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Variational principles for symplectic eigenvalues

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Abstract. If A is a real $2n \times 2n$ positive definite matrix, then there exists a symplectic matrix M such that $M^T AM = \mathrm{diag}(D, D)$, where D is a positive diagonal matrix with diagonal entries $d_1(A) \leqslant \cdots \leqslant d_n(A)$. We prove a maxmin principle for $d_k(A)$ akin to the classical Courant–Fisher–Weyl principle for Hermitian eigenvalues and use it to derive an analogue of the Weyl inequality $d_{i+j-1}(A + B) \geqslant d_i(A) + d_j(B)$.

1 Introduction

Let $\mathbb{M}(2n)$ be the space of $2n \times 2n$ real matrices. An element M of this space is said to be *symplectic* if $M^T JM = J$, where $J = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$. If $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product on $\mathbb{R}^{2n}$, and $(\cdot, \cdot)$ the bilinear form $(x, y) = \langle x, Jy \rangle$, then M is symplectic if and only if $(Mx, My) = (x, y)$ for all $x, y \in \mathbb{R}^{2n}$. The collection of symplectic matrices will be denoted by $Sp(2n, \mathbb{R})$. This is a real Lie group.

Let $\mathbb{P}(2n)$ be the set of all $2n \times 2n$ real positive definite matrices. Williamson's theorem says that for every A in $\mathbb{P}(2n)$, there exists a symplectic matrix M such that

$$M^T AM = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}, \tag{1.1}$$

where D is a diagonal matrix with positive diagonal entries

$$d_1(A) \leqslant d_2(A) \leqslant \cdots \leqslant d_n(A). \tag{1.2}$$

The numbers $d_j(A)$ are uniquely determined by A and characterise the orbits of $\mathbb{P}(2n)$ under the action of the group $Sp(2n, \mathbb{R})$. We call these numbers the *symplectic eigenvalues* of A. They are also called *Williamson parameters*. These play an important role in classical mechanics, quantum mechanics, symplectic topology, and the newer area of quantum information that has driven much of the recent work on the topic; see [1, 7–10, 12].

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For over a century, two recurring themes in the classical eigenvalue problem for Hermitian matrices have been variational principles (like the Courant–Fischer–Weyl minmax principle) and relationships between eigenvalues of A, B and $A + B$ (beginning with Weyl's inequalities and culminating in the solution of Horn's problem). We refer the reader to [2, Chapter III] and to the expository article [3] for a summary of these topics. Though there are major differences between the classical eigenvalue and the symplectic eigenvalue problems, it is very remarkable that, with appropriate modifications and suitable interpretations, many standard results in the classical problem have been extended to the symplectic case; see [5, 6, 8, 10, 11]. In this paper, we carry this program further by proving a maxmin principle for symplectic eigenvalues and, as a corollary, an analogue of the famous Weyl inequalities.

Let A be any $2n \times 2n$ real positive definite matrix. Then the matrix $A^{1/2}JA^{1/2}$ is skew-symmetric, and its eigenvalues occur in conjugate imaginary pairs. The absolute values of these are the numbers $d_j(A)$. Thus, in principle, the known theorems about Hermitian eigenvalues can be used to get information about symplectic eigenvalues. In practice, this often leads to cumbersome expressions from which it is difficult to extract elegant formulas. Our approach here is to work directly with the matrix A.

Theorem 1.1 (The maxmin principle) *Let A be any element of $\mathbb{P}(2n)$ with symplectic eigenvalues $d_1(A) \leqslant \cdots \leqslant d_n(A)$. Then for $1 \leqslant k \leqslant n$,*

$$d_k(A) = \max_{\substack{\mathcal{M}\subset\mathbb{R}^{2n} \\ \dim \mathcal{M}=2(n-k+1)}} \min_{\substack{u\in\mathcal{M}, v\in\mathbb{R}^{2n} \\ \langle u,Jv\rangle=1}} \frac{\langle u, Au\rangle + \langle v, Av\rangle}{2}. \tag{1.3}$$

(Here, it is understood that $\mathcal{M}$ is a linear subspace of $\mathbb{R}^{2n}$, and $\dim \mathcal{M}$ is its dimension.)

We use this to prove the following theorem.

Theorem 1.2 (A "Weyl inequality") *Let A and B be two elements of $\mathbb{P}(2n)$ and let $1 \leqslant i, j \leqslant n$ be two indices such that $i + j - 1 \leqslant n$. Then*

$$d_i(A) + d_j(B) \leqslant d_{i+j-1}(A + B). \tag{1.4}$$

One form of the maxmin principle known to us [5, 7] says

$$\frac{1}{d_k(A)} = \max_{\substack{\mathcal{M}\subset\mathbb{C}^{2n} \\ \dim \mathcal{M}=k}} \min_{\substack{x\in\mathcal{M} \\ \langle x,Ax\rangle=1}} \langle x, iJx\rangle. \tag{1.5}$$

The formula (1.3) has the advantage of involving expressions linear in A, and is more in line with the classical maxmin principle ([2, Cor. III.1.2]). Our proof of Theorem 1.2 will make good use of this. The special case of (1.4) with $j = 1$ has been considered in [4] and established under an additional assumption.

