Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2016.m1421

Vol. **9**, No. 2, pp. 262-288
May 2016

A Multistep Scheme for Decoupled Forward-Backward Stochastic Differential Equations

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Received 31 July 2014; Accepted 6 March 2015

Abstract. Upon a set of backward orthogonal polynomials, we propose a novel multi-step numerical scheme for solving the decoupled forward-backward stochastic differential equations (FBSDEs). Under Lipschtiz conditions on the coefficients of the FBSDEs, we first get a general error estimate result which implies zero-stability of the proposed scheme, and then we further prove that the convergence rate of the scheme can be of high order for Markovian FBSDEs. Some numerical experiments are presented to demonstrate the accuracy of the proposed multi-step scheme and to numerically verify the theoretical results.

AMS subject classifications: 60H35, 65C20, 60H10

Key words: Decoupled forward-backward stochastic differential equations, backward orthogonal polynomials, multi-step numerical scheme, error estimate, numerical analysis.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete, filtered probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ ($T > 0$ a finite time) satisfying the usual conditions and $\mathcal{F}_0$ containing all the P-null sets of $\mathcal{F}$. On the probability space, a standard d-dimensional Brownian motion W_t is defined. Let $L^2 = L^2_{\mathcal{F}}(0,T)$ be the set of all $\mathcal{F}_t$-adapted and mean-square-integrable vector/matrix processes. We consider the system of decoupled forward-backward stochastic differential equations (FBSDEs)

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad t \in [0,T], \tag{1.1}$$

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0,T], \tag{1.2}$$

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where (1.1) is a forward stochastic differential equation (SDE) and (1.2) is a backward stochastic differential equation (BSDE). Assume that the coefficients b: $\Omega\times[0,T]\times\mathbb{R}^q \to \mathbb{R}^q$ and σ: $\Omega\times[0,T]\times\mathbb{R}^q \to \mathbb{R}^{q\times d}$, the generator f: $\Omega\times[0,T]\times\mathbb{R}^q\times\mathbb{R}^m\times\mathbb{R}^{m\times d} \to \mathbb{R}^m$, and the terminal condition ξ is mean-square integrable and $\mathcal{F}_T$-measurable. We note that the integrals with respect to W_s in (1.1) and (1.2) are the Itô-type integrals. A process $(X_t, Y_t, Z_t) : [0,T]\times\Omega \to \mathbb{R}^q\times\mathbb{R}^m\times\mathbb{R}^{m\times d}$ is called an L^2 solution of the decoupled FBSDEs (1.1) and (1.2) if, in the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, it is $\{\mathcal{F}_t\}$-adapted, square integrable and satisfies the Eqs. (1.1) and (1.2) [25]. In this paper, we assume that the system of decoupled FBSDEs (1.1) and (1.2) has a unique solution (X_t, Y_t, Z_t).

Under some standard conditions on the coefficients b and σ of the SDE (1.1) and the generator f of the BSDE (1.2), such as Lipschtiz conditions on b, σ and f, Pardoux and Peng first proved the existence and uniqueness of the solution of nonlinear BSDEs in [25] in 1990. Since then the theory of BSDEs and FBSDEs has been extensively studied, and important applications of them have been found in many fields, such as finance, risk measure, stochastic control, and so on [11, 13, 26–29]. Consequently it has become very important to solve the FBSDEs analytically or numerically for practical purposes. On the other hand it is often difficult to obtain analytic solutions of FBSDEs in explicitly closed form, or it is too complicated to compute the values of the solutions even we have the closed form of the solutions. Thus numerical methods for solving FBSDEs are highly desired, especially the methods with high efficiency and accuracy. There exist quite a few numerical methods for solving BSDEs and FBSDEs. Some of them are derived from the relationship between the FBSDEs and the corresponding parabolic PDEs, such as in [9, 17, 18, 21–23, 34], and others are obtained directly from discretizing BSDEs or FBSDEs, such as in [1–3, 5–8, 10, 12, 16, 19, 24, 29, 32, 33, 35–40]. Most of these algorithms are Euler-type methods with half order convergence rate [3, 4, 8–10, 17, 18, 21, 34]. In [12], a method with weak first order convergence was studied. In [16, 32, 35–40], some methods with convergence rates up to two were proposed and studied.

