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Dissipation, surface energy and defects

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A model for Poiseuille flow of nematic liquid crystals is examined where a layered structure of defects parallel to the flow is present. Constant gradient flows are discussed, together with symmetric and chevron configurations for such flow problems.

1 Introduction

In this article, we examine non-Newtonian aspects of flow of polymeric liquid crystals and their relationship with the presence of defects in the flow region. Non-Newtonian phenomena become specially relevant in flow with a larger Ericksen number, $\mathscr{E}$. The latter measures the ratio of the viscous torque of the flow with respect to the elastic torque.

Flows corresponding to large $\mathscr{E}$ present a high density of defects which increases with increasing values of this parameter (cf. Larson & Mead, 1992). While this seems to consistently be the case for polymeric materials, it has also been observed in flows of low molecular weight liquid crystals (cf. Gleeson *et al.* 1991; Wahl & Fischer, 1973). Other non-Newtonian aspects of polymeric liquid crystal flow include *hysteresis* as well as *stress jumps* (cf. Berry, 1988; Marrucci, 1981; Marrucci & Maffetone, 1989; Viola *et al.* 1985).

The condition of $\mathscr{E}$ being large renders rigorous the notion of *fast flow*, and it is a physically relevant condition for polymeric materials. We employ the model developed by Ericksen (1991), and emphasize the role of the order parameter in describing non-Newtonian properties of the flow.

We carry out the present analysis for the case of Poiseuille flow regimes. We consider plane aligning configurations, and construct solutions with a layered structure of defects parallel to the flow. The presence of the order parameter in the model is a cause for non-uniqueness of solutions of boundary value problems for the governing equations. We take the principle of minimum dissipation as the selection mechanism between the presence of either a defect or a stress jump along a line of the flow domain.

In the modelling of steady state flow phenomena according to the Leslie–Ericksen equations, $\mathscr{E}$ is the relevant non-dimensional parameter of the problem. In the present model, we find an additional quantity playing a significant role in the analysis: it is the ratio $\mathscr{I} = \nu/K$ of the non-gradient free energy contribution represented by a parameter $\nu > 0$ with respect to $K > 0$ corresponding to a typical Frank constant. We refer to $\mathscr{I}$ as the *interface* number, and it is associated with the free energy that is required to maintain defects in the flow configuration. So, in the present context, such energy penalization translates into a predominance of the non-gradient type energy with respect to its gradient counterpart.

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The properties of Poiseuille flow that we explore here are in fact part of a larger mathematical structure shared by liquid crystal systems with very diverse degrees of order (Calderer, 1995). The main components of such a structure may be presented as follows:

(1) Existence of a *Lyapunov* function for selected boundary conditions on the domain.
(2) Existence of a positive definite and non-convex *dissipation* function.

Although the principle of *minimum dissipation* has found ample applications in the study of liquid crystal flow, it is perhaps the attempt of justifying its limitations as well as its applicability as a variational principle that might bring out some relevant properties of the theory. On the other hand, it is a well-known fact that non-convex variational problems often do not posses a solution, but rather a *minimizing sequence* approaching the infimum of the functional. Such a minimizing sequence often realizes itself in terms of an ever finer layer structure associated with the physical nature of the problem. In the present case, the fine defect structure associated with *chevronning* results in a value of the total dissipation less than that corresponding to the three-dimensional out-of-plane configuration realized in low molecular weight liquid crystal flow with a large Ericksen number. The analysis of Poiseuille flow given here intends to provide an example of the mathematically rich structure of stationary states.

We conclude this introduction by returing to the question of the limitations and validity of the variational principle minimizing the dissipation. From one point of view, it turns out that an unsteady liquid crystal flow with low Reynolds number converges to a stationary state asymptotically in time (Calderer, 1995; Lin & Liu, 1995). On the other hand, stationary states of liquid crystal flow posses a high degree of non-uniqueness, and more so with increasing order of the phases. Minimizing the dissipation then provides a rule for selecting limiting stationary states. However, the physical significance of such stationary states stands only within the dynamics of a small Reynolds number (Calderer, 1995).

