FINITE-BY-NILPOTENT GROUPS

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1. *Introduction.* 1·1. *Notation.* Let

$$1 = Z_0 \leqslant Z_1 \leqslant Z_2 \leqslant \dots \tag{1}$$

and

$$G = \Gamma_1 \geqslant \Gamma_2 \geqslant \Gamma_3 \geqslant \dots \tag{2}$$

be, respectively, the upper and lower central series of a group G. By definition, Z_{i+1}/Z_i is the centre of G/Z_i and $\Gamma_{j+1} = [\Gamma_j, G]$ is the commutator subgroup of Γ_j with G. When necessary for clearness, we write $Z_i(G)$ for Z_i and $\Gamma_j(G)$ for Γ_j.

If H is a normal subgroup of G, it is also convenient to define

$$H^{(0)} = H, \quad H^{(k+1)} = [H^{(k)}, G] \quad (k = 0, 1, 2, \dots). \tag{3}$$

In this notation, $\Gamma_{j+1} = G^{(j)}$ and Z_i is the largest normal subgroup H of G such that $H^{(i)} = 1$.

1·2. We say that G is a *finite-by-nilpotent* or *f.n. group* if it has a finite normal subgroup N such that G/N is nilpotent. The condition that G/N be nilpotent is equivalent to $\Gamma_{k+1} \leqslant N$ for some finite value of k. Hence f.n. groups are precisely those groups in which some term of the lower central series is finite.

We owe to R. Baer ((1), p. 396 or (2)) the following

THEOREM 1. *Let G be a group such that, for some k, the index $(G: Z_k) = m$ is finite. Then Γ_{k+1} is also finite and its order divides some power of m.*

(The case $k = 1$ is due to I. Schur and is an important part of his theory of the 'Multiplicator'.)

According to Baer's theorem, all groups in which some term of the upper central series is of finite index are f.n. groups. The main object of the present note is to prove the converse of this, that is, to show that f.n. groups may be defined alternatively as those groups G for which some $(G: Z_l)$ is finite.

It follows that with any f.n. group G we may associate two numerical invariants $k = k(G)$ and $l = l(G)$, these being by definition the smallest integers such that Γ_{k+1} and G/Z_l are finite. From Baer's theorem we always have $k \leqslant l$. We shall show that $l \leqslant 2k$; and, more precisely, that if k and l are any positive integers such that

$$k \leqslant l \leqslant 2k, \tag{4}$$

then there exists an f.n. group, indeed a nilpotent group, having these numbers as its invariants.

1·3. Before stating our main theorem, we first remark that in any group G, and for all $k = 1, 2, \dots$, we have

$$[Z_k, \Gamma_k] = 1. \tag{5}$$

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Thus Z_{k+1}, and so *a fortiori* Z_k, is contained in the centralizer H of Γ_{k+1}. Note also that, in the notation (3) above, $H^{(k)} \leqslant \Gamma_{k+1}$. Also, if Γ_{k+1} is finite, then G/H is also finite, since it is faithfully represented by automorphisms of the finite group Γ_{k+1}.

Our main result is

THEOREM 2. *Let $\Gamma_{k+1} = \Gamma_{k+1}(G)$ be finite, let H be its centralizer in G and let*

$$G = \{H, a_1, a_2, \ldots, a_r\},$$

where r may be taken to be the minimum number of generators of the finite group G/H. Define

$$B_s = H^{(k-s)}/Z_{k+s} \cap H^{(k-s)} \quad (s = 0, 1, \ldots, k).$$

Then B_s is isomorphic with a subgroup of the direct product of r^s copies of the finite Abelian group B_0. In particular, $B_k = H/Z_{2k} \cap H$ is finite and hence $(G : Z_{2k})$ is finite.

Thus for an f.n. group the inequality $l \leqslant 2k$ holds, as stated above.

COROLLARY. *Let M be a normal subgroup of a group G such that both $(G : M)$ and $M^{(k)}$ are finite. Then Γ_{k+1} and $(G : Z_{2k})$ are finite.*

For $M/M^{(k)} \leqslant Z_k(G/M^{(k)})$. Baer's theorem then gives that

$$\Gamma_{k+1}(G/M^{(k)}) = M^{(k)}\Gamma_{k+1}(G)/M^{(k)}$$

is finite. Hence $\Gamma_{k+1}(G)$ is finite. Theorem 2 then tells us that G/Z_{2k} is also finite.

