1}^{s}b_i^{[\nu]}f^{[\nu]}(t_n+c_ih,Y_{n,i}),\\ Y_{n,i}=y_n+h\displaystyle\sum_{\nu=1}^{N}\sum_{j=1}^{s}a_{ij}^{[\nu]}f^{[\nu]}(t_n+c_jh,Y_{n,j}),\end{cases}\tag{1.5}$$

and the coefficients of the methods (1.5) may be organized in the Butcher tableau

$$\begin{array}{c|c|c|c|c} c & A^{[1]} & A^{[2]} & \cdots & A^{[N]}\\ \hline & b^{[1]T} & b^{[2]T} & \cdots & b^{[N]T}\end{array},$$

where

$$\begin{aligned}&c=[c_1,c_2,\cdots,c_s]^T, && b^{[\nu]}=[b_1^{[\nu]},b_2^{[\nu]},\cdots,b_s^{[\nu]}]^T,\\ &A=[A^{[1]},A^{[2]},\cdots,A^{[N]}], && A^{[\nu]}=(a_{ij}^{[\nu]})_{i,j=1}^{s},\quad \nu=1,2,\cdots,N,\end{aligned}$$

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h is any given stepsize, the numerical solutions y_n and $Y_{n,i}$ are approximations to the true solutions $y(t_n)$ and $y(t_n+c_ih)$, $i=1,2,\cdots,s$, $t_n=nh$, $n\geq 0$, respectively.

In recent years, main researches into the ARK methods (1.5) for the problems (1.1) are focused on their linear stability analysis, order conditions, efficient method construction and applications (cf. [2–6,20–24,26–28,31]). Moreover, there exist some results in nonlinear stability (such as strong stability (cf. [18])), contractivity/monotonicity for the class $K_{0,\tau}$ with $\tau=[0,0,\cdots,0,2\nu_{l+1},2\nu_{l+2},\cdots,2\nu_N]^T$ (cf. [16]), B-stability and algebraic-stability for the class $K_{0,0}$ (cf. [1]), and convergence of IMEX RK methods for semilinear reaction-diffusion equations by using a technique of B-convergence analysis (cf. [22]). In particular, the concepts and properties of B-stability, contractivity and monotonicity of standard RK methods can be found in [10,11,14,17]. Now, we further consider this subject.

The rest of this paper is organized as follows. In Section 1, we introduce the stiff problem $K_{\sigma,\tau}$ and give its some properties. In Section 2, for the ARK methods (1.5) applied to the problem class $K_{\sigma,\tau}$, we generalize the algebraic-stability to $(\theta,\bar{p},\bar{q})$-algebraic-stability, and it is proven that this stability implies some contractive properties of the ARK methods. In Sections 3 and 4, some results on optimal B-convergence of the ARK methods (1.5) including IMEX RK methods for the class $K_{\sigma,0}$ are given. These new results extend the ones of RK methods and ARK methods in the references (such as [1,10,11,14,16,25]). In Section 5, some numerical examples are presented to verify the given theoretical results.

2 $(\theta,\bar{p},\bar{q})$-algebraic stability

Consider any two parallel integration steps of the ARK methods (1.5) for the problems (1.1)-(1.2)

$$(t_n,y_n)\to(t_{n+1},y_{n+1}):\quad\begin{cases} y_{n+1}=y_n+h\sum\limits_{\nu=1}^{N}\sum\limits_{i=1}^{s}b_i^{[\nu]}f^{[\nu]}(t_n+c_ih,Y_{n,i}),\\ Y_{n,i}=y_n+h\sum\limits_{\nu=1}^{N}\sum\limits_{j=1}^{s}a_{ij}^{[\nu]}f^{[\nu]}(t_n+c_jh,Y_{n,j}),\quad i=1,2,\cdots,s,\end{cases}\tag{2.1a}$$

$$(t_n,z_n)\to(t_{n+1},z_{n+1}):\quad\begin{cases} z_{n+1}=z_n+h\sum\limits_{\nu=1}^{N}\sum\limits_{i=1}^{s}b_i^{[\nu]}f^{[\nu]}(t_n+c_ih,Z_{n,i}),\\ Z_{n,i}=z_n+h\sum\limits_{\nu=1}^{N}\sum\limits_{j=1}^{s}a_{ij}^{[\nu]}f^{[\nu]}(t_n+c_jh,Z_{n,j}),\quad i=1,2,\cdots,s.\end{cases}\tag{2.1b}$$

We introduce some expressions:

$$\begin{aligned}&Y_{n,0}=y_n,\quad Z_{n,0}=z_n,\quad Y_{n,s+1}=y_{n+1},\quad Z_{n,s+1}=z_{n+1},\\ &W_{n,i}=Y_{n,i}-Z_{n,i},\quad i=0,1,\cdots,s,s+1,\\ &Q_{n,i}^{[\nu]}=h(f^{[\nu]}(t_n+c_ih,Y_{n,i})-f^{[\nu]}(t_n+c_ih,Z_{n,i})),\quad \nu=1,2,\cdots,N,\quad i=1,2,\cdots,s.\end{aligned}$$

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It is well known that for RK methods, the concept of algebraic stability was introduced first in [10] and extended to (θ,p,q)-algebraic stability in [10,14].

Definition 2.1. The ARK methods (1.5) are said to be $(\theta,\bar{p},\bar{q})$-algebraically-stable if there exist the real vectors

$$\theta=[\theta_1,\theta_2,\cdots,\theta_s]^T,\quad \bar{p}=[\bar{p}_1,\bar{p}_2,\cdots,\bar{p}_N]^T,\quad \bar{q}=[\bar{q}_1,\bar{q}_2,\cdots,\bar{q}_N]^T,$$

here $\sum_{i=1}^{s}\theta_i\leq 1$, $\bar{p}_i\bar{q}_i\leq 1$, $(i=1,2,\cdots,N)$, such that

$$\begin{cases} (i)\ \ b^{[\nu]}-A^{[\nu]T}\theta\geq 0,\\ (ii)\ \ \theta+\sum_{\nu=1}^{N}\bar{p}_\nu(b^{[\nu]}-A^{[\nu]T}\theta)\leq 0,\\ (iii)\ \ M^{[\nu\nu]}=M^{[\nu\nu](s+1)}-\sum_{k=1}^{s}\theta_k M^{[\nu\nu](k)}-\bar{q}_\nu diag(b^{[\nu]}-A^{[\nu]T}\theta)\geq 0,\\ \qquad M^{[\nu\mu]}=M^{[\nu\mu](s+1)}-\sum_{k=1}^{s}\theta_k M^{[\nu\mu](k)}\geq 0,\quad \nu\neq\mu, \end{cases} \tag{2.2}$$

where

$$\begin{cases} M^{[\nu\mu](l)}=diag\big(a_l^{[\nu]}\big)A^{[\mu]}+A^{[\nu]T}diag\big(a_l^{[\mu]}\big)-a_l^{[\nu]}a_l^{[\mu]T},\\ a_l^{[\nu]}=[a_{l1}^{[\nu]},a_{l2}^{[\nu]},\cdots,a_{ls}^{[\nu]}]^T,\quad l=1,2,\cdots,s+1,\\ a_{s+1,i}^{[\nu]}=b_i^{[\nu]},\quad i=1,2,\cdots,s,\quad \nu,\mu=1,2,\cdots,N. \end{cases} \tag{2.3}$$

If $\theta=0$, $\bar{q}=\bar{p}=0$, then $(0,0,0)$-algebraic-stability becomes algebraic-stability introduced in [1]. If $\theta=0$, $\bar{p}=0$, $\bar{q}=[0,0,\cdots,0,\frac{-1}{r_{l+1}},\cdots,\frac{-1}{r_N}]^T$, here r_i $(i=l+1,l+2,\cdots,N)$ are positive real numbers, then $(0,0,\bar{q})$-algebraic stability implies the algebraic stability introduced in [16]. Let

$$M^{[\nu\mu]}=(m_{ij}^{[\nu\mu]})_{i,j=1}^{s},\quad M^{[\nu\mu](l)}=(m_{ij}^{[\nu\mu](l)})_{i,j=1}^{s},$$

where $\nu,\mu=1,2,\cdots,N$, $l=1,2,\cdots,s+1$.