2 Poiseuille flow of nematic liquid crystals

2.1 Modelling

We assume that the *Helmholtz free energy*, $\mathscr{F}$, and the Cauchy stress tensor, σ, are of the form (Ericksen, 1991)

$$\mathscr{F} = \mathscr{F}(s, \nabla \boldsymbol{n}, \nabla s) = k_1|\nabla s|^2 + k_2 s^2|\nabla \boldsymbol{n}|^2 + \nu f(s), \tag{1}$$

$$\sigma = -pI - \nabla \boldsymbol{n}^{\mathrm{T}} \frac{\partial \mathscr{F}}{\partial \nabla \boldsymbol{n}} - \nabla s \otimes \frac{\partial \mathscr{F}}{\partial \nabla s} + \hat{\sigma}, \tag{2}$$

respectively. Here $\boldsymbol{n}$ is the usual nematic director, the order parameter s represents the degree of orientation of the nematic sample and k_1, k_2 and ν denote positive constants. $f(s)$ denotes the scalar contribution to the free energy, and it typically exhibits a double-well structure. We let $-\frac{1}{2} < s_1 \leqslant s_2 < s_3 < 1$, with $s_1 s_2 = 0$, denote the critical points of $f(s)$. When s is constant, the free energy becomes the usual Frank free energy (see equation (3) in the article by Faelti and Virga in this volume).

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Also, $\hat{\sigma}$ denotes the viscous part of the stress tensor which we will assume to be linear in the velocity gradient:

$$\hat{\sigma} = (\beta_1 \dot{s} + \alpha_1 \boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{n})\,\boldsymbol{n}\otimes\boldsymbol{n} + \alpha_2 \boldsymbol{N}\otimes\boldsymbol{n} + \alpha_3 \boldsymbol{n}\otimes\boldsymbol{N} + \alpha_4 \boldsymbol{A} + \alpha_5 \boldsymbol{A}\boldsymbol{n}\otimes\boldsymbol{n} + \alpha_6 \boldsymbol{n}\otimes\boldsymbol{A}\boldsymbol{n}, \tag{3}$$

$$\boldsymbol{N} = \dot{\boldsymbol{n}} - \Omega\boldsymbol{n}, \tag{4}$$

with A and Ω representing the symmetric and skew part of $\nabla\boldsymbol{v}$, respectively; $p(\boldsymbol{x}, t)$ denotes the pressure. The α_i are the ususal Leslie viscosity coefficients.

The balance laws associated with the dependent variables $\boldsymbol{v}$, $\boldsymbol{n}$ and s are given by

$$\rho\dot{\boldsymbol{v}} = \nabla\cdot\sigma, \tag{5}$$

$$\beta_2 \dot{s} = \nabla\cdot\left(\frac{\partial\mathscr{F}}{\partial\nabla s}\right) - \frac{\partial\mathscr{F}}{\partial s} - \beta_3 \boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{n}, \tag{6}$$

$$\gamma_1(s)\,\dot{\boldsymbol{n}}\times\boldsymbol{n} = \nabla\cdot\left(\frac{\partial\mathscr{F}}{\partial\nabla\boldsymbol{n}}\right)\times\boldsymbol{n} - \frac{\partial\mathscr{F}}{\partial\boldsymbol{n}}\times\boldsymbol{n} + \gamma_1\,\Omega\boldsymbol{n}\times\boldsymbol{n} - \gamma_2\,\boldsymbol{A}\boldsymbol{n}\times\boldsymbol{n}. \tag{7}$$

The constant $\rho > 0$ denotes the density and $\alpha_i(s)$, $\beta_i(s)$ and $\gamma_i(s)$ are constitutive functions.