A multi-step scheme was first proposed for solving the BSDEs only (not combined together with forward SDEs). The high order of the scheme was numerically demonstrated and was theoretically proved for BSDEs with the generator f independent of Z_t. In [5, 6], the authors introduced high-order multi-step schemes for FBSDEs with general generators. They got the high-order error estimates for their schemes, in which the forward SDEs were not discretized.

In this paper, we propose a novel multi-step scheme for solving the decoupled FBSDEs (1.1) and (1.2). We first introduce a new set of orthogonal polynomials, which we call backward orthogonal polynomials and study some of their simple properties. Based on the theory of numerical integrals and polynomial approximation, we develop the new multi-step scheme by using the backward orthogonal polynomials and a special Gaussian process. Under not strong conditions, such as Lipschitz conditions on the coefficients b and σ of SDE (1.1) and the generator f of BSDE (1.2), we obtain error estimate of the proposed multi-step scheme in a very general form, which also implies

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the stability of the scheme; and on this error estimate results, for the decoupled FBSDEs (1.2) of Markovian type with the terminal condition $\xi = \varphi(X_T)$, we rigorously prove that the convergence rate of the scheme can be of high order when the coefficients b, σ, f and φ are smooth enough. The high-order error estimate result clearly shows the influence of the accuracy of numerical schemes for solving the forward SDEs on our scheme for solving FBSDEs. Our numerical results are consistent with the theoretical results.

For simple representations, let us first introduce the following notations:

- $|\cdot|$: the standard Euclidean norm in the Euclidean space $\mathbb{R}^q$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$.
- $\mathcal{F}_s^{t,x}(t \le s \le T)$: the σ-field generated by the diffusion process $\{X_r : t \le r \le s, X_t = x\}$ starting from the time-space point (t, x). When $s = T$, we use $\mathcal{F}^{t,x}$ to denote $\mathcal{F}_T^{t,x}$.
- $\mathbb{E}[X]$: the mathematical expectation of the random variable X.
- $\mathbb{E}_s^{t,x}[X]$: the conditional expectation of the random variable X under the σ-field $\mathcal{F}_s^{t,x}(t \le s \le T)$, i.e., $\mathbb{E}_s^{t,x}[X] = \mathbb{E}[X|\mathcal{F}_s^{t,x}]$. Let $\mathbb{E}_t^x[X] = \mathbb{E}[X|\mathcal{F}_t^{t,x}]$.
- $\mathbb{P}_i[a, b]$: the set of polynomials of degree i defined on $[a, b]$.
- C_b^k: the set of continuously differential functions $\phi : x \in R^q \to R$ with uniformly bounded partial derivatives $\partial_x^{k_1}\phi$ for $k_1 \le k$.
- $C_b^{l,k}$: the set of functions $\phi : (t, x) \in [0, T] \times R^q \to R$ with uniformly bounded partial derivatives $\partial_t^{l_1}\partial_x^{k_1}\phi$ for $l_1 \le l$ and $k_1 \le k$.
- $C_b^{l,k,k}$: the set of continuously differential functions $\phi : (t, x, y) \in [0, T]\times R^q \times R \to R$ with uniformly bounded partial derivatives $\partial_t^{l_1}\phi$ and $\partial_x^{k_1}\partial_y^{k_2}\phi$ for $l_1 \le l$ and $k_1 + k_2 \le k$.
- C_p^k: the set of k times continuously differentiable functions $\phi : x \in R^q \to R$ for which ϕ and all of its partial derivatives of order up to and including k have polynomial growth.