1·4. We prove Theorem 2 with the help of the following

LEMMA 1. *Let K, L and G be normal subgroups of any group. Then, in the notation* (3) *above, we have*

$$[K^{(a)}, L^{(b)}]^{(c)} \leqslant \prod_{i=0}^{c} [K^{(a+i)}, L^{(b+c-i)}].$$

From this, a very simple deduction yields

LEMMA 2. *In any group G, let H and K be the centralizers of Γ_{k+1} and Γ_{l+1}, respectively. Then*

$$[H, K] \leqslant Z_{k+l-1} \quad (k, l = 1, 2, 3, \ldots)$$

and

$$[H^{(a)}, H^{(b)}] \leqslant Z_c,$$

provided $a + b + c \geqslant 2k - 1$. In particular, the derived group H' of H is contained in Z_{2k-1}.

It is a corollary of this lemma that, in a group whose centre is 1, all the centralizers of the terms of the lower central series are Abelian.

1·5. Suppose that G is an f.n. group with invariants k and l. It is easy to show that if either (i) $\Gamma_{k+1} \cap Z_1 = 1$ or (ii) G has an Abelian subgroup of finite index or (iii) G is finitely generated, then $k = l$. The case (i) is trivial. For by an elementary property of the upper central series $\Gamma_{k+1} \cap Z_1 = 1$ implies $\Gamma_{k+1} \cap Z_l = 1$ for all l. But $Z_l^{(k)} \leqslant \Gamma_{k+1} \cap Z_l$. Hence $Z_l^{(k)} = 1$ or $Z_l \leqslant Z_k$. Since in any case $k \leqslant l$ by Baer's theorem, we deduce that $k = l$.

The case (ii) is particularly interesting because we shall give examples of f.n. groups with $l = 2k$ which have a nilpotent subgroup of class 2 (i.e. just non-Abelian) and of finite index. But both (ii) and (iii) are special cases of more general theorems and we shall therefore not deal with them further here.

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2. *Proof of the main theorem.* 2·1. Let the A's, B and C be normal subgroups of some containing group. Then the following formulae are well known:

$$[A_1 A_2 \dots A_n, B] = \prod_{i=1}^{n} [A_i, B] \tag{6}$$

and

$$[A, B, C] \leqslant [B, C, A] . [C, A, B], \tag{7}$$

where we use the convention $[A, B, C] = [[A, B], C]$ and recall that $[A, B] = [B, A]$. For (7), see Zassenhaus ((3), p. 60, Satz 14).

We first prove Lemma 1.

When $c = 0$, the two sides of the inequality coincide. Suppose the result holds for some $c \geqslant 0$, and argue by induction on c. Commuting both sides with G and using (6), we obtain

$$[K^{(a)}, L^{(b)}]^{(c+1)} \leqslant \prod_{i=0}^{c} [K^{(a+i)}, L^{(b+c-i)}, G].$$

But by (7), and the definition (3), we have

$$[K^{(a+i)}, L^{(b+c-i)}, G] \leqslant [K^{(a+i+1)}, L^{(b+c-i)}] [K^{(a+i)}, L^{(b+c-i+1)}].$$

Since all the subgroups involved are normal and therefore permutable, the result follows at once for the value $c+1$.

Next, we deduce Lemma 2.

By Lemma 1,

$$[H, K]^{(k+l-1)} \leqslant \prod_{i=0}^{k+l-1} [H^{(i)}, K^{(k+l-1-i)}]. \tag{8}$$

Either $i \geqslant l$ or else $k+l-1-i \geqslant k$. In the former case, $H^{(i)} \leqslant \Gamma_{l+1}$ and so $[H^{(i)}, K] = 1$ since K is the centralizer of Γ_{l+1}. But $K^{(b)} \leqslant K$ for all $b = 0, 1, \dots$, and so

$$[H^{(i)}, K^{(k+l-1-i)}] = 1.$$

Similarly in the other case. Thus all the factors on the right of (8) reduce to 1. This gives $[H, K]^{(k+l-1)} = 1$ or $[H, K] \leqslant Z_{k+l-1}$.

Similarly, taking $H = K = L$ in Lemma 1, we have

$$[H^{(a)}, H^{(b)}]^{(c)} \leqslant \prod_{i=0}^{c} [H^{(a+i)}, H^{(b+c-i)}].$$

If $a+b+c \geqslant 2k-1$, at least one of the numbers $a+i$ and $b+c-i$ must be $\geqslant k$. But $H^{(k)} \leqslant \Gamma_{k+1}$, $H^{(d)} \leqslant H$ for all $d = 0, 1, 2, \dots$, and so $[H^{(d)}, H^{(e)}] = 1$ whenever $\max(d, e) \geqslant k$. Thus again we have all terms on the right reducing to 1 and deduce that $[H^{(a)}, H^{(b)}] \leqslant Z_c$.