The Clausius–Duhem inequality requires the coefficients to satisfy:

$$\alpha_4 > 0, \quad \beta_2 > 0, \tag{8}$$

$$\alpha_1 + \frac{3}{2}\alpha_4 + \alpha_5 + \alpha_6 - \frac{\beta_1^2}{\beta_2} > 0, \quad 2\alpha_4 + \alpha_5 + \alpha_6 - \frac{\gamma_2^2}{\gamma_1} > 0, \tag{9}$$

$$\gamma_1 = \alpha_3 - \alpha_2 > 0, \quad \gamma_2 = \alpha_6 - \alpha_5. \tag{10}$$

Ericksen's theory imposes additional requirements on the stress coefficients:

$$\alpha_6 - \alpha_5 = \alpha_2 + \alpha_3, \quad \beta_1 = \beta_3. \tag{11}$$

We now list additional hypotheses to be placed on the Leslie coefficients. Their motivation lies on the properties of the approximation formulas of such coefficients as obtained from the molecular theory of lyotropic liquid crystals developed in by Kuzuu & Doi (1983–1984). Their derivation is given in Calderer & Mukherjee (1995):

$$\lim_{s\to 0}\frac{\gamma_1}{\gamma_2}(s) = 0, \quad \gamma_1(0) = 0 = \gamma_2(0), \tag{12}$$

$$\lim_{s\to 1^-}\frac{\gamma_1}{\gamma_2}(s) = -1, \quad \lim_{s\to -\frac{1}{2}^+}\frac{\gamma_1}{\gamma_2}(s) = 1, \tag{13}$$

$$-\beta(s) \equiv \beta_1(s) < 0, \quad s\gamma_2(s) \leqslant 0, \quad s\in\left(-\frac{1}{2}, 1\right). \tag{14}$$

Moreover, we consider flows satisfying the *alignment condition*

$$\left|\frac{\gamma_1}{\gamma_2}(s)\right| \leqslant 1, \quad \text{for all} \quad s\in\left(-\frac{1}{2}, 1\right). \tag{15}$$

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The dependence of the Leslie coefficients on s derived in the present theory for lyotropic liquid crystals turns out to be consistent with the dependence of such coefficients on the temperature in the case of thermotropic materials (Pieransky & Guyon, 1976).

The following non-dimensional quantities become relevant in the study of *fast flow* regimes:

$$\mathscr{E} = \frac{\eta}{K}, \quad \mathscr{I} = \frac{\nu}{K} \quad \text{and} \quad \mathscr{J} = \frac{\mathscr{E}}{\mathscr{I}}. \tag{16}$$

Here η, K represent typical viscosity and elasticity coefficients, respectively. Since the current model does not include a surface energy term, $\mathscr{I}$ takes on such a role: its contribution becomes relevant whenever the solution presents an interface. In particular, conditions ensuring the stability of boundary layers are formulated in terms of such a quantity (Calderer & Mukherjee, 1995).

2.2 Constant gradient flows

We now consider the class of plane steady state flows such that

$$\boldsymbol{v} = (0, 0, v_3), \quad p = p(x, y, z),$$
$$n_1 = \sin\phi, \quad n_2 = 0, \quad n_3 = \cos\phi. \tag{17}$$

We let
$$v_3 = v(x), \quad \phi = \phi(x), \quad s = s(x), \tag{18}$$

$x \in (-1, 1)$. We denote the derivative with respect to x with a prime.

Constant solutions $\{\phi, s\}$ of the governing equations (corresponding to constant $v'(x) = \kappa$) satisfy:

(1) If $\mathscr{E} = 0$ then $s = 0$ solves the equations. Moreover, in such a case ϕ is undertermined.
(2) Solutions with $s \neq 0$ satisfy the algebraic equations

$$f'(s) = e\mathscr{J}\beta(s)\left(1 - \frac{\gamma_1^2}{\gamma_2^2}(\sigma)\right)^{\frac{1}{2}}$$
$$\equiv eH(s, \mathscr{J}), \tag{19}$$

with
$$e = \pm 1.$$

Furthermore, for each $s \in (-\frac{1}{2}, 1)$ satisfying (19),

$$\cos 2\phi = -\frac{\gamma_1(s)}{\gamma_2(s)}. \tag{20}$$

For a given value of $\mathscr{J}$ equation (19) admits multiple solutions, as illustrated in Figure 1.