In Section 2, we recall some basic facts about symplectic eigenvalues. General references for some of the basic facts used here are [1, 7]. Proofs of the theorems are given in Section 3.

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2 Preliminaries

A *symplectic basis* for $\mathbb{R}^{2n}$ is a collection of $2n$ vectors $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ such that

$$\langle x_i, Jx_j \rangle = \langle y_i, Jy_j \rangle = 0, \quad \langle x_i, Jy_j \rangle = \delta_{ij}, \quad 1 \leqslant i, j \leqslant n.$$

If such a basis is also orthonormal with respect to the usual inner product on $\mathbb{R}^{2n}$, we call it an *orthosymplectic basis*. The standard basis $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}\}$ is such a basis.

Associated with each symplectic eigenvalue d_j of A is a pair of vectors u_j and v_j such that

$$Au_j = d_j Jv_j, \quad Av_j = -d_j Ju_j, \quad \langle u_j, Jv_j \rangle = 1, \quad 1 \leqslant j \leqslant n. \tag{2.1}$$

We call these *symplectic eigenvector pairs* of A. The collection $\{u_1, \dots, u_n, v_1, \dots, v_n\}$ is a symplectic basis for $\mathbb{R}^{2n}$. The columns of the matrix M in the Williamson decomposition (1.1) are symplectic eigenvector pairs of A.

Let $\mathcal{V}$ be any even-dimensional subspace of $\mathbb{R}^{2n}$. Its *symplectic complement* is the subspace

$$\mathcal{V}^{\perp_s} = \left\{ y \in \mathbb{R}^{2n} : \langle x, Jy \rangle = 0 \text{ for all } x \in V \right\}.$$

The subspace $\mathcal{V}$ is called a *symplectic subspace* if $\mathcal{V} \cap \mathcal{V}^{\perp_s} = \{0\}$. This is equivalent to saying that for every nonzero vector x in $\mathcal{V}$, there exists a vector y in $\mathcal{V}$ such that $\langle x, Jy \rangle \neq 0$. It can be seen that $\dim \mathcal{V} + \dim \mathcal{V}^{\perp_s} = 2n$. The span of symplectic eigenvector pairs corresponding to any collection of symplectic eigenvalues of A is a symplectic subspace.

Let $\mathcal{V}$ be a $2k$ dimensional symplectic subspace and let $\mathfrak{B} = \{x_1, \dots, x_k, y_1, \dots, y_k\}$ be a symplectic basis for $\mathcal{V}$. We define an (ordinary) inner product on $\mathcal{V}$ as follows. If

$$x = \sum_{j=1}^{k} (\alpha_j x_j + \beta_j y_j) \quad \text{and} \quad y = \sum_{j=1}^{k} (\gamma_j x_j + \delta_j y_j)$$

are any two vectors in $\mathcal{V}$, then

$$\langle x, y \rangle_{\mathfrak{B}} = \sum_{j=1}^{k} \left(\alpha_j \gamma_j + \beta_j \delta_j \right).$$

With respect to this inner product, $\mathfrak{B}$ is also an orthonormal basis for $\mathcal{V}$. We will need the notion of a $\mathfrak{B}$-*complementary vector* x' for a given vector $x \in \mathcal{V}$. Let $x = \sum_{j=1}^{k} \left(\alpha_j x_j + \beta_j y_j \right)$, and assume $\sum_{j=1}^{k} \alpha_j^2 + \beta_j^2 = 1$. Put $x' = \sum_{j=1}^{k} \left(-\beta_j x_j + \alpha_j y_j \right)$. Clearly, then

$$\langle x, x' \rangle_{\mathfrak{B}} = 0 \quad \text{and} \quad \langle x, Jx' \rangle = \sum_{j=1}^{k} \left(\alpha_j^2 + \beta_j^2 \right) = 1.$$

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3 Proofs

Throughout this section, A is a real $2n \times 2n$ positive definite matrix, $d_j(A)$ its symplectic eigenvalues enumerated as in (1.2), (u_j, v_j) the corresponding symplectic eigenvector pairs. From (2.1), it follows that

$$\langle u_j, Au_j \rangle = \langle v_j, Av_j \rangle = d_j(A). \tag{3.1}$$

Hence,

$$d_j(A) = \frac{\langle u_j, Au_j \rangle + \langle v_j, Av_j \rangle}{2}. \tag{3.2}$$

The special case $k = 1$ of Theorem 1.1 is already known [5]. We give a proof to serve as a warm up and to introduce the ideas that occur in later proofs.