2·2. Now let Γ_{k+1} be finite, so that its centralizer H is of finite index and we may write $G = \{H, a_1, \dots, a_r\}$, as in the statement of Theorem 2. Clearly $H \cap \Gamma_{k+1}$ is the centre of Γ_{k+1}. Since $H^{(k)} \leqslant \Gamma_{k+1} \cap H$, it is clear that $B_0 = H^{(k)}/Z_k \cap H^{(k)}$ is a finite Abelian group. We prove Theorem 2 by showing that, for $0 \leqslant s < k$, B_{s+1} is isomorphic with a subgroup of the direct product of r copies of $B_s = H^{(k-s)}/Z_{k+s} \cap H^{(k-s)}$.

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By Lemma 2, $[H^{(k-s-1)}, H] \leqslant Z_{k+s}$. Since it is also contained in $H^{(k-s)} = [H^{(k-s-1)}, G]$, it follows that $H^{(k-s-1)}/M$ is a central factor of H, where $M = Z_{k+s} \cap H^{(k-s)}$. Now let $a \in G$, and let x and y be any elements of $H^{(k-s-1)}$. Then

$$[a, xy] = [a, y]\,[a, x]^y \equiv [a, x]\,[a, y] \mod M.$$

Moreover $[a, x] \in H^{(k-s)}$. For fixed a, the mapping

$$x \to [a, x]\,M$$

is therefore a homomorphism of $H^{(k-s-1)}$ into B_s. Let $K(a)$ be the kernel of this mapping. Then $H^{(k-s-1)}/K(a)$ is isomorphic with a subgroup of B_s. If $K = \bigcap_{i=1}^{r} K(a_i)$, it follows that $H^{(k-s-1)}/K$ is isomorphic with a subgroup of the direct product of r copies of B_s. If $x \in K$, we have $[a_i, x] \in M$ for $i = 1, 2, \ldots, r$. Also, since

$$M \leqslant K \leqslant H^{(k-s-1)} \quad \text{and} \quad [H^{(k-s-1)}, H] \leqslant M,$$

we have $[h, x] \in M$ for all $h \in H$. But H together with $a_1, \ldots, a_r$ generate G. Hence $[K, G] \leqslant M \leqslant Z_{k+s}$. Finally, therefore, $K \leqslant H^{(k-s-1)} \cap Z_{k+s+1}$ and therefore B_{s+1} is a homomorphic image of $H^{(k-s-1)}/K$. We may assume inductively that B_s is finite. Since every homomorphic image of a finite Abelian group A is isomorphic with a subgroup of A, we obtain the required result, viz. that B_{s+1} is isomorphic with a subgroup of the direct product of r copies of B_s. Thus Theorem 2 is completely proved.

It may be observed that, without any finiteness assumptions about Γ_{k+1}, our argument shows that in any group each of the exponents of the Abelian groups $B_0, B_1, \ldots, B_k$ is a multiple of the next.

3. *Counter-examples.* 3·1. Let p be any prime. We begin by constructing a p-group G, which is nilpotent of class $2p$, such that

$$Z_i = \Gamma_{2p-i+1} \quad (i = 0, 1, \ldots, 2p),$$

and such that Z_i/Z_{i-1} is of order p for $i = 1, 2, \ldots, p$, but of infinite order for $i = p+1$, $p+2, \ldots, 2p$. For this group, therefore, $k = p$ but $l = 2p$.

From this it is easy to form a nilpotent p-group having any assigned invariants k, l subject only to $k \leqslant l \leqslant 2k$, by choosing p sufficiently large. All these groups will possess a normal subgroup of index p which is nilpotent of class 2. As was remarked in § 1·5, it would not be possible to construct an f.n. group having $k < l$ and having an *Abelian* subgroup of finite index.