An interpretation of the solution branches previously obtained in terms of the first normal stress difference

$$\sigma_N \equiv \sigma_{33} - \sigma_{11}$$
$$= -C\frac{\sin 2\phi}{2\gamma_2}(\alpha_1\gamma_1 + \gamma_2^2), \quad C > 0 \text{ constant},$$

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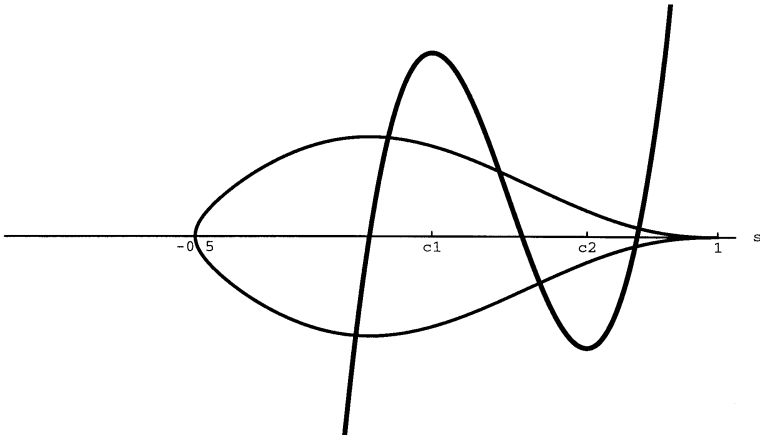


FIGURE 1. The graph of $f'(s)$ together with the graphs of $\pm H(s, \mathscr{J})$. The upper part of the figure shows the points of intersection, $S_i^+, i = 1, 2, 3$, of $f'(s)$ with $H(s, \mathscr{J})$. Likewise, the lower part shows the points $S_i^-, i = 1, 2, 3$.

gives the following classification:

(1) If $s_1 = 0$ then solutions such that $s = S_i^+, 1 \leqslant i \leqslant 3$, $s = S_1^-$ satisfy

$$\sigma_N > 0. \tag{21}$$

Otherwise, $\sigma_N < 0$ holds.

(2) If $s_2 = 0$ then solutions such that $s = S_1^-, S_3^+$ satisfy (21). Moreover the remaining ones satisfy $\sigma_N < 0$.

From another point of view, Ericksen showed that, under time-dependent perturbations of the director, aligning flows of the Leslie–Ericksen equations are stable provided

$$\kappa \frac{\gamma_2}{\gamma_1} \sin 2\phi < 0.$$

Such a condition becomes necessary for the stability of the analogous flows in the present theory. (In the current non-dimensionalized context, the parameter $\mathscr{J}$ plays the role of the former shear rate constant $\kappa \geqslant 0$.)

It follows from (12) and (14) that *the above stability condition holds if and only if the corresponding normal stress difference associated with such flow is positive*.

2.3 Symmetric Poiseuille flow configurations

Next we consider flow regimes with large Ericksen number,

$$0 < \mathscr{E}^{-1} \equiv \mu \ll 1. \tag{22}$$

The goal is to construct solutions with line defects parallel to the velocity field. We restrict ourselves to symmetric solutions. The first type of solutions that we obtain exhibit one line defect only located at the centre of the region. The second class present one or more *isotropic internal layers* away from the centre. The order parameter takes the value zero across such layers, while the director experiences a jump discontinuity of the order ± 45 degrees. While the first type corresponds to those flows predicted from the Leslie–Ericksen

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theory (Leslie, 1979), the second ones seem to be a novelty of modelling flow problems with the inclusion of the order parameter.

Next we consider non-constant flows as in equation (17). They satisfy

$$0 = \mu(a_1 s'' - k_2 s(\phi')^2) - \mathscr{J}^{-1} f'(s) - g_3(s) v' \sin\phi\cos\phi, \tag{23}$$

$$0 = (g_1(s)\sin^2\phi - g_2(s)\cos^2\phi) v' + \mu a_2 (s^2\phi')', \tag{24}$$

$$v'(x) = g^{-1}(s,\phi)\, x, \quad x \in (-1,1), \tag{25}$$

with $$g(s,\phi) = \frac{1}{2}\alpha_4 + \alpha_1 \sin^2\phi\cos^2\phi + \frac{1}{2}(\alpha_5 - \alpha_2)\sin^2\phi + \frac{1}{2}(\alpha_3 + \alpha_6)\cos^2\phi > 0,$$

$$g_1(s) = \frac{1}{2\eta}(-\gamma_1 + \gamma_2), \quad g_2(s) = \frac{1}{2\eta}(\gamma_1 + \gamma_2),$$

$$g_3(s) = \frac{\beta_3}{\eta} \quad \text{and} \quad a_i = \frac{k_i}{K}, \quad i = 1,2.$$

Also, boundary conditions for $s, \phi, \partial p/\partial x$ are prescribed at $x = \pm 1$ together with vanishing values of $v(\pm 1)$. The dependence of the velocity on the flow alignment shown by equation (25) may be in agreement with experimental observations on velocity profiles reported in, for example, Sardi *et al.* (1992).