The rest of the paper is organized as follows. In Section 2, we introduce the backward orthogonal polynomials and discuss their properties. In Section 3, based on the backward orthogonal polynomials, we propose a novel multi-step scheme for solving the decoupled forward-backward differential stochastic Eqs. (1.1) and (1.2). We present a new Gronwall-type inequality, and then derive error estimate of the multi-step scheme in a general form in Section 4. In Section 5, in the case of the Markovian decoupled FBSDEs, under some reasonable regularity conditions on b, σ, f and $\xi = \varphi(X_T)$, we further prove high-order convergence of the scheme. Numerical examples and some conclusions are given in Section 6 and Section 7.

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2. Backward orthogonal polynomials

We first give the following definition.

Definition 2.1. *We call a set of polynomials* $\{Q_i(s)\}_{i=0}^{L}$ *defined on the interval* $[0,1]$ *the* backward orthogonal polynomials*, if for each* $i = 0, 1, \ldots, L$*, it holds that* $Q_i(s) \in \mathbb{P}_i[0,1]$ *and*

$$\int_0^1 Q_i(s)ds = 1, \qquad \int_0^1 Q_i(s)s^j ds = 0, \quad 1 \le j \le i. \tag{2.1}$$

Remark 2.1. The classic orthogonal polynomials $\{V_i(s)\}_{i=0}^{L}$ defined on the interval $[0,1]$ with $V_i(s) \in \mathbb{P}_i[0,1]$ and a weight function $w(s)$ is defined by

$$\int_0^1 w(s)V_i(s)V_j(s)ds = \delta_{ij}, \quad 0 \le i, j \le L,$$

which is almost equivalent to

$$\int_0^1 w(s)V_i(s)s^i ds \neq 0, \quad 0 \le i \le L, \qquad \int_0^1 w(s)V_i(s)s^j ds = 0, \quad 0 \le j < i \le L.$$

This is the main reason we call the polynomial set $\{Q_i(s)\}_{i=0}^{L}$ the backward orthogonal polynomials on $[0,1]$.

Note that the backward orthogonal polynomials with respect to a given weight function also could be defined similarly.

By Definition 2.1, we easily have the following lemma.

Lemma 2.1. *There is a unique set of polynomials* $\{Q_i(s)\}_{i=0}^{L}$ *on* $[0,1]$ *satisfying the conditions stated in* (2.1)*. Furthermore, let* $Q_i(s) = \sum_{j=0}^{i} a_{i,j}s^j$*, then for each* $i = 0, 1, \ldots, L$*, the* $a_{i,j}$ *is determined by*

$$\sum_{j=0}^{i} \frac{a_{i,j}}{j+1} = 1, \qquad \sum_{j=0}^{i} \frac{a_{i,j}}{j+m+1} = 0, \quad 1 \le m \le i. \tag{2.2}$$

We list the first seven of them in the following.

$$\begin{aligned}
&Q_0(s) = 1,\\
&Q_1(s) = -6\,s + 4,\\
&Q_2(s) = 30\,s^2 - 36\,s + 9,\\
&Q_3(s) = -140\,s^3 + 240\,s^2 - 120\,s + 16,\\
&Q_4(s) = 630\,s^4 - 1400\,s^3 + 1050\,s^2 - 300\,s + 25,\\
&Q_5(s) = -2772\,s^5 + 7560\,s^4 - 7560\,s^3 + 3360\,s^2 - 630\,s + 36,\\
&Q_6(s) = 12012\,s^6 - 38808\,s^5 + 48510\,s^4 - 29400\,s^3 + 8820\,s^2 - 1176\,s + 49.
\end{aligned}$$

Define the new polynomial set $\{P_i(s)\}_{i=0}^{L}$ on $[a,b]$, where $P_i(s) = Q_i\left(\frac{s-a}{b-a}\right)$. The polynomials $\{P_i(s)\}_{i=0}^{L}$ are then called the backward orthogonal polynomials on $[a,b]$. It is easy to verify that the polynomials $\{P_i(s)\}_{i=0}^{L}$ have the following properties: for each $i = 0, 1, \ldots, L$,

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1. $P_i(a) = Q_i(0)$,
2. $\int_a^b P_i(s)ds = b - a$,
3. $\int_a^b P_i(s)(s-a)^j ds = 0$, for $1 \le j \le i$,
4. $\int_a^b P_i(s)q(s)ds = q(a)(b-a)$ for any polynomial $q(s) \in \mathbb{P}_i[a,b]$.