3·2. Let $C = C(p)$ be the cyclic group of order p. Let $D = D(p)$ be the non-Abelian group of order p^3 and exponent p if p is odd, and let $D(2)$ be the quaternion group. Let $F = F(p)$ be the central product of a countable infinity of groups each isomorphic with D. This is the group obtained from the direct product by identifying the centres of the direct factors D. Thus $F' = Z_1(F)$ is cyclic of order p and F/F' is an elementary Abelian p-group of infinite order. Finally, let $G = G(p)$ be the wreath product $F \wr C$, which is defined as follows. $G = \{\bar{F}, c\}$, where $c^p = 1$, $\bar{F} = F_1 \times F_2 \times \ldots \times F_p$ is the direct

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product of p copies of F; and if $u \in F$ and u_i is the corresponding element of F_i, we have $c^{-1}u_i c = u_{i+1}$, the suffixes being taken modulo p. We note that $\overline{F}$ is nilpotent of class 2, $\overline{F}' = Z_1(\overline{F}) = F_1' \times F_2' \times \ldots \times F_p'$ is an elementary Abelian group of order p^p with basis $z_1, z_2, \ldots, z_p$, where $F' = \{z\}$.

If we represent the element $z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_p^{\alpha_p}$ of $\overline{F}'$ by the vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ where the coordinates α_i are integers mod p, the group operation in $\overline{F}'$ becomes vector addition. Also $\alpha^c = (\alpha_p, \alpha_1, \ldots, \alpha_{p-1})$ and therefore

$$[\alpha, c] = (\alpha_p - \alpha_1, \alpha_1 - \alpha_2, \ldots, \alpha_{p-1} - \alpha_p) = \Delta\alpha, \quad \text{say.}$$

Thus $(\Delta^k\alpha)_{i+1} = (-1)^k \sum_{r=0}^{k} (-1)^r \binom{k}{r} \alpha_{k-r+i}$, where the suffixes are taken mod p. In particular,

$$\Delta^{p-1}\alpha = (\sigma, \sigma, \ldots, \sigma),$$

where $\sigma = \alpha_1 + \alpha_2 + \ldots + \alpha_p$ and $\Delta^p\alpha = 0$. Since $\overline{F}'$ is the centre of $\overline{F}$ and $G = \{\overline{F}, c\}$, it follows that $\alpha \in Z_k$ if and only if $\Delta^k\alpha = 0$. If we choose α so that $\sigma \not\equiv 0 \bmod p$, it follows that, for $k \leqslant p$,

$$\overline{F}' \cap Z_k = \{\Delta^{p-k}\alpha, \Delta^{p-k+1}\alpha, \ldots, \Delta^{p-1}\alpha\}.$$

Now let H be a normal subgroup of G not contained in $\overline{F}'$. Since $G/\overline{F}'$ is not the direct product of $\overline{F}/\overline{F}'$ with a group of order p, H must contain an element $u_1 v_2 \ldots w_p \in \overline{F} - \overline{F}'$, so that at least one factor x_i of this product must belong to $F_i - F_i'$. We can then choose $y \in F$ so that $[x, y] = z$. Then $[u_1 v_2 \ldots w_p, y_i] = z_i$, and all its conjugates $z_j = c^{i-j} z_i c^{j-i}$ must belong to $[H, G]$. Hence $[H, G] \geqslant \overline{F}'$, and therefore, if $k \leqslant p$, we have $Z_k = \overline{F}' \cap Z_k$ of order p^k. In particular $Z_p = \overline{F}'$.

Since $G/\overline{F}' \simeq E \wr C$, where E is an elementary p-group of infinite order, a similar argument shows that, for $k < p$, Z_{k+p} is generated mod $\overline{F}'$ by all elements of the form $\Delta^{p-k}u, \Delta^{p-k+1}u, \ldots, \Delta^{p-1}u$, where $u \in F$ and

$$\Delta^l u = [u_p, c, c, \ldots, c]$$

with l terms c; while $Z_{2p} = G$. Thus G is nilpotent of class $2p$, Z_i/Z_{i-1} is of order p for $i \leqslant p$, but of infinite order for $p < i \leqslant 2p$. It is now equally clear that $Z_i = \Gamma_{2p-i+1}$ for $0 \leqslant i \leqslant 2p$. The invariants k and l of G are therefore p and $2p$ respectively.