The goal is to find solutions which, away from the boundary of the flow region, admit the following asymptotic expansions:

$$s(x,\mu) = \sum_{n=0}^{N} s_n(x)\mu^n + O(\mu^{N+1}), \tag{26}$$

$$\phi(x,\mu) = \sum_{n=0}^{N} \phi_n(x)\mu^n + O(\mu^{N+1}). \tag{27}$$

We neglect the boundary layer terms that should otherwise appear in the previous expansions to satisfy the boundary conditions. We denote the leading terms

$$\{s_0, \phi_0\} \quad \text{with} \quad \{S, \Phi\}, \tag{28}$$

respectively.

A standard asymptotic argument shows that $\{S, \Phi\}$ satify algebraic equations analogous to those in (19) and (20), but exhibit an explicit dependence on x as a result of (25). The next proposition results from the pointwise solvability of such equations, and gives solution pairs, $(S^+(x), \Phi^+(x))$ and $(S^-(x), \Phi^-(x))$, that present a line defect at $x = 0$ and are regular elsewhere.

Proposition 1 *Suppose that* $\alpha_i, \beta_i, \gamma_i$ *are as in* §2.1. *Suppose* $s_1 = 0$. *Then,*

(1) *the equations for the leading terms admit even continuous solutions* $S = S^+(x)$ *and* $S = S^-(x)$, $x \in (-1,1)$. *Moreover, they are monotonic on the subintervals* $(0,1)$ *and* $(-1,0)$.
(2) $S^+(\cdot)$ *and* $S^-(\cdot)$ *satisfy* $S^+(0) = 0 = S^-(0)$.

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(3) *the solutions* $\Phi^+(\cdot)$ *and* $\Phi^-(\cdot)$ *associated with* $S^+(\cdot)$ *and* $S^-(\cdot)$, *respectively*, *are discontinuous at* $x = 0$ *and satisfy*

$$\Phi^{\pm}(x) = -\Phi^{\pm}(-x), \quad x \in (-\alpha, \alpha), \tag{29}$$

$$0 \leqslant \Phi^+(x) < \frac{\pi}{4}, \quad -\frac{\pi}{4} < \Phi^-(x) < 0, \quad x \in (0, \alpha), \tag{30}$$

$$\lim_{x \to 0^+} \Phi^+(x) = \frac{\pi}{4} = \lim_{x \to 0^-} \Phi^-(x), \tag{31}$$

$$\lim_{x \to 0^-} \Phi^+(x) = -\frac{\pi}{4} = \lim_{x \to 0^+} \Phi^-(x). \tag{32}$$

We point out that the first normal stress difference associated with each of the fields $\{S^{\pm}(x), \Phi^{\pm}(x)\}$ is, in fact, positive. (However, a construction analogous to the previous one yields solutions with negative values of the normal stress difference, in the case that $s_2 = 0$.)

2.4 Chevron configurations in Poiseuille flow

Let $0 < X < 1$ and set

$$s(x) = S_1^+, \quad x \in (0, X),$$
$$s(x) = S_1^-, \quad x \in [X, 1),$$
$$s(x) = s(-x), \quad x \leqslant 0,$$
$$\phi(x) \text{ satisfies equation (4.9)}, \quad \text{for} \quad x \in (0, 1),$$

and

$$\phi(x) = -\phi(-x), \quad \text{for} \quad x < 0.$$

$S_1^{\pm}$ are as in §2.2. It is easy to check, however, that in the case $\mu = 0$, $\{S, \Phi\}$ do not solve the governing equations even in a weak sense. In particular, the σ_{12} component of the stress tensor corresponding to such a construction is not continuous. The presence of an internal layer around X becomes necessary to achieve the balance of the relevant stress component about $x = X$.