Theorem 2.1. *If the ARK methods* (1.5) *with* $(\theta,\bar{p},\bar{q})$*-algebraic-stability are applied to the problem class* $K_{\sigma,\tau}$ *under the stepsize conditions* $h>0$, $\sigma_i h\leq\bar{p}_i$ *and* $\tau_i/h\leq\bar{q}_i$, $(i=1,2,\cdots,N)$, *then the estimation*

$$\|y_{n+1}-z_{n+1}\|\leq\sqrt{1-\sum_{i=1}^{s}\theta_i}\|y_n-z_n\| \tag{2.4}$$

holds for any two parallel integration steps (2.1a) *and* (2.1b). *Moreover, when* $\sum_{i=1}^{s}\theta_i\geq 0$, *these ARK methods preserve the contractivity of the problem class* $K_{\sigma,\tau}$ *with* $\sigma,\tau\leq 0$, *which is shown in Lemma 1.3.*

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Proof. Subtraction of (2.1b) from (2.1a) leads to

$$W_{n,k}=W_{n,0}+\sum_{\nu=1}^{N}\sum_{j=1}^{s}a_{kj}^{[\nu]}Q_{n,j}^{[\nu]},\quad k=1,2,\cdots,s+1. \tag{2.5}$$

It follows from (2.5) that

$$\begin{aligned}\|W_{n,k}\|^2&=\Big\langle W_{n,0}+\sum_{\nu=1}^{N}\sum_{i=1}^{s}a_{ki}^{[\nu]}Q_{n,i}^{[\nu]},W_{n,0}+\sum_{\mu=1}^{N}\sum_{j=1}^{s}a_{kj}^{[\mu]}Q_{n,j}^{[\mu]}\Big\rangle\\&=\|W_{n,0}\|^2+\sum_{\nu=1}^{N}\sum_{i=1}^{s}a_{ki}^{[\nu]}\langle Q_{n,i}^{[\nu]},W_{n,0}\rangle+\sum_{\mu=1}^{N}\sum_{j=1}^{s}a_{kj}^{[\mu]}\langle W_{n,0},Q_{n,j}^{[\mu]}\rangle\\&\quad+\sum_{\nu,\mu=1}^{N}\sum_{i,j=1}^{s}a_{ki}^{[\nu]}a_{kj}^{[\mu]}\langle Q_{n,i}^{[\nu]},Q_{n,j}^{[\mu]}\rangle.\end{aligned} \tag{2.6}$$

From (2.5), we have

$$\langle Q_{n,i}^{[\nu]},W_{n,0}\rangle=\langle Q_{n,i}^{[\nu]},W_{n,i}\rangle-\sum_{\mu=1}^{N}\sum_{j=1}^{s}a_{ij}^{[\mu]}\langle Q_{n,i}^{[\nu]},Q_{n,j}^{[\mu]}\rangle, \tag{2.7a}$$

$$\langle W_{n,0},Q_{n,j}^{[\mu]}\rangle=\langle W_{n,j},Q_{n,j}^{[\mu]}\rangle-\sum_{\nu=1}^{N}\sum_{i=1}^{s}a_{ji}^{[\nu]}\langle Q_{n,i}^{[\nu]},Q_{n,j}^{[\mu]}\rangle. \tag{2.7b}$$

It follows from (2.6) and (2.7) that

$$\begin{aligned}\|W_{n,k}\|^2&=\|W_{n,0}\|^2+2\sum_{\nu=1}^{N}\sum_{i=1}^{s}a_{ki}^{[\nu]}\langle W_{n,i},Q_{n,i}^{[\nu]}\rangle-\sum_{\nu,\mu=1}^{N}\sum_{i,j=1}^{s}(a_{ki}^{[\nu]}a_{ij}^{[\mu]}+a_{kj}^{[\mu]}a_{ji}^{[\nu]}-a_{ki}^{[\nu]}a_{kj}^{[\mu]})\langle Q_{n,i}^{[\nu]},Q_{n,j}^{[\mu]}\rangle\\&=\|W_{n,0}\|^2+2\sum_{\nu=1}^{N}\sum_{i=1}^{s}a_{ki}^{[\nu]}\langle W_{n,i},Q_{n,i}^{[\nu]}\rangle-\sum_{\nu,\mu=1}^{N}\sum_{i,j=1}^{s}m_{ij}^{[\nu\mu](k)}\langle Q_{n,i}^{[\nu]},Q_{n,j}^{[\mu]}\rangle,\end{aligned} \tag{2.8}$$

where $k=1,2,\cdots,s+1$. In particular, we have

$$\|W_{n,s+1}\|^2=\|W_{n,0}\|^2+2\sum_{\mu=1}^{N}\sum_{j=1}^{s}b_{j}^{[\mu]}\langle W_{n,j},Q_{n,j}^{[\mu]}\rangle-\sum_{\nu,\mu=1}^{N}\sum_{i,j=1}^{s}m_{ij}^{[\nu\mu](s+1)}\langle Q_{n,i}^{[\nu]},Q_{n,j}^{[\mu]}\rangle. \tag{2.9}$$

Moreover, from (2.8), we have

$$\begin{aligned}\sum_{k=1}^{s}\theta_k\|W_{n,k}\|^2&=\sum_{k=1}^{s}\theta_k\|W_{n,0}\|^2+2\sum_{\nu=1}^{N}\sum_{i=1}^{s}\Big(\sum_{k=1}^{s}\theta_k a_{ki}^{[\nu]}\Big)\langle W_{n,i},Q_{n,i}^{[\nu]}\rangle\\&\quad-\sum_{\nu,\mu=1}^{N}\sum_{i,j=1}^{s}\Big(\sum_{k=1}^{s}\theta_k m_{ij}^{[\nu\mu](k)}\Big)\langle Q_{n,i}^{[\nu]},Q_{n,j}^{[\mu]}\rangle.\end{aligned} \tag{2.10}$$

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Subtraction of (2.10) from (2.9) yields

$$\|W_{n,s+1}\|^2=\Big(1-\sum_{k=1}^{s}\theta_k\Big)\|W_{n,0}\|^2+\sum_{j=1}^{s}\Big[\theta_j\|W_{n,j}\|^2+2\sum_{\mu=1}^{N}\Big(b_j^{[\mu]}-\sum_{k=1}^{s}\theta_k a_{kj}^{[\mu]}\Big)\langle W_{n,j},Q_{n,j}^{[\mu]}\rangle\Big]$$
$$-\sum_{\nu,\mu=1}^{N}\sum_{i,j=1}^{s}\Big(m_{ij}^{[\nu\mu](s+1)}-\sum_{k=1}^{s}\theta_k m_{ij}^{[\nu\mu](k)}\Big)\langle Q_{n,i}^{[\nu]},Q_{n,j}^{[\mu]}\rangle. \tag{2.11}$$

It follows from (1.2), Definition 2.1 and the assumption conditions that

$$\begin{aligned}\|W_{n,s+1}\|^2\le&\Big(1-\sum_{k=1}^{s}\theta_k\Big)\|W_{n,0}\|^2\\&+\sum_{j=1}^{s}\Big[\theta_j\|W_{n,j}\|^2+h\sum_{\mu=1}^{N}\Big(b_j^{[\mu]}-\sum_{k=1}^{s}\theta_k a_{kj}^{[\mu]}\Big)\Big(\sigma_\mu\|W_{n,j}\|^2+\frac{\tau_\mu}{h^2}\|Q_{n,j}^{[\mu]}\|^2\Big)\Big]\\&-\sum_{\nu,\mu=1}^{N}\sum_{i,j=1}^{s}\Big(m_{ij}^{[\nu\mu](s+1)}-\sum_{k=1}^{s}\theta_k m_{ij}^{[\nu\mu](k)}\Big)\langle Q_{n,i}^{[\nu]},Q_{n,j}^{[\mu]}\rangle\\\le&\Big(1-\sum_{k=1}^{s}\theta_k\Big)\|W_{n,0}\|^2+\sum_{j=1}^{s}\Big[\theta_j+\sum_{\mu=1}^{N}\bar{p}_\mu\Big(b_j^{[\mu]}-\sum_{k=1}^{s}\theta_k a_{kj}^{[\mu]}\Big)\Big]\|W_{n,j}\|^2\\&+\sum_{j=1}^{s}\sum_{\mu=1}^{N}\bar{q}_\mu\Big(b_j^{[\mu]}-\sum_{k=1}^{s}\theta_k a_{kj}^{[\mu]}\Big)\langle Q_{n,j}^{[\mu]},Q_{n,j}^{[\mu]}\rangle\\&-\sum_{\nu,\mu=1}^{N}\sum_{i,j=1}^{s}\Big(m_{ij}^{[\nu\mu](s+1)}-\sum_{k=1}^{s}\theta_k m_{ij}^{[\nu\mu](k)}\Big)\langle Q_{n,i}^{[\nu]},Q_{n,j}^{[\mu]}\rangle\\\le&\Big(1-\sum_{k=1}^{s}\theta_k\Big)\|W_{n,0}\|^2-\sum_{\nu,\mu=1}^{N}\sum_{i,j=1}^{s}m_{ij}^{[\nu\mu]}\langle Q_{n,i}^{[\nu]},Q_{n,j}^{[\mu]}\rangle\\\le&\Big(1-\sum_{k=1}^{s}\theta_k\Big)\|W_{n,0}\|^2.\end{aligned}\tag{2.12}$$