3·3. We now prove that

$$[Z_{p+i}, Z_{p+j}] = Z_{i+j-1} \tag{9}$$

if $0 < i, j < p$ and $i + j - 1 \leqslant p$. For Z_{p+i} and Z_{p+j} are both contained in $\overline{F}$ and so $[Z_{p+i}, Z_{p+j}] \leqslant \overline{F}'$. But the only subgroups of $\overline{F}'$ which are normal in G are the Z_k with $0 \leqslant k \leqslant p$. The elements of Z_{p+1} have the form $u_1 u_2 \ldots u_p \bmod Z_p = \overline{F}'$, where $u \in F$. The commutator of two such elements is a power of $z_1 z_2 \ldots z_p$ and so belongs to Z_1. Thus (9) holds when $i + j = 2$. We may argue by induction on $i + j$. Since $\overline{F}$ is of class 2 and Z_{p+i} is obtained from Z_{p+i-1} by adjoining all elements of the form $\Delta^{p-i}u$, $[Z_{p+i}, Z_{p+j}]$ is obtained from $[Z_{p+i-1}, Z_{p+j}][Z_{p+i}, Z_{p+j-1}]$ by adjoining all commutators of the form $[\Delta^{p-i}u, \Delta^{p-j}v]$. By the induction hypothesis, both terms of this product equal Z_{i+j-2}. But mod $\overline{F}'$,

$$(\Delta^l u)^{(-1)^l} \equiv u_p u_{p-1}^{-l} u_{p-2}^{\binom{l}{2}} \ldots u_{p-l}^{(-1)^l}.$$

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Also $[u, v]$ is a power of z and without loss of generality we may suppose $[u, v] = z$. Then in the vector notation we shall have $[\Delta^{p-i}u, \Delta^{p-j}v]^{\pm 1} = \alpha.\beta = (\alpha_1\beta_1, \ldots, \alpha_p\beta_p)$, where $\alpha_{p-r} = (-1)^r\binom{p-i}{r}$ and $\beta_{p-r} = (-1)^r\binom{p-j}{r}$. Hence $\alpha \in Z_i - Z_{i-1}$ and $\beta \in Z_j - Z_{j-1}$. But for any pair of vectors α, β the rth coordinate of $\Delta(\alpha.\beta)$ is

$$\alpha_{r-1}\beta_{r-1} - \alpha_r\beta_r = \alpha_{r-1}(\beta_{r-1} - \beta_r) + (\alpha_{r-1} - \alpha_r)\beta_r.$$

Thus
$$\Delta(\alpha.\beta) = \alpha^c.\Delta\beta + \Delta\alpha.\beta.$$

Since $(\Delta\alpha)^c = \Delta(\alpha^c)$, we deduce that

$$\Delta^k(\alpha.\beta) = \sum_{r=0}^{k}\binom{k}{r}\Delta^r\alpha^{c^r}.\Delta^{k-r}\beta.$$

But if $\alpha \in Z_i - Z_{i-1}$, so do all the α^{c^r}. In our case, therefore, $\Delta^r\alpha^{c^r} = 0$ for $r \geqslant i$ but $\Delta^{i-1}\alpha^{c^{i-1}} \neq 0$ and is of the form $(\sigma, \sigma, \ldots, \sigma)$; and similarly $\Delta^{k-r}\beta = 0$ if $k - r \geqslant j$ but $\Delta^{j-1}\beta \neq 0$ and has the form $(\tau, \tau, \ldots, \tau)$. We conclude that, if $k \geqslant i + j - 1$, $\Delta^k(\alpha.\beta) = 0$, but

$$\Delta^{i+j-2}(\alpha.\beta) = \binom{i+j-2}{i-1}\Delta^{i-1}\alpha^{c^{i-1}}.\Delta^{j-1}\beta \neq 0.$$

Hence $\alpha.\beta \in Z_{i+j-1} - Z_{i+j-2}$, as required. Thus (9) is proved.

Now let k and l be positive integers such that $k \leqslant l \leqslant 2k$ and choose any prime $p > 2k$. In $G = G(p)$, consider the subgroup $H = \{Z_{p+k}, c\}$. For $i < k$, we have $\Gamma_{i+1}(H).\overline{F}' \geqslant Z_{p+k-i}$. Since $\overline{F}'$ is finite, this makes $\Gamma_{i+1}(H)$ infinite. But $\Gamma_{k+1}(H) \leqslant \overline{F}'$, so that k is in fact the first invariant of H. From (9) we deduce at once that, for $i \leqslant k$, $Z_i(H) = Z_i(G)$, while $Z_{k+i}(H)\,\overline{F}' = Z_{p+i}(G)$. In particular, $Z_{2k-1}(H) \leqslant Z_{p+k-1}(G)$. Since $(Z_{p+k} : Z_{p+k-1})$ is infinite, the second invariant of H is $2k$. Finally, it is easy to see that the invariants of the group H/Z_{2k-l} are precisely k and l.

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