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A note on Cusick's theorem on units in totally real cubic fields

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Let $\epsilon = \epsilon_1$, with conjugates ϵ_2, ϵ_3, be a unit in a totally real cubic field, and let $T(\epsilon) = \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2$. Let ϵ be a unit for which $T(\epsilon)$ is least and let η be a unit, not a power of ϵ, for which $T(\eta)$ is least. It was shown by Cusick [1] that ϵ, η form a pair of fundamental units under certain conditions. The purpose of the present note is to show that these conditions are unnecessary and that ϵ, η form a pair of fundamental units in all cases.

LEMMA. *Let* $\alpha = \alpha_1$ (*with conjugates* α_2, α_3), *and* $\beta = \beta_1$ (*with conjugates* β_2, β_3) *be any two units. Then* $T(\alpha\beta) - 2T(\alpha) + T(\alpha\beta^{-1}) > 0$.

Proof.
$$\begin{aligned} &T(\alpha\beta) - 2T(\alpha) + T(\alpha\beta^{-1}) \\ &= \alpha_1^2\beta_1^2 + \alpha_2^2\beta_2^2 + \alpha_3^2\beta_3^2 - 2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) + (\alpha_1^2\beta_1^{-2} + \alpha_2^2\beta_2^{-2} + \alpha_3^2\beta_3^{-2}) \\ &= \alpha_1^2(\beta_1 - \beta_1^{-1})^2 + \alpha_2^2(\beta_2 - \beta_2^{-1})^2 + \alpha_3^2(\beta_3 - \beta_3^{-1})^2 \\ &> 0. \end{aligned}$$

Equality cannot arise since no component of a unit can be $0, \pm 1$.

THEOREM. ϵ, η *form a pair of fundamental units.*

Proof. As shown by Cusick, if $T(\eta) \geqslant 12$ then the theorem is true except possibly when there is a unit $\epsilon^{\frac{1}{2}}\eta^{\frac{1}{2}}$ or $\epsilon^{\frac{3}{2}}\eta^{\frac{1}{2}}$. Let $U = T(\epsilon^{\frac{1}{2}}\eta^{\frac{1}{2}})$, $V = T(\epsilon^{\frac{3}{2}}\eta^{\frac{1}{2}})$, $W = T(\epsilon^{\frac{1}{2}}\eta^{\frac{3}{2}})$. By the Lemma we have $T(\epsilon) + T(\eta) > 2U$, $T(\epsilon) + W > 2V$, $V + T(\eta) > 2W$, whence

$$T(\epsilon) + T(\eta) > V + W.$$

But by the construction of ϵ, η we have $U, V, W \geqslant T(\eta) \geqslant T(\epsilon)$ and the contradictions show that such cases cannot arise.

It is easy to verify that the only fields for which $T(\eta) < 12$ are those with discriminants 49, 81, 148 and 169, and that the theorem is true in these cases also.

It may be noted that the lemma does not hold for the function

$$S(\epsilon) = \tfrac{1}{2}((\epsilon_1 - \epsilon_2)^2 + (\epsilon_2 - \epsilon_3)^2 + (\epsilon_3 - \epsilon_1)^2)$$

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that I used some years ago [2]: e.g. for the field with discriminant 257 defined by $\theta^3 - 7\theta^2 + 12\theta - 1 = 0$, with $\alpha = \theta^2 - 4\theta$, $\beta = \theta - 4$, we have $S(\alpha) = 15$, $S(\alpha\beta) = 15$, $S(\alpha\beta^{-1}) = 13$.

REFERENCES

[1] T. W. CUSICK. Finding fundamental units in cubic fields. *Math. Proc. Cambridge Philos. Soc.* **92** (1982), 385–389.

[2] H. J. GODWIN. The determination of units in totally real cubic fields. *Proc. Cambridge Philos. Soc.* **56** (1960), 318–321.

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