Thus, we complete the proof. □

Remark 2.1. If $\theta=0$, $\bar{p}=0$, $\bar{q}_i=0$ for $i=1,2,\cdots,l$, $\bar{q}_i<0$ for $i=l+1,l+2,\cdots,N$, then Theorem 2.1 becomes Theorem 1 in [14], here $\bar{q}_i=-1/r_i$, $\tau_i=2\nu_i$ $(i=l+1,\cdots,N)$. In particular, if $l=N$, then Theorem 2.1 becomes Theorem 1 in [1]. In fact, the algebraic stability of ARK methods (1.5) ensures their unconditional numerical contractivity. For non-algebraically-stable ARK methods (1.5), $(\theta,\bar{p},\bar{q})$-algebraic stability ensures their conditional numerical contractivity under certain stepsize restrictions (also see some special cases in [11,16]).

We can easily obtain the following corollaries from Definition 2.1 and Theorem 2.1.

Corollary 2.1. *Assume that $\bar{p}\le\breve{p}$, $\bar{q}\le\breve{q}$ and $\bar{p}_i\bar{q}_i\le 1$ $(i=1,2,\cdots,N)$. If the ARK methods* (1.5) *are $(\theta,\breve{p},\breve{q})$-algebraically-stable, then these methods also are $(\theta,\bar{p},\bar{q})$-algebraically stable.*

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Corollary 2.2. *If the* $(\theta,\bar{p},0)$*-algebraically-stable ARK methods* (1.5) *are applied to the class* $K_{\sigma,0}$*, then the inequality* (2.4) *holds under the conditions* $\sigma_\nu h\le\bar{p}_\nu$ $(\nu=1,2,\cdots,N)$*. If the* $(\theta,0,\bar{q})$*-algebraically-stable ARK methods* (1.5) *are applied to the class* $K_{0,\tau}$*, then the inequality* (2.4) *holds under the conditions* $\tau_\nu/h\le\bar{q}_\nu$ $(\nu=1,2,\cdots,N)$.

Corollary 2.3. *If* $b_i^{[\nu]}>0$ $(i=1,2,\cdots,s,\ \nu=1,2,\cdots,N)$*,* $M^{[\nu\mu](s+1)}\ge 0$ $(\nu,\mu=1,2,\cdots,N,\nu\neq\mu)$*, then the ARK methods* (1.5) *are* $(0,0,\bar{q}^*)$*-algebraically-stable, where*

$$\bar{q}^*=[\bar{q}_1^*,\bar{q}_2^*,\cdots,\bar{q}_N^*]^T, \tag{2.13a}$$

$$\bar{q}_\nu^*=-\max_{1\le i\le s}\Big[\Big(\sum_{j=1,\ j\neq i}^{s}|m_{ij}^{[\nu\nu](s+1)}|-m_{ii}^{[\nu\nu](s+1)}\Big)_+\Big/b_i^{[\nu]}\Big], \tag{2.13b}$$

the symbol $(\cdot)_+$ *is defined by*

$$(x)_+=\begin{cases}x, & \text{if } x\ge 0,\\ 0, & \text{if } x<0.\end{cases}$$

Thus, for the ARK methods (1.5) *applied to the class* $K_{0,\tau}$ $(\tau<0)$*, if* $h\le\min_{1\le\nu\le N}\tau_\nu/\bar{q}_\nu^*$ $(\nu=1,2,\cdots,N)$*, then the contractivity inequality*

$$\|y_{n+1}-z_{n+1}\|\le\|y_n-z_n\|$$

holds.

Proof. In view of $\bar{p}=0$, $\theta=0$, and $b_i^{[\nu]}>0$ $(i=1,2,\cdots,s,\ \nu=1,2,\cdots,N)$, the conditions (i) and (ii) of Definition 2.1 are apparently satisfied. So we only prove that the condition (iii) is satisfied. By the definition of $\bar{q}_\nu^*$, we have

$$-b_i^{[\nu]}\bar{q}_\nu^*\ge\Big(\sum_{j=1,\ j\neq i}^{s}|m_{ij}^{[\nu\nu](s+1)}|-m_{ii}^{[\nu\nu](s+1)}\Big)_+,\quad i=1,2,\cdots,s,\quad \nu=1,2,\cdots,N.$$

Moreover,

$$\begin{aligned}m_{ii}^{[\nu\nu]}&=m_{ii}^{[\nu\nu](s+1)}-b_i^{[\nu]}\bar{q}_\nu^*\\&\ge m_{ii}^{[\nu\nu](s+1)}+\Big(\sum_{j=1,\ j\neq i}^{s}|m_{ij}^{[\nu\nu](s+1)}|-m_{ii}^{[\nu\nu](s+1)}\Big)_+\\&\ge\sum_{j=1,\ j\neq i}^{s}|m_{ij}^{[\nu\nu](s+1)}|=\sum_{j=1,\ j\neq i}^{s}|m_{ij}^{[\nu\nu]}|.\end{aligned}$$

It follows from the Gerschgorin disk theorem and the symmetry of $M^{[\nu\nu]}$ that the eigenvalues of $M^{[\nu\nu]}$ are nonnegative real numbers. Thus, $M^{[\nu\nu]}\ge 0$ $(\nu=1,2,\cdots,N)$. □

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When $N=1$, the above Theorem 2.1 and Corollaries 2.1-2.3 become Theorem 4.3.1 and Corollaries 4.3.2-4.3.5 in [25], respectively; moreover, in [30], we have proved that if non-confluent Runge-Kutta methods with stage-order not less than two are $(\theta,\bar{p},\bar{q})$-algebraically-stable, then $\bar{q}\leq 0$.

Now we present some examples of $(\theta,\bar{p},\bar{q})$-algebraically-stable ARK methods (1.5).

Example 2.1. The first-order IMEX Euler methods (cf. [3,16,31])

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 0 & 1 \\ \hline (A^{[1]},b^{[1]}) & 0 & 1 \end{array} \qquad \begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline (A^{[2]},b^{[2]}) & 1 & 0 \end{array}$$

are $(0,0,\bar{q})$-algebraically-stable, where $\bar{q}=(0,-1)^T$.

Example 2.2. The second-order IMEX trapezoidal methods (cf. [16,20])

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline (A^{[1]},b^{[1]}) & 1/2 & 1/2 \end{array} \qquad \begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline (A^{[2]},b^{[2]}) & 1/2 & 1/2 \end{array}$$

are $(0,0,\bar{q})$-algebraically-stable, where $\bar{q}=(-1,-1)^T$.

Example 2.3. The ARK methods

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline (A^{[1]},b^{[1]}) & 1/2 & 1/2 \end{array} \qquad \begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 0 & 1 \\ \hline (A^{[2]},b^{[2]}) & 0 & 1 \end{array}$$

are $(0,0,\bar{q})$-algebraically-stable, where $\bar{q}=(-1,0)^T$.