So, we proceed to construct solutions admitting an internal layer around $X \in (0, 1)$. For a given integer $N \geqslant 0$, we seek solutions that admit the following asymptotic expansion:

$$s(x, \mu) = \sum_{n=0}^{N} s_n(x)\mu^n + \sum_{n=0}^{N} IS_n(\lambda)\mu^n + o(\mu^N), \tag{33}$$

$$\phi(x, \mu) = \sum_{n=0}^{N} \phi_n(x)\mu^n + \sum_{n=0}^{N} I\Phi_n(\lambda)\mu^n + o(\mu^N), \tag{34}$$

where

$$\lambda = \frac{1}{\epsilon}(x - X), \quad \text{with} \quad \mu = \epsilon^2. \tag{35}$$

The terms IS_n and $I\Phi_n, n \geqslant 0$, represent internal layers around $x = X$. We introduce the notation

$$IS \equiv IS_0, \quad I\Phi \equiv I\Phi_0. \tag{36}$$

The following proposition summarizes properties of fields $\{S, \Phi\}$ away from the layer $x = X$ and the boundaries $x = \pm 1$:

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Proposition 2 *Suppose that the hypotheses of Proposition* 1 *hold. We assumed that* $f(s)$ *satisfies growth conditions at* $s = -\frac{1}{2}, 1$, *i.e.*

$$\lim_{s \to \{-\frac{1}{2}, 1\}} \frac{F(s)}{|f'(s)|} = 0. \tag{37}$$

Then,

(1) *there exists an even solution* $S = S(x), x \in (-1, 1)$ *that is discontinuous at* $x = \pm X$. *Moreover, it is monotonic on the subintervals* $(0, X), (X, 1), (-1, -X)$ *and* $(-X, 0)$. *It satisfies* $S(0) = 0$.

(2) $\Phi(\cdot)$ *is odd, discontinuous at* $x = \pm X$ *and satisfies*

$$0 \leqslant \Phi(x) < \frac{\pi}{4}, \quad x \in (-X, 0) \cup (X, 1)$$

and

$$-\frac{\pi}{4} < \Phi(x) < 0, \quad x \in (-1, -X) \cup (0, X). \tag{38}$$

Moreover,

$$\lim_{x \to 0^+} \Phi(x) = -\frac{\pi}{4} \quad \text{and} \quad \lim_{x \to 0^-} \Phi(x) = \frac{\pi}{4}.$$

We denote

$$\hat{S}_-(x) = S(x), \quad x \in (0, X) \quad \text{and} \quad \hat{S}_+(x) = S(x), \quad x \in (X, 1), \tag{40}$$

and, likewise, for $\hat{\Phi}_\pm(x)$.

An alternate construction in the case that $f(s)$ is non-convex allows the coexistence of distinct solution branches in the region such as S_1^+ and S_3^+ of §2.2. We observe that matching of such branches does not involve passing through the isotropic state. On the other hand, the corresponding solution at X experiences spatially rapid changes on s and ϕ which result into the analogous behaviour of the shear and normal stress components. One may interpret it in terms of local *melt-fracture* or *stress jumps* (cf. Viola *et al*. 1985).

Next we study the internal layer terms. For a given function $\omega(\lambda), \lambda \in (-\infty, \infty)$ differentiable, we denote

$$\frac{d\omega}{d\lambda} \equiv \dot{\omega}.$$

The governing system for $\{IS(\lambda), I\Phi(\lambda)\}, \lambda \in (-\infty, \infty)$ is:

$$0 = a_1(I\ddot{S}) - (\mathscr{J}^{-1}f''(S) + \frac{1}{2}g_3'(S)\,v'(X)\sin(2\Phi))(IS) - g_3(S)\,v'(X)\cos 2\Phi(I\Phi) + q_1(IS, I\Phi), \tag{41}$$

$$0 = \frac{1}{2}(g_1(S) + g_2(S))\,v'(X)\sin 2\Phi \sin 2(I\Phi) + (g_1'(S)\sin^2\Phi - g_2'(S)\cos^2\Phi)\,v'(X)(IS) + a_2 S^2(I\ddot{\Phi}) + q_2(IS, I\Phi), \tag{42}$$

where $\quad S = \hat{S}_-(X), \quad \Phi = \hat{\Phi}_-(X), \quad$ whenever $\quad \lambda < 0,$