Proposition 3.1

$$d_1(A) = \min_{\substack{u,v \in \mathbb{R}^{2n} \\ \langle u, Jv \rangle = 1}} \frac{\langle u, Au \rangle + \langle v, Av \rangle}{2}.$$

Proof Expand u, v as

$$u = \sum_{j=1}^{n} \left(\alpha_j u_j + \beta_j v_j \right) \quad \text{and} \quad v = \sum_{j=1}^{n} \left(\gamma_j u_j + \delta_j v_j \right). \tag{3.3}$$

The condition $\langle u, Jv \rangle = 1$ is equivalent to the condition $\sum_{j=1}^{n} \left(\alpha_j \delta_j - \beta_j \gamma_j \right) = 1$. From (2.1), (3.1), and (3.3), it follows that

$$\begin{aligned} \frac{1}{2}\left(\langle u, Au \rangle + \langle v, Av \rangle \right) &= \frac{1}{2} \sum_{j=1}^{n} d_j(A) \left(\alpha_j^2 + \beta_j^2 + \gamma_j^2 + \delta_j^2 \right) \\ &\geqslant \frac{1}{2} d_1(A) \sum_{j=1}^{n} \left(\alpha_j^2 + \beta_j^2 + \gamma_j^2 + \delta_j^2 \right) \\ &\geqslant d_1(A) \sum_{j=1}^{n} \left(\alpha_j \delta_j - \beta_j \gamma_j \right) \\ &= d_1(A). \end{aligned}$$

This shows that

$$d_1(A) \leqslant \min_{\substack{u,v \in \mathbb{R}^{2n} \\ \langle u, Jv \rangle = 1}} \frac{\langle u, Au \rangle + \langle v, Av \rangle}{2}.$$

The proposition now follows from (3.2). ∎

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Proposition 3.2 *Let $1 \leqslant k \leqslant n$ and let $\mathcal{V}_k$ be the space spanned by the vectors $u_k, \dots, u_n, v_k, \dots, v_n$. Then*

$$d_k(A) = \min_{\substack{u \in \mathcal{V}_k, v \in \mathbb{R}^{2n} \\ \langle u, Jv \rangle = 1}} \frac{\langle u, Au \rangle + \langle v, Av \rangle}{2}.$$

Proof Let u be any vector in $\mathcal{V}_k$ and v any vector in $\mathbb{R}^{2n}$. Let

$$u = \sum_{j=k}^{n} (\alpha_j u_j + \beta_j v_j) \quad \text{and} \quad v = \sum_{j=1}^{n} \left(\gamma_j u_j + \delta_j v_j \right). \tag{3.4}$$

The condition $\langle u, Jv \rangle = 1$ is equivalent to

$$\sum_{j=k}^{n} (\alpha_j \delta_j - \beta_j \nu_j) = 1. \tag{3.5}$$

From (2.1), (3.1), (3.4), and (3.5), we obtain

$$\begin{aligned}
\frac{1}{2}\big(\langle u, Au \rangle + \langle v, Av \rangle \big) &= \frac{1}{2} \sum_{j=k}^{n} d_j(A)(\alpha_j^2 + \beta_j^2) + \frac{1}{2} \sum_{j=1}^{n} d_j(A)(\gamma_j^2 + \delta_j^2) \\
&\geqslant \frac{1}{2} \sum_{j=k}^{n} d_j(A)(\alpha_j^2 + \beta_j^2 + \gamma_j^2 + \delta_j^2) \\
&\geqslant \frac{1}{2} d_k(A) \sum_{j=k}^{n} (\alpha_j^2 + \beta_j^2 + \gamma_j^2 + \delta_j^2) \\
&\geqslant d_k(A) \sum_{j=k}^{n} (\alpha_j \delta_j - \beta_j \gamma_j) \\
&= d_k(A).
\end{aligned}$$

The two extreme sides of this chain of inequalities are equal when $u = u_k$ and $v = v_k$. This proves the proposition. ∎

Proposition 3.3 *For $1 \leqslant k \leqslant n$, let $\mathfrak{B} = \{u_1, \dots, u_k, v_1, \dots, v_k\}$ and let $\mathcal{N}_k$ be the linear span of the vectors in $\mathfrak{B}$. Let $x = \sum_{j=1}^{k} (\alpha_j u_j + \beta_j v_j)$ be any vector in $\mathcal{N}_k$ with $\sum_{j=1}^{k} (\alpha_j^2 + \beta_j^2) = 1$. Let x' be the $\mathfrak{B}$-complementary vector for x. Then*

$$\frac{\langle x, Ax \rangle + \langle x', Ax' \rangle}{2} \leqslant d_k(A).$$

Proof Using relations (2.1), one can check that $\langle x, Ax \rangle \leqslant d_k(A)$ and $\langle x', Ax' \rangle \leqslant d_k(A)$. ∎