Example 2.4. The second-order IMEX methods

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ a_1 & a_1 & 0 & 0 & 0 \\ b_{21} & b_{21} & 0 & 0 & 0 \\ c_{21}+a_2 & c_{21} & 0 & 0 & 0 \\ \hline (A^{[2]},b^{[2]}) & \omega_1 & 0 & \omega_2 & 0 \end{array} \qquad \begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & a_1 & 0 & 0 \\ b_{21} & 0 & b_{21} & 0 & 0 \\ c_{21}+a_2 & 0 & c_{21} & 0 & a_2 \\ \hline (A^{[1]},b^{[1]}) & 0 & \omega_1 & 0 & \omega_2 \end{array}$$

are considered in [3,31], where

$$\omega_1=\frac{1}{2},\quad \omega_2=\frac{1}{2},\quad b_{21}=1,\quad a_1=\frac{1}{4},\quad a_2=\frac{1}{3},\quad c_{21}=\frac{5}{12}.$$

They are $(0,0,\bar{q})$-algebraically-stable, where

$$\bar{q}=\Big(-\frac{\sqrt{2}-1}{12},-1\Big)^T.$$

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Example 2.5. The second-order Peaceman-Rachford ARK methods (cf. [16,20])

0	0	0	0	0	0	0	0
1/2	0	1/2	0	1/2	1/2	0	0
1	0	1	0	1	1/2	0	1/2
$(A^{[2]},b^{[2]})$	0	1	0	$(A^{[2]},b^{[2]})$	1/2	0	1/2

are $(0,0,\bar{q})$-algebraically-stable, where $\bar{q}=(-1,-1)^T$.

Example 2.6. The first-order ARK methods

2/5	2/5	0	2/5	2/5	0
1	1	0	1	2/3	1/3
$(A^{[1]},b^{[1]})$	1	0	$(A^{[2]},b^{[2]})$	1	0

are $(\theta,\bar{p},0)$-algebraically-stable, where

$$\theta=\left(-\frac{5}{4},0\right)^T,\quad \bar{p}=\left(\frac{5}{6},0\right)^T.$$

Example 2.7. The first-order ARK methods

2/3	2/3	0	2/3	2/3	0
1	1	0	1	3/4	1/4
$(A^{[1]},b^{[1]})$	1	0	$(A^{[2]},b^{[2]})$	1/2	1/2

are $(\theta,\bar{p},0)$-algebraically-stable, where

$$\theta=\left(\frac{3}{4},0\right)^T,\quad \bar{p}=\left(-\frac{3}{2},0\right)^T.$$

3 B-convergence of ARK methods

For (1.1), let

$$y^{[\nu]'}(t)=f^{[\nu]}(t,y(t)),\quad \nu=1,2,\cdots,N,\quad t\in[0,T],\quad \sum_{\nu=1}^{N}y^{[\nu]}(0)=y_0.$$

Thus, $y(t)=\sum_{\nu=1}^{N}y^{[\nu]}(t)$, $y^{[\nu]}(t)$ is sufficiently smooth and its derivatives used later are bounded by

$$\|y^{[\nu](l)}(t)\|\le M_l,\quad t\in[0,T],\quad v=1,2,\cdots,N,\quad l=0,1,2,\cdots.$$

The B-convergence concept of RK methods was first introduced in [13] and discussed in [8,9,15,17,25,29]. Now we extend this concept to the ARK methods (1.5) for the class $K_{\sigma,0}$. We introduce some notations:

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(i) We denote a $l\times l$-identity matrix by I_l for $l=1,2,\cdots$, and if no confusion rises, simply by I.

(ii) $\overline{G}=G\otimes I_m$ for a given $N_1\times M_1$ matrix G, where the symbol $\otimes$ denotes Kronecker product. $e=[1,1,\cdots,1]\in R^s$.

(iii) $D_{\sigma_\nu}=\{Z^{[\nu]}:Z^{[\nu]}=diag(Z_1^{[\nu]},Z_2^{[\nu]},\cdots,Z_s^{[\nu]})\in R^{ms\times ms},\ Z_i^{[\nu]}\in R^{m\times m},\ \mu(Z_i^{[\nu]})\le\sigma_\nu/2,\ i=1,2,\cdots,s\}$, $\nu=1,2,\cdots,N$.

(iv) For

$$\Phi(Z_n)=\Phi(Z_n^{[1]},Z_n^{[2]},\cdots,Z_n^{[N]})=I-h\sum_{\mu=1}^{N}\overline{A}^{[\mu]}Z_n^{[\mu]},$$

$$K(Z_n)=K(Z_n^{[1]},Z_n^{[2]},\cdots,Z_n^{[N]})=I+h\sum_{\nu=1}^{N}\overline{b}^{[\nu]T}Z_n^{[\nu]}\Phi^{-1}(Z_n)\overline{e},$$

$$L^{[\nu]}(Z_n)=\overline{d}_0^{[\nu]}+h\sum_{\xi=1}^{N}\overline{b}^{[\xi]T}Z_n^{[\xi]}\Phi^{-1}(Z_n)\overline{d}^{[\nu]},$$

$$d_0^{[\nu]}=(q!)^{-1}((q+1)^{-1}-b^{[\nu]T}c^q)\in R,\quad d^{[\nu]}=(q!)^{-1}((q+1)^{-1}c^{q+1}-A^{[\nu]}c^q)\in R^s,$$

where $Z_n=[Z_n^{[1]},Z_n^{[2]},\cdots,Z_n^{[N]}]$, $Z_n^{[\nu]}\in D_{\sigma_\nu}$, $\nu=1,2,\cdots,N$.

(v) $F^{[\nu]}(t_n,Y_n)=(f^{[\nu]T}(t_n+c_1h,Y_{n,1}),f^{[\nu]T}(t_n+c_2h,Y_{n,2}),\cdots,f^{[\nu]T}(t_n+c_sh,Y_{n,s}))^T$, where $Y_n=(Y_{n1}^T,Y_{n2}^T,\cdots,Y_{ns}^T)^T$.

The methods (1.5) can be rewritten as

$$\begin{cases} Y_n=e\otimes y_n+h\sum_{\nu=1}^{N}\overline{A}^{[\nu]}F^{[\nu]}(t_n,Y_n),\\ y_{n+1}=y_n+h\sum_{\nu=1}^{N}\overline{b}^{[\nu]T}F^{[\nu]}(t_n,Y_n). \end{cases}\tag{3.1}$$

We now extend the concepts of stage order from RK methods to ARK methods.

Definition 3.1. The methods (1.5) have stage order q if the simplifying conditions

$$B^{[\nu]}(q):\ b^{[\nu]T}c^{j-1}=1/j,\qquad j=1,2,\cdots,q,$$
$$C^{[\nu]}(q):\ A^{[\nu]}c^{j-1}=c^j/j,\qquad j=1,2,\cdots,q,$$

hold, where $\nu=1,2,\cdots,N$, $c^j=[c_1^j,c_2^j,\cdots,c_s^j]^T$, $j=0,1,2,\cdots$.

Definition 3.2. The ARK methods (1.5) are said to be optimally B-convergent of order p for the given problem class $K_{\sigma,0}$, if there exists a real number $\tilde{h}>0$ and a function $\gamma:(0,T]\to(0,+\infty)$ such that

$$\|y(t_n)-y_n\|\le\gamma(t_n)h^p,\quad 0\le h\le\tilde{h},\quad t_{n-1}+\tilde{c}h\le T,\quad n=1,2,\cdots,$$

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where $\tilde{c}=\max\{1,c_1,c_2,\cdots,c_s\}$, $\tilde{h}$ and $\gamma(t_n)$ are independent of the stiffness of the considered problems, $\gamma(t_n)$ only depends on the methods, σ and some derivative bounds of the true solution $y(t)$, $\tilde{h}$ only depends on the methods and σ.

Definition 3.3. The ARK methods (1.5) are called ANS-stable, if $\Phi^{-1}(Z_n)$ exists, $\Phi^{-1}(Z_n)$ and $h\sum_{\nu=1}^{N}\overline{b}^{[\nu]T}Z_n^{[\nu]}\Phi^{-1}(Z_n)$ are uniformly bounded, i.e., there exist real numbers $\widetilde{D}_1>0$, $\widetilde{D}_2>0$ such that

$$\|\Phi^{-1}(Z_n)\|\le\widetilde{D}_1,\quad \Big\|h\sum_{\nu=1}^{N}\overline{b}^{[\nu]T}Z_n^{[\nu]}\Phi^{-1}(Z_n)\Big\|\le\widetilde{D}_2, \tag{3.2}$$

hold for given h, all $Z_n^{[\nu]}\in D_{\sigma_\nu}$ $(\nu=1,2,\cdots,N)$.