About the polynomials $\{P_i(s)\}_{i=0}^L$, we have the following two lemmas.

Lemma 2.2. *Let $\{P_i(s)\}_{i=0}^L$ be the backward orthogonal polynomials on $[a,b]$. Then we have*

$$P_0(a) = 1 \quad \text{and} \quad P_i(a) > 1 \ \text{ if } \ 1 \le i \le L.$$

Proof. First obviously $P_0(a) = 1$. According to the properties of the polynomials $\{P_i(s)\}_{i=0}^L$, it holds that

$$\int_a^b P_i(s)ds = b - a \quad \text{and} \quad \int_a^b P_i^2(s)ds = P_i(a)(b-a).$$

Then for any i with $1 \le i \le L$, we have

$$\begin{aligned} 0 < \int_a^b (P_i(s)-1)^2 ds &= \int_a^b (P_i^2(s) - 2P_i(s) + 1)ds \\ &= P_i(a)(b-a) - 2(b-a) + (b-a) \\ &= (P_i(a) - 1)(b-a), \end{aligned}$$

which leads to the conclusion of the lemma. □

Lemma 2.3. *Let $\{P_i(s)\}_{i=0}^L$ be the backward orthogonal polynomials on $[a,b]$. Assume that the function $g(s)$ is differentiable with the $(i+1)$th-order derivarives $g^{(i+1)}$ bounded for $0 \le i \le L$. Then we have*

$$\int_a^b P_i(s)g(s)ds = g(a)(b-a) + O((b-a)^{i+2}).$$

Proof. Under the condition of the lemma, we obtain by the Taylor expansion,

$$g(s) = \sum_{j=0}^{i} \frac{g^{(j)}(a)}{j!}(s-a)^j + O((s-a)^{i+1}).$$

Then by using the definition of $P_i(s)$, we have

$$\begin{aligned} \int_a^b P_i(s)g(s)ds =& \sum_{j=0}^{i} \frac{g^{(j)}(a)}{j!} \int_a^b P_i(s)(s-a)^j ds + \int_a^b O((s-a)^{i+1})ds \\ =& g(a)(b-a) + O((b-a)^{i+2}). \end{aligned}$$

The proof is complete. □

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3. A multi-step scheme for the decoupled FBSDEs

For the time interval $[0, T]$, we introduce the following partition:

$$0 = t_0 < t_1 < \ldots < t_N = T. \tag{3.1}$$

Let $\Delta t_n = t_{n+1} - t_n, n = 0, 1, \ldots, N-1$ and $\Delta t = \max\limits_{0\le n\le N-1} \Delta t_n$. We also assume that the time partition have the following regularity:

$$\frac{\max\limits_{0\le n\le N-1} \Delta t_n}{\min\limits_{0\le n\le N-1} \Delta t_n} \le c_0, \tag{3.2}$$

where $c_0 \ge 1$ is a constant.

Let $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ be the solution of (1.1) and (1.2) starting from time t with $X_t = x$, that is, $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ satisfies

$$\begin{cases} X_s^{t,x} = x + \displaystyle\int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dW_r, & s \in [t, T], \\ Y_s^{t,x} = \varphi(X_T^{t,x}) + \displaystyle\int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x} dW_r, & s \in [t, T]. \end{cases} \tag{3.3}$$

Let $Y_{t_N}^{t_n,X^n} = \varphi(X_T^{t_n,X^n})$ and $(X_t^{t_n,X^n}, Y_t^{t_n,X^n}, Z_t^{t_n,X^n})$ be the solution of (3.3) starting from t_n with $x = X^n$ for $t \in [t_n, T]$. Denote $f(s, X_s^{t_n,X^n}, Y_s^{t_n,X^n}, Z_s^{t_n,X^n})$ by $f_s^{t_n,X^n}$.