Proof of Theorem 1.1 Let $\mathcal{M}$ be any subspace of $\mathbb{R}^{2n}$ with $\dim \mathcal{M} = 2(n - k + 1)$. Let $\mathcal{N}_k$ be linear span of the family $\mathfrak{B} = \{u_1, \dots, u_k, v_1, \dots, v_k\}$. Then

$$\dim \mathcal{M} + \dim \mathcal{N}_k = 2n + 2.$$

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Hence, the intersection $\mathcal{M} \cap \mathcal{N}_k$ contains a nonzero vector. Choose such a vector $x = \sum_{j=1}^{k} (\alpha_j u_j + \beta_j v_j)$ with $\sum_{j=1}^{k} (\alpha_j^2 + \beta_j^2) = 1$, and let x' be its $\mathfrak{B}$-complementary vector. By Proposition 3.3,

$$\frac{\langle x, Ax\rangle + \langle x', Ax'\rangle}{2} \leqslant d_k(A).$$

This shows that

$$\min_{\substack{u \in \mathcal{M}_k, v \in \mathbb{R}^{2n} \\ \langle u, Jv\rangle = 1}} \frac{\langle u, Au\rangle + \langle v, Av\rangle}{2} \leqslant d_k(A).$$

Combining this with Proposition 3.2, we get Theorem 1.1. ∎

As an immediate corollary, we obtain the following theorem.

Theorem 3.4 (Monotonicity Principle) *Let A and B be two elements of $\mathbb{P}(2n)$. If $A \leqslant B$, then $d_k(A) \leqslant d_k(B)$ for $1 \leqslant k \leqslant n$. (This result is already known; see [5] .)*

Proof of Theorem 1.2 Let $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$, $\{w_1, \ldots, w_n, x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n, z_1, \ldots, z_n\}$ be symplectic eigenbases for A, B, and $A + B$, respectively, corresponding to their symplectic eigenvalues $d_j(A), d_j(B)$, and $d_j(A + B), 1 \leqslant j \leqslant n$. Let $1 \leqslant i, j \leqslant n$ be such that $i + j - 1 \leqslant n$. Let

$$\begin{aligned} \mathcal{U} &= \text{span}\{u_i, \ldots, u_n, v_i, \ldots, v_n\}, \\ \mathcal{V} &= \text{span}\{w_j, \ldots, w_n, x_j, \ldots, z_n\}, \\ \mathcal{W} &= \text{span}\{y_1, \ldots, y_{i+j-1}, z_i, \ldots, z_{i+j-1}\}. \end{aligned}$$

These spaces have dimensions $2(n - i + 1), 2(n - j + 1)$, and $2(i + j - 1)$, respectively. The sum of these dimensions is $4n + 2$. Hence, the three subspaces have an nonzero intersection. (If the sum of the dimensions of three subspaces of an m-dimensional vector space is bigger than $2m$, then they have a nonzero intersection.) Choose a nonzero vector x in this intersection. Let $\mathfrak{B}$ be the symplectic basis for $\mathcal{W}$ chosen above. Assume $\|x\|_{\mathfrak{B}} = 1$ and let x' be the $\mathfrak{B}$-complementary vector for x. Using Proposition 3.3, and then Proposition 3.2, we get

$$\begin{aligned} d_{i+j-1}(A + B) &\geqslant \frac{\langle x, (A + B)x\rangle + \langle x', (A + B)x'\rangle}{2} \\ &= \frac{\langle x, Ax\rangle + \langle x', Ax'\rangle}{2} + \frac{\langle x, Bx\rangle + \langle x', Bx'\rangle}{2} \\ &\geqslant d_i(A) + d_j(B). \end{aligned}$$

This proves Theorem 1.2. ∎

We remark that the vector x' in the last proof belongs to $\mathcal{W}$ but may not belong to $\mathcal{U}$ and $\mathcal{V}$. Proposition 3.2 does not require this to be so.

In the case of the classical Hermitian eigenvalue problem, the maxmin principle is complemented by a minmax principle. That is not so for the symplectic eigenvalue problem, because the symplectic group is not compact like the orthogonal group. The

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role that is played by "Rayleigh quotients"—products $\langle x, Ax\rangle$ subject to $\langle x, x\rangle = 1$—in the classical problem is played here by the expressions $\frac{1}{2}(\langle u, Au\rangle + \langle v, Av\rangle)$ subject to $\langle u, Jv\rangle = 1$. The set of pairs u, v satisfying this last constraint is unbounded.

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