The concept of ANS-stability of (1.5) extends the concepts of AS-stability and ASI-stability of RK methods (cf. [8]).

Lemma 3.1. *Assume that the ARK methods* (1.5) *are algebraically-stable and ANS-stable. Then there exists a real number* $\widetilde{D}_4>0$ *such that*

$$\|K(Z_n)\|\le 1+\widetilde{D}_4h. \tag{3.3}$$

Proof. By a similar way used in the proof of Lemma 3.2 of [10], for $\forall u\in R^m$, we get

$$\begin{aligned}\langle K(Z_n)u,K(Z_n)u\rangle=\langle u,u\rangle&+2\sum_{\nu=1}^{N}\sum_{i=1}^{s}b_i^{[\nu]}\langle\widetilde{Y}_{ni}u,hZ_{ni}^{[\nu]}\widetilde{Y}_{ni}u\rangle\\&-\sum_{\nu,\mu=1}^{N}\sum_{i,j=1}^{s}(b_i^{[\nu]}a_{ij}^{[\mu]}+b_j^{[\mu]}a_{ji}^{[\nu]}-b_i^{[\nu]}b_j^{[\mu]})\langle hZ_{ni}^{[\nu]}\widetilde{Y}_{ni}u,hZ_{nj}^{[\mu]}\widetilde{Y}_{nj}u\rangle,\end{aligned}$$

where $\widetilde{Y_n}=(\widetilde{Y}_{n1}^T,\widetilde{Y}_{n2}^T,\cdots,\widetilde{Y}_{ns}^T)^T=\Phi^{-1}(Z_n)\overline{e}$, $\widetilde{Y}_{ni}\in R^{m\times m}$, $\|\widetilde{Y}_{ni}\|\le\widetilde{D}_1$ $(i=1,2,\cdots,s)$.

By the given assumption conditions, we obtain

$$2\sum_{\nu=1}^{N}\sum_{i=1}^{s}b_i^{[\nu]}\langle\widetilde{Y}_{ni}u,hZ_{ni}^{[\nu]}\widetilde{Y}_{ni}u\rangle\le\widetilde{D}_3h\|u\|,$$

where $\widetilde{D}_3=\widetilde{D}_1^2\sum_{\nu=1}^{N}\sigma_\nu\delta(\sigma_\nu)\sum_{i=1}^{s}b_i^{[\nu]}$, $\delta(\sigma_\nu)=0$ for $\sigma_\nu\le0$, $\delta(\sigma_\nu)=1$ for $\sigma_\nu>0$. Moreover, we obtain

$$\|K(Z_n)\|\le\sqrt{1+\widetilde{D}_3h}\le1+\widetilde{D}_4h,$$

where $\widetilde{D}_4=\widetilde{D}_3/2$ only depends on the methods, σ and $\widetilde{D}_1$. □

Let ε_0, ε_n denote the initial error $y(0)-y_0$ and the global error $y(t_n)-y_n$, $n\ge1$, respectively.

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Theorem 3.1. *Assume that ARK methods* (1.5) *with stage order q are algebraically-stable and ANS-stable for the problem class $K_{\sigma,0}$, and $\varepsilon_0=\mathcal{O}(h^q)$. Then these methods are optimally B-convergent of order q.*

Proof. Inserting $y(t)$ in (3.1), we obtain

$$\begin{cases} \widetilde{Y}_n = e\otimes y_n + h\sum\limits_{\nu=1}^{N}\overline{A}^{[\nu]}F^{[\nu]}(t_n,\widetilde{Y}_n)+Q_n, \\ y(t_{n+1}) = y(t_n)+h\sum\limits_{\nu=1}^{N}\overline{b}^{[\nu]T}F^{[\nu]}(t_n,\widetilde{Y}_n)+r_n, \end{cases} \tag{3.4}$$

where $\widetilde{Y}_n=(\widetilde{Y}_{n,1}^T,\widetilde{Y}_{n,2}^T,\cdots,\widetilde{Y}_{n,s}^T)^T$, $Q_n=(Q_{n,1}^T,Q_{n,2}^T,\cdots,Q_{n,s}^T)^T\in R^{sm}$ and r_n are local residual errors, here $\widetilde{Y}_{ni}=y(t_n+c_ih)$ $(i=1,2,\cdots,s)$.

By a Taylor series expansion, it follows that

$$r_n=\sum_{\xi=1}^{N}\overline{d}_0^{[\xi]}h^{q+1}y^{[\xi](q+1)}(t_n)+h^{q+2}R_0^{(n)},\quad Q_n=\sum_{\xi=1}^{N}\overline{d}^{[\xi]}h^{q+1}y^{[\xi](q+1)}(t_n)+h^{q+2}R_n,$$

where $R_n=(R_1^{(n)T},R_2^{(n)T},\cdots,R_s^{(n)T})^T$, $\|R_i^{(n)}\|\le C_0$, $i=0,1,\cdots,s$, $C_0>0$ only depends on the methods and some derivative bounds of the true solution.

Now consider the global error ε_n. Subtraction of (3.2) from (3.4) yields

$$\varepsilon_{n+1}=K(Z_n)\varepsilon_n+h\sum_{\nu=1}^{N}\overline{b}^{[\nu]T}Z_n^{[\nu]}\Phi^{-1}(Z_n)Q_n+r_n, \tag{3.5}$$

where $Z_n^{[\nu]}=diag(Z_{n,1}^{[\nu]},Z_{n,2}^{[\nu]},\cdots,Z_{n,s}^{[\nu]})$, $Z_{n,i}^{[\nu]}=\int_0^1 f_y^{[\nu]}(t_n+c_ih,Y_{n,i}+\theta(\widetilde{Y}_{n,i}-Y_{n,i}))d\theta$.

From (3.5) and ANS-stability, we have

$$\varepsilon_{n+1}=K(Z_n)\varepsilon_n+\widetilde{r}_n, \tag{3.6}$$

where $\widetilde{r}_n=h\sum_{\nu=1}^{N}\overline{b}^{[\nu]T}Z_n^{[\nu]}\Phi^{-1}(Z_n)Q_n+r_n$, $\|\widetilde{r}_n\|\le C_1h^{q+1}$, C_1 only depends on the methods, σ and some derivative bounds of the solutions. Moreover,

$$\varepsilon_{n+1}=\prod_{k=0}^{n}K(Z_k)\varepsilon_0+\sum_{j=0}^{n-1}\Big(\prod_{k=j+1}^{n}K(Z_k)\widetilde{r}_j\Big)+\widetilde{r}_n. \tag{3.7}$$

It follows from Lemma 3.1 and $\|\varepsilon_0\|\le C_2h^q$ that

$$\|\varepsilon_{n+1}\|\le\gamma(t_{n+1})h^q,$$

where

$$\gamma(t_{n+1})=C_2e^{\widetilde{D}_3t_{n+1}}+C_1(1+t_{n+1}e^{\widetilde{D}_3t_{n+1}}).$$

Thus, we complete the proof. □

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Theorem 3.2. *Assume that ARK methods* (1.5) *with stage order q are algebraically stable and ANS-stable for the problem class $K_{\sigma,0}$, $\varepsilon_0=\mathcal{O}(h^{q+1})$, $d_0^{[\nu]}=0$, all components of $d^{[\nu]}$ are equal $(\nu=1,2,\cdots,N)$. Then these methods are optimally B-convergent of order $q+1$.*

Proof. If $d_0^{[\xi]}=0$, and all components of $d^{[\xi]}$ are equal, i.e., $d^{[\xi]}=\lambda^{[\xi]}e$, $\lambda^{[\xi]}$ is a real constant, then

$$L^{[\xi]}(Z_n)=h\sum_{\nu=1}^{N}\overline{b}^{[\nu]T}Z_n^{[\nu]}\Phi^{-1}(Z_n)\lambda^{[\xi]}\overline{e}=\lambda^{[\xi]}(K(Z_n)-1).$$

And (3.5) becomes

$$\varepsilon_{n+1}=K(Z_n)\varepsilon_n+\sum_{\xi=1}^{N}L^{[\xi]}(Z_n)h^{q+1}y^{[\xi](q+1)}(t_n)+\widehat{r}_n, \tag{3.8}$$

where

$$\widehat{r}_n=h^{q+2}\Big(\sum_{\xi=1}^{N}\overline{b}^{[\xi]T}Z_n^{[\xi]}\Phi^{-1}(Z_n)R_n+R_0^{(n)}\Big).$$

From ANS-stability, we obtain

$$\|\widehat{r}_n\|\le C_3h^{q+2}, \tag{3.9}$$

where C_3 only depends on the methods, σ and some derivative bounds of the true solution.