3.1. Reference equation for Y_t

By (3.3), we have

$$Y_{t_n}^{t_n,X^n} = Y_{t_{n+1}}^{t_n,X^n} + \int_{t_n}^{t_{n+1}} f_s^{t_n,X^n} ds - \int_{t_n}^{t_{n+1}} Z_s^{t_n,X^n} dW_s. \tag{3.4}$$

Taking the conditional expectation $\mathbb{E}_{t_n}^{X^n}[\cdot]$ on both sides of (3.4), we get

$$Y_{t_n}^{t_n,X^n} = \mathbb{E}_{t_n}^{X^n}[Y_{t_{n+1}}^{t_n,X^n}] + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n}[f_s^{t_n,X^n}]ds. \tag{3.5}$$

Note that the integrand $\mathbb{E}_{t_n}^{X^n}[f_s^{t_n,X^n}]$ is a deterministic smooth function of time s under $\mathcal{F}_{t_n}$. To approximate the integral, any numerical methods for computing integrals with deterministic integrands can be used. When the values of (X_t, Y_t, Z_t) on the time levels t_{n+1}, t_{n+2}, …, t_{n+K_y} are available, where K_y is a positive integer with $1 \le K_y \le N - n$, it is often a natural idea to construct an approximation of the integrand $\mathbb{E}_{t_n}^{X^n}[f_s^{t_n,X^n}]$ based on these values. For better stability and accuracy implicit methods are often used [37–40] to solve $(Y_{t_n}^{t_n,X^n}, Z_{t_n}^{t_n,X^n})$, in which the values on the time level $t = t_n$ are also utilized for the approximation.

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As done in [40] we here choose the Lagrange interpolating polynomial $p_{K_y}^{t_n,X^n}(s)$ based on the support points $(t_{n+i},\ \mathbb{E}_{t_n}^{X^n}[f_{t_{n+i}}^{t_n,X^n}])$, $i=0,1,\ldots,K_y$, to approximate the deterministic integrand $\mathbb{E}_{t_n}^{X^n}[f_s^{t_n,X^n}]$. Thus we have

$$\begin{aligned}\int_{t_n}^{t_{n+1}}\mathbb{E}_{t_n}^{X^n}[f_s^{t_n,X^n}]ds &= \int_{t_n}^{t_{n+1}} p_{K_y}^{t_n,X^n}(s)ds + R_y^n \\ &= \Delta t_n\sum_{i=0}^{K_y} b_{K_y,i}^n\mathbb{E}_{t_n}^{X^n}[f_{t_{n+i}}^{t_n,X^n}] + R_y^n,\end{aligned} \tag{3.6}$$

where

$$b_{K_y,i}^n = \frac{1}{\Delta t_n}\int_{t_n}^{t_{n+1}}\prod_{\substack{j=0\\ j\neq i}}^{K_y}\left(\frac{s-t_{n+j}}{t_{n+i}-t_{n+j}}\right)ds. \tag{3.7}$$

It is easy to verify that

$$|b_{K_y,i}^n| \le \frac{1}{\Delta t_n}\int_{t_n}^{t_{n+1}}\frac{K_y!\Big(\max\limits_{n\le j\le n+K_y-1}\Delta t_j\Big)^{K_y}}{i!(K_y-i)!\Big(\min\limits_{n\le j\le n+K_y-1}\Delta t_j\Big)^{K_y}}ds = \frac{c_0^{K_y}\ K_y!}{i!(K_y-i)!}. \tag{3.8}$$

We set $B_{K_y} = \max\limits_{0\le i\le K_y}\frac{c_0^{K_y}\ K_y!}{i!(K_y-i)!}$. For the uniform time partition, that is $\Delta t_n = \Delta t$, $b_{K_y,i}^n$ can be explicitly presented as

$$b_{K_y,i}^n = \int_0^1\prod_{\substack{j=0\\ j\neq i}}^{K_y}\left(\frac{s-j}{i-j}\right)ds. \tag{3.9}$$