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and
$$S = \hat{S}_+(X), \quad \Phi = \hat{\Phi}_+(X) \quad \text{whenever} \quad \lambda > 0. \tag{44}$$

$q_1(IS, I\Phi), q_2(IS, I\Phi)$ denote the nonlinear terms of the equations. We note that the coefficients of $(IS, I\Phi)$ in equations (41) and (42) are discontinuous at $\lambda = 0$.

We seek solutions $(IS(\lambda), I\Phi(\lambda)), \lambda \in (-\infty, \infty)$ of (41)–(42) satisfying the following conditions:

$$\lim_{\lambda \to 0^+} IS(\lambda) = -S_+(X), \quad \lim_{\lambda \to 0} IS(\lambda) = -S_-(X),$$

$$\lim_{\lambda \to 0^+} I\Phi(\lambda) = \frac{\pi}{4} - \Phi_+(X), \quad \lim_{\lambda \to 0^-} I\Phi(\lambda) = -\frac{\pi}{4} - \Phi_-(X),$$

and
$$\lim_{\lambda \to \pm\infty} IS(\lambda) = 0, \quad \lim_{\lambda \to \pm\infty} I\Phi(\lambda) = 0. \tag{45}$$

The first two sets of limiting relations establish the singularity of the line $x = X$, with the director jumping from $\frac{1}{4}\pi$ to $-\frac{1}{4}\pi$ across such a line. The last pair of limits involve the customary requirement that, away from the boundary layer region, the contribution of the boundary layer terms becomes negligible.

Existence of solutions of the problem (41)–(45) follow under the assumptions of Proposition 1, together with the additional condition

$$\mathscr{I} \gg 1. \tag{46}$$

We point out that the significance of such a condition may be explained in terms of a large contribution of the elastic energy of non-gradient type (with respect to the Frank–Oseen counterpart). We may regard such an energy input as a requirement to maintain internal layer configurations.

3 Selection of points X

For a given integer $M > 0$, we consider points $X_i, 1 \leqslant i \leqslant M$, such that $0 < X_i < 1$, $X_i < X_{i+1}$. We then construct a $2M$-layered solution following the method developed in the previous sections.

We point out that the method previously shown does not determine M as well as some of the points X. However, for the functions that we construct to be approximations of solutions of the boundary value problem (23)–(35), the data M and X_i cannot be arbitrarily prescribed, but must be determined from μ.

There are two mechanisms of selection of the layer position X that reveal the different nature of such points.

First, let us assume that $f(\cdot)$ is non-convex and $0 < c_1 < c_2 < 1$ denote the critical points of $f'(\cdot)$ as in §2.2. Let $x_1 > 0$ denote the points such that $s(x_1) = c_1$. Then for $\mathscr{J} > 0$ such that $c_1 J < 1$, we take $X = x_1$.

Moreover, for such a choice of X, we observe that another solution construction is available by letting

$$s(x) = S_3^+, \quad x \in [X, 1),$$

instead of S_1^- in (29). Out of both such solutions, we select the one that minimizes the rate of dissipation.

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The second mechanism is associated with the fact that the first term of the right-hand side of (3) changes sign at $s = 0$, and consequently, the forcing term of the equation becomes oscillatory. (Moreover, points where s vanishes are associated with infinite values of $|\phi'|$.) We constructed a numerical algorithm to get approximations of such points.

4 Conclusions

In plane steady state flows there are multiple solutions to the governing equations. A classification and interpretation of the solution branches has been given, and a stability condition in §2.2 holds if and only if the normal stress difference associated with the flow is positive. In symmetric Poiseuille flow configurations, solutions away from the boundary of the flow region are shown to have asymptotic expansions which incorporate discontinuities. Asymptotic solutions for chevron configurations in Poiseuille flow are also found to include discontinuities. A method for determining the points of discontinuity has been outlined in §3.

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