For all $n\ge 0$, let

$$\widehat{\varepsilon}_n=\varepsilon_n+\sum_{\nu=1}^{N}\lambda^{[\nu]}h^{q+1}y^{[\nu](q+1)}(t_n). \tag{3.10}$$

It follows from (3.8)-(3.10) that

$$\widehat{\varepsilon}_{n+1}=K(Z)\widehat{\varepsilon}_n+\overline{\overline{Q}}_n=\prod_{k=0}^{n}K(Z_k)\widehat{\varepsilon}_0+\sum_{j=0}^{n-1}\Big(\prod_{k=j+1}^{n}K(Z_k)\Big)\overline{\overline{r}}_j+\overline{\overline{r}}_n, \tag{3.11}$$

where

$$\overline{\overline{r}}_n=\widehat{r}_n+\sum_{\nu=1}^{N}\lambda^{[\nu]}h^{q+1}(y^{[\nu](q+1)}(t_{n+1})-y^{[\nu](q+1)}(t_n)),\quad \|\overline{\overline{r}}_n\|\le C_4h^{q+2},$$

and $C_4>0$ does not depend the stiffness of the problems.

It follows from (3.11) and Lemma 3.1 that

$$\|\widehat{\varepsilon}_{n+1}\|\le e^{\widetilde{D}_3t_{n+1}}\|\widehat{\varepsilon}_0\|+(1+t_{n+1}e^{\widetilde{D}_3t_{n+1}})C_4h^{q+1}.$$

Moreover, it follows from the formula

$$\Big\|\sum_{\nu=1}^{N}\lambda^{[\nu]}h^{q+1}y^{[\nu](q+1)}(t_{n+1})\Big\|\le C_5h^{q+1},\quad \|\widehat{\varepsilon}_0\|\le C_6h^{q+1},$$

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that

$$\begin{aligned}\|\varepsilon_{n+1}\| &= \Big\|\widehat{\varepsilon}_{n+1} - \sum_{\nu=1}^{N} \lambda^{[\nu]} h^{q+1} y^{[\nu](q+1)}(t_{n+1})\Big\| \\ &\le \|\widehat{\varepsilon}_{n+1}\| + \Big\| \sum_{\nu=1}^{N} \lambda^{[\nu]} h^{q+1} y^{[\nu](q+1)}(t_{n+1})\Big\| \\ &\le e^{\widetilde{D}_3 t_{n+1}} \|\widehat{\varepsilon}_0\| + ((1+t_{n+1}e^{\widetilde{D}_3 t_{n+1}})C_4 + C_5)h^{q+1} \\ &\le \overline{\gamma}(t_{n+1})h^{q+1},\end{aligned}$$

where $\overline{\gamma}(t_{n+1}) = e^{\widetilde{D}_3 t_{n+1}} C_6 + (1+t_{n+1}e^{\widetilde{D}_3 t_{n+1}})C_4 + C_5$, C_5 and C_6 do not depend the stiffness of the problems. □

Remark 3.1. If $c_i - c_j$ is not an integer for $1 \le i < j \le s$, and the optimal B-convergence order of the ARK methods (1.5) is $q+1$, then as in [9], we can show similarly that $d_0^{[\nu]} = 0$, and all components of $d^{[\nu]}$ are equal, $\nu = 1,2,\cdots,N$.

Remark 3.2. In general, it is difficult to verify whether an ARK method is ANS-stable. But we can show ANS-stability of FSRK methods and ARK methods with $N = 2$. The FSRK methods can be expressed in the condensed form (cf. [1,5])

$$\begin{array}{c|c} & \widetilde{\theta}^T \\ \hline \mathrm{c} & \widetilde{A} \\ \hline & b^T \end{array}, \tag{3.12}$$

where

$$\widetilde{\theta} = [\widetilde{\theta}_1, \widetilde{\theta}_2, \cdots, \widetilde{\theta}_s]^T, \qquad \widetilde{\theta}_j \in \{1,2,\cdots,N\}, \qquad \widetilde{A} = (a_{ij}^{[\widetilde{\theta}_j]})_{i,j=1}^{s} = \sum_{\nu=1}^{N} A^{[\nu]},$$

$$b = (b_j^{[\widetilde{\theta}_j]})_{j=1}^{s} = \sum_{\nu=1}^{N} b^{[\nu]}, \qquad \sum_{\nu=1,\nu\neq\widetilde{\theta}_j}^{N} \Big(|b_j^{[\nu]}| + \sum_{i=1}^{s} |a_{ij}^{[\nu]}|\Big) = 0, \qquad j = 1,2,\cdots,s.$$

And the methods (1.5)-(3.12) may be written in the form

$$\begin{cases} y_{n+1} = y_n + h\sum_{i=1}^{s} b_i^{[\widetilde{\theta}_i]} f^{[\widetilde{\theta}_i]}(t_n + c_i h, Y_{n,i}), \\ Y_{n,i} = y_n + h\sum_{j=1}^{s} a_{ij}^{[\widetilde{\theta}_j]} f^{[\widetilde{\theta}_j]}(t_n + c_j h, Y_{n,j}). \end{cases} \tag{3.13}$$

The algebraic stability of the methods (1.5)-(3.12) can be reduced as [1]

$$b \ge 0, \quad diag(b)\widetilde{A} + \widetilde{A}^T diag(b) - bb^T \ge 0.$$

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The methods (1.5)-(3.12) are diagonally-stable if there exists an $s\times s$ diagonal matrix $D>0$ such that

$$D\widetilde{A}+\widetilde{A}^TD>0.$$

As shown in Lemma 4.1, the diagonal stablility of (1.5)-(3.12) implies ANS-stability.

4 B-convergence of ARK methods with $N=2$

Consider the problems (1.1) with $N=2$

$$y'(t)=f^{[1]}(t,y(t))+f^{[2]}(t,y(t)),\quad y(0)=y_0, \tag{4.1}$$

where $f^{[1]}$ is stiff and satisfies (1.2) with $\tau_1=0$, $f^{[2]}$ is not stiff and satisfies the Lipschitz condition

$$\|f^{[2]}(t,y)-f^{[2]}(t,\widetilde{y})\|\le L_2\|y-\widetilde{y}\|,\quad \forall y,\widetilde{y}\in R^m, \tag{4.2}$$

where the real number L_2 is a Lipschitz constant. It is obvious that the problems (4.1)-(4.2) belong to $K_{\sigma,0}$, where $\sigma=[\sigma_1,2L_2]$. We apply the methods (1.5) with $N=2$ to the problems (4.1)-(4.2) and consider that the method $(A^{[1]},b^{[1]},c)$ is implicit and the method $(A^{[2]},b^{[2]},c)$ is explicit or implicit. If the method $(A^{[2]},b^{[2]},c)$ is explicit, then the methods (1.5) become the IMEX RK methods.

We now introduce some Lemmas.

Lemma 4.1 (see [29]). *If the RK method $(A^{[1]},b^{[1]},c)$ is diagonally-stable, then there exists the real number $h_0>0$ such that this method is ANS-stable for any given $h<h_0$.*

Lemma 4.2. *If the method $(A^{[1]},b^{[1]},c)$ in ARK methods* (1.5) *with $N=2$ for the problems* (4.1)-(4.2) *is diagonally-stable, then there exists the real number $h_1>0$ such that these ARK methods are ANS-stable for any given $h<h_1$, i.e.,*

$$\left(I-h\sum_{\mu=1}^{2}\overline{A}^{[\mu]}Z_n^{[\mu]}\right)^{-1}$$

exists, and

$$\left\|\left(I-h\sum_{\mu=1}^{2}\overline{A}^{[\mu]}Z_n^{[\mu]}\right)^{-1}\right\|\le\widetilde{D}_5,\quad \left\|h\sum_{\nu=1}^{2}\overline{b}^{[\nu]T}Z_n^{[\nu]}\left(I-h\sum_{\mu=1}^{2}\overline{A}^{[\mu]}Z_n^{[\mu]}\right)^{-1}\right\|\le\widetilde{D}_6, \tag{4.3}$$

where $h<h_1$, and $\widetilde{D}_5,\widetilde{D}_6>0$ are real constants. Moreover, if the method $(A^{[1]},b^{[1]},c)$ is also algebraically-stable, then there exists the real number $\widetilde{D}_4>0$, such that for any given $h<h_1$,

$$\|K(Z_n)\|\le 1+\widetilde{D}_4h.$$

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Proof. Let $A_1=\overline{A}^{[1]}Z_n^{[1]}$, $A_2=\overline{A}^{[2]}Z_n^{[2]}$. If the method $(A^{[1]},b^{[1]},c)$ is diagonally-stable, then $(A^{[1]})^{-1}$ exists, and it is ANS-stable by Lemma 4.1, i.e., there exist real numbers $h_0,\widetilde{D}_1,\widetilde{D}_2>0$ such that $(I-hA_1)^{-1}$ exists, and

$$\|(I-hA_1)^{-1}\|\le\widetilde{D}_1,\quad \|h\overline{b}^{[1]T}Z_n^{[1]}(I-hA_1)^{-1}\|\le\widetilde{D}_2, \tag{4.4}$$

hold, where $h<h_0$. Apparently, there exists $\widetilde{h}_0\le h_0$ such that for any given $0<h<\widetilde{h}_0$, $(I-hA_1-hA_2)^{-1}$ exists.

Let $(I-hA_1-hA_2)^{-1}u=v$ for $\forall u\in R^{ms}$. We have

$$\begin{aligned}&\|(I-hA_1)^{-1}u\|=\|v-(I-hA_1)^{-1}hA_2v\|\\&\ge(1-\|(I-hA_1)^{-1}hA_2\|)\|v\|\\&\ge(1-h\widetilde{D}_1\|A^{[2]}\|L_2)\|v\|,\end{aligned} \tag{4.5}$$

where $h<\widehat{h}_0=1/(\widetilde{D}_1\|A^{[2]}\|L_2)$. From (4.4) and (4.5), we obtain

$$\left\|\left(I-h\sum_{\nu=1}^{2}\overline{A}^{[\nu]}Z_n^{[\nu]}\right)^{-1}\right\|=\|(I-hA_1-hA_2)^{-1}\|\le\frac{\widetilde{D}_1}{1-h\widetilde{D}_1\|A^{[2]}\|L_2}<\widetilde{D}_5,$$

where

$$h<h_1<\min\{\widetilde{h}_0,\widehat{h}_0\},\quad \widetilde{D}_5=\frac{\widetilde{D}_1}{1-h_1\widetilde{D}_1\|A^{[2]}\|L_2}.$$

Moreover, for $\forall u\in R^{ms}$,

$$\begin{aligned}&h\sum_{\mu=1}^{2}\overline{b}^{[\mu]T}Z_n^{[\mu]}\left(I-h\sum_{\nu=1}^{2}\overline{A}^{[\nu]}Z_n^{[\nu]}\right)^{-1}u\\&=h\overline{b}^{[2]T}Z_n^{[2]}\left(I-h\sum_{\nu=1}^{2}\overline{A}^{[\nu]}Z_n^{[\nu]}\right)^{-1}u+h\overline{b}^{[1]T}Z_n^{[1]}\left(I-h\sum_{\nu=1}^{2}\overline{A}^{[\nu]}Z_n^{[\nu]}\right)^{-1}u\\&=h\overline{b}^{[2]T}Z_n^{[2]}\left(I-h\sum_{\nu=1}^{2}\overline{A}^{[\nu]}Z_n^{[\nu]}\right)^{-1}u-\overline{b}^{[1]T}\overline{A}^{[1]-1}u\\&\quad+\overline{b}^{[1]T}\overline{A}^{[1]-1}(I-hA_2)(I-hA_1-hA_2)^{-1}u.\end{aligned}$$

Thus

$$\left\|h\sum_{\mu=1}^{2}\overline{b}^{[\mu]T}Z_n^{[\mu]}\left(I-h\sum_{\nu=1}^{2}\overline{A}^{[\nu]}Z_n^{[\nu]}\right)^{-1}\right\|\le\widetilde{D}_6,$$

where $h<h_1$,

$$\widetilde{D}_6=(1+\widetilde{D}_5)\|\overline{b}^{[1]T}\overline{A}^{[1]-1}\|+h_1L_2\widetilde{D}_5(\|\overline{b}^{[1]T}\overline{A}^{[1]-1}\|\cdot\|\overline{A}^{[2]}\|+\|\overline{b}^{[2]}\|).$$

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Therefore these ARK methods are ANS-stable for any given $h<h_1$. Moreover, it follows from Lemma 3.1 and the above proven ANS-stability that if the method $(A^{[1]},b^{[1]},c)$ is also algebraically-stable, then there exists the real number $\widetilde{D}_4>0$, such that for any given $h<h_1$,

$$\|K(Z_n)\|\leq 1+\widetilde{D}_4 h.$$

So, the lemma is proved. □

It follows from Lemma 4.2, Theorems 3.1 and 3.2 that we obtain the following theorems.

Theorem 4.1. *Assume that the method $(A^{[1]},b^{[1]},c)$ in the ARK methods* (1.5) *with $N=2$ and stage-order q for the problems* (4.1)-(4.2) $\in K_{\sigma,0}$ *with $N=2$ is diagonally-stable and algebraically-stable, $\varepsilon_0=\mathcal{O}(h^q)$. Then these ARK methods are optimally B-convergent of order q.*

Theorem 4.2. *Assume that the method $(A^{[1]},b^{[1]},c)$ in the ARK methods* (1.5) *with $N=2$ and stage-order q for the problems* (4.1)-(4.2) $\in K_{\sigma,0}$ *with $N=2$ is diagonally-stable and algebraically-stable, $\varepsilon_0=\mathcal{O}(h^{q+1})$, $d_0^{[\nu]}=0$, all components of $d^{[\nu]}$ are equal $(\nu=1,2)$. Then these ARK methods are optimally B-convergent of order $q+1$.*

Remark 4.1. In fact, IMEX RK methods are of stage-order only 1, and all components of $d^{[2]}$ cannot be equal. Therefore, in general, IMEX RK methods for the problems (4.1)-(4.2) are optimally B-convergent of order 1. This shows that the order-reduction phenomenon will appear.

Method 4.1. The second-order ARK methods

0	1/2	−1/2	0	0	0
1	1/2	1/2	1	1	0
$(A^{[1]},b^{[1]})$	1/2	1/2	$(A^{[2]},b^{[2]})$	1/2	1/2

are of stage-order 1, and $(A^{[1]},b^{[1]},c)$ is algebraically-stable and diagonally-stable. Therefore, these ARK methods for the problems (4.1)-(4.2) are optimally B-convergent of order 1 by Theorem 4.1.

Method 4.2. The second-order Lobatto IIIC-IIIA RK pair

0	1/2	−1/2	0	0	0
1	1/2	1/2	1	1/2	1/2
$(A^{[1]},b^{[1]})$	1/2	1/2	$(A^{[2]},b^{[2]})$	1/2	1/2

is of stage-order 1, and $(A^{[1]},b^{[1]},c)$ is algebraically-stable and diagonally-stable. Therefore, this ARK pair for the problems (4.1)-(4.2) is optimally B-convergent of order 1 by Theorem 4.1.

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Method 4.3. The second-order midpoint-rule pair

1/2	1/2	0	1/2	1/2	0
1/2	0	1/2	1/2	1/2	0
$(A^{[1]},b^{[1]})$	1/2	1/2	$(A^{[2]},b^{[2]})$	1/2	1/2

is of stage-order $q=1$, $d^{[1]}=d^{[2]}=-\frac{1}{8}e$, $d_0^{[\nu]}=0$, i.e., $B^{[\nu]}(2)$ $(\nu=1,2)$ hold, and $(A^{[1]},b^{[1]},c)$ is algebraically-stable and diagonally-stable. Therefore, this pair for the problems (4.1)-(4.2) is optimally B-convergent of order 2 by Theorem 4.2.

5 Numerical examples

In this section, we use Methods 4.1-4.3 to solve the following problems and obtain the corresponding results in Tables 1-6, respectively. "err1, err2, err3" denote the maxium norm errors under three different stepsizes. The reference exact solutions are obtained by the code RAUAU5 with stepsize $h=0.001$. The observed orders confirm our results given in Theorems 4.1 and 4.2.

Example 5.1. Consider the Robertson reaction problem (cf. [1,17])

$$\begin{cases} y_1'(t)=-0.04y_1+10^4y_2y_3, \\ y_2'(t)=0.04y_1-10^4y_2y_3-3\times10^7y_2^2, \\ y_3'(t)=3\times10^7y_2^2, \\ y_1(0)=1, \quad y_2(0)=y_3(0)=0, \end{cases} \tag{5.1}$$

where $0\le t\le 1$, the right function $f(t,y)$ is separated into the two parts:

$$f^{[1]}=\begin{pmatrix} -0.04y_1 \\ 0.04y_1-3\times10^7y_2^2 \\ 3\times10^7y_2^2 \end{pmatrix}, \quad f^{[2]}=\begin{pmatrix} 10^4y_2y_3 \\ -10^4y_2y_3 \\ 0 \end{pmatrix}.$$

Example 5.2. Consider the chemical recation problem (cf. [17])

$$\begin{cases} y_1'(t)=-\widehat{A}y_1-\widehat{B}y_1y_3, \\ y_2'(t)=\widehat{A}y_1-\widehat{M}\widehat{C}y_2y_3, \\ y_3'(t)=\widehat{A}y_1-\widehat{B}y_1y_3-\widehat{M}\widehat{C}y_2y_3+\widehat{C}y_4, \\ y_4'(t)=\widehat{B}y_1y_3-\widehat{C}y_4, \\ y_1(0)=1.76\times10^{-3}, \quad y_2(0)=y_3(0)=y_4(0)=0, \end{cases} \tag{5.2}$$

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Table 1: Method 4.1 for the problem (5.1).

t	err1 ($h=0.01$)	err2 ($h=0.01/2$)	err3 ($h=0.01/4$)	$\log_2$ (err1/err2)	$\log_2$ (err2/err3)
0.1	1.405513e-007	6.451758e-008	2.772225e-008	1.123333	1.218648
0.3	1.286798e-007	5.891544e-008	2.520981e-008	1.127068	1.224660
0.5	1.186927e-007	5.420737e-008	2.310299e-008	1.130671	1.230409
0.7	1.102191e-007	5.021629e-008	2.132048e-008	1.134148	1.235915
0.9	1.029770e-007	4.680785e-008	1.980075e-008	1.137501	1.241195

Table 2: Method 4.2 for the problem (5.1).

t	err1 ($h=0.01$)	err2 ($h=0.01/2$)	err3 ($h=0.01/4$)	$\log_2$ (err1/err2)	$\log_2$ (err2/err3)
0.1	2.711217e-008	1.323327e-008	5.698417e-009	1.121412	1.215536
0.3	2.415578e-008	1.172313e-008	5.02665e-009	1.125218	1.221688
0.5	2.164425e-008	1.044867e-008	4.461972e-009	1.128861	1.227563
0.7	2.052818e-008	9.363898e-009	3.983172e-009	1.132450	1.233191
0.9	1.853212e-008	8.433393e-009	3.573977e-009	1.135842	1.238582

Table 3: Method 4.3 for the problem (5.1).

t	err1 ($h=0.01$)	err2 ($h=0.01/2$)	err3 ($h=0.01/4$)	$\log_2$ (err1/err2)	$\log_2$ (err2/err3)
0.1	6.2115849e-008	1.028078e-008	2.218761e-009	2.595011	2.212123
0.3	3.965908e-008	1.104130e-008	2.344352e-009	1.844741	2.235648
0.5	4.745249e-008	1.121332e-008	2.415444e-009	2.081270	2.214852
0.7	4.770344e-008	1.124767e-008	2.448678e-009	2.084466	2.199552
0.9	4.731072e-008	1.118791e-008	2.455505e-009	2.080227	2.187848

Table 4: Method 4.1 for the problem (5.2).

t	err1 ($h=0.01$)	err2 ($h=0.01/2$)	err3 ($h=0.01/4$)	$\log_2$ (err1/err2)	$\log_2$ (err2/err3)
0.1	7.787217e-014	3.860401e-014	1.741570e-014	1.012357	1.148362
0.3	7.787000e-014	3.538693e-014	1.675721e-014	1.137851	1.078435
0.5	7.786783e-014	3.743909e-014	1.689632e-014	1.056482	1.147836
0.7	7.786501e-014	3.515316e-014	1.642959e-014	1.147321	1.097358
0.9	7.786133e-014	3.821171e-014	1.785916e-015	1.026892	1.097351

where $\widehat{A}=7.89\times10^{-10}$, $\widehat{B}=1.1\times10^{7}$, $\widehat{C}=1.13\times10^{3}$ and $\widehat{M}=1.0\times10^{6}$, the right function $f(t,y)$ is separated into the two parts:

$$f^{[1]}=\begin{pmatrix} -\widehat{A}y_1-\widehat{B}y_1y_3 \\ -\widehat{M}\widehat{C}y_2y_3 \\ -\widehat{B}y_1y_3-\widehat{M}\widehat{C}y_2y_3 \\ -\widehat{C}y_4 \end{pmatrix}, \quad f^{[2]}=\begin{pmatrix} 0 \\ \widehat{A}y_1 \\ \widehat{A}y_1+\widehat{C}y_4 \\ \widehat{B}y_1y_3 \end{pmatrix}.$$

6 Conclusions

The nonlinear stability and B-convergence properties of ARK methods for nonlinear stiff problems with the vector function f partitioned into N parts with different stiffness

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Table 5: Method 4.2 for the problem (5.2).

t	err1 ($h=0.01$)	err2 ($h=0.01/2$)	err3 ($h=0.01/4$)	$\log_2$ (err1/err2)	$\log_2$ (err2/err3)
0.1	3.584372e-016	1.773755e-016	8.738670e-017	1.014914	1.021321
0.3	3.606056e-016	1.797607e-016	8.977194e-017	1.004344	1.001741
0.5	3.610393e-016	1.810618e-016	9.107298e-017	0.995674	0.991387
0.7	3.621235e-016	1.827965e-016	9.085614e-017	0.986244	1.008582
0.9	3.629909e-016	1.838807e-016	9.085614e-017	0.981163	1.017114

Table 6: Method 4.3 for the problem (5.2).

t	err1 ($h=0.01$)	err2 ($h=0.01/2$)	err3 ($h=0.01/4$)	$\log_2$ (err1/err2)	$\log_2$ (err2/err3)
0.1	5.009014e-017	1.246705e-017	3.141670e-018	2.006407	1.988516
0.3	3.577867e-017	8.957517e-018	2.248869e-018	1.997929	1.993899
0.5	2.515349e-017	6.301414e-018	1.579783e-018	1.997011	1.995949
0.7	1.734723e-017	4.344459e-018	1.088707e-018	1.997457	1.996561
0.9	1.301043e-017	3.257172e-018	8.161282e-019	1.997977	1.996752

are considered. By using different RK methods for different parts of the function f, the computational cost will be heavily reduced, and the computation becomes more efficient.

In this paper, the concept of $(\theta,\bar{p},\bar{q})$-algebraic stability of ARK methods for the stiff problem class $K_{\sigma,\tau}$ is introduced, and it is proven that this stability implies some contractive properties of the ARK methods. Some results on optimal B-convergence of ARK methods for $K_{\sigma,0}$ are given. $(\theta,\bar{p},\bar{q})$-algebraic stability of ARK methods generalize the algebraic stability of ARK methods and (θ,p,q)-algebraic stability of RK methods in the literature. For difficulty in verifying the ANS-stability of general ARK methods, we further consider FSRK methods and ARK methods with $N=2$, whose diagonal stability ensures ANS-stability. For ARK methods with $N=2$, we obtain some similar results on optimal B-convergence to those of RK methods and show that the IMEX RK methods for the problems (4.1)-(4.2) are optimally B-convergent of order 1. Thus the order-reduction phenomenon appears. Some numerical examples also verify the theoretical results given in Theorems 4.1 and 4.2. Moreover, it is not difficult to extend the new results in this paper from the fixed stepsize to the variable stepsizes.

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