Inserting (3.6) into (3.5) gives us the following reference equation for solving $Y_{t_n}^{t_n,X^n}$:

$$Y_{t_n}^{t_n,X^n} = \mathbb{E}_{t_n}^{X^n}[Y_{t_{n+1}}^{t_n,X^n}] + \Delta t_n\sum_{i=0}^{K_y} b_{K_y,i}^n\mathbb{E}_{t_n}^{X^n}[f_{t_{n+i}}^{t_n,X^n}] + R_y^n \tag{3.10}$$

with the residual $R_y^n = \displaystyle\int_{t_n}^{t_{n+1}}\mathbb{E}_{t_n}^{X^n}[f_s^{t_n,X^n}]\,ds - \Delta t_n\sum_{i=0}^{K_y} b_{K_y,i}^n\mathbb{E}_{t_n}^{X^n}[f_{t_{n+i}}^{t_n,X^n}]$.

3.2. Reference equation for Z_t

Let K_z be a positive integer with $0\le K_z\le N$, and $\{P_i(s)\}_{i=0}^{K_z}$ be the set of backward orthogonal polynomials on $[t_n,t_{n+1}]$ with order up to K_z defined in Section 2. For $s\in[t_n,T]$, we define the *Gaussian process* $\Delta\tilde{W}_{t_n,s}$ by

$$\Delta\tilde{W}_{t_n,s} = \int_{t_n}^{s} P_{K_z}(r)dW_r, \tag{3.11}$$

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where $P_{K_z}(r) = Q_{K_z}\left(\frac{r-t_n}{t_{n+1}-t_n}\right)$.

If $P_{K_z}(r) = 1$, $\Delta\tilde{W}_{t_n,s} = \Delta W_{t_n,s}$, where $\Delta W_{t_n,s} = W_s - W_{t_n}(t_n \le s \le t_N)$, which is a standard Brownain motion with mean zero and variance $s - t_n$. If $s = t_{n+i}$, we use $\Delta W_{n,i}$ to denote $W_{t_{n+i}} - W_{t_n}$. Similarly, if $s = t_{n+i}$, we use $\Delta\tilde{W}_{n,i}$ to denote $\Delta\tilde{W}_{t_n,t_{n+i}}$. By the definition (3.11), $\Delta\tilde{W}_{t_n,s} = (\Delta\tilde{W}^1_{t_n,s}, \Delta\tilde{W}^2_{t_n,s}, \cdots, \Delta\tilde{W}^d_{t_n,s})^*$ is a Gaussian process with the following *special properties*:

1. $\mathbb{E}^{X^n}_{t_n}[\Delta\tilde{W}_{n,1}] = 0$,
2. $\mathbb{E}^{X^n}_{t_n}[(\Delta\tilde{W}^j_{n,1})^2] = \int_{t_n}^{t_{n+1}} P^2_{K_z}(s)ds = P_{K_z}(t_n)\Delta t_n = Q_{K_z}(0)\Delta t_n$,
3. $\mathbb{E}^{X^n}_{t_n}[\Delta W^j_{n,1}\Delta\tilde{W}^j_{n,1}] = \int_{t_n}^{t_{n+1}} P_{K_z}(s)ds = \Delta t_n$.

Now multiply the transpose of $\Delta\tilde{W}_{n,1}$ to both sides of (3.4), and subsequently take the conditional mathematical expectation $\mathbb{E}^{X^n}_{t_n}[\cdot]$ to both sides of the derived equation, we obtain by the Itô isometry formula

$$0 = \mathbb{E}^{X^n}_{t_n}[Y^{t_n,X^n}_{t_{n+1}}\Delta\tilde{W}^*_{n,1}] + \int_{t_n}^{t_{n+1}} \mathbb{E}^{X^n}_{t_n}[f^{t_n,X^n}_s\Delta\tilde{W}^*_{t_n,s}]\,ds - \int_{t_n}^{t_{n+1}} \mathbb{E}^{X^n}_{t_n}[P_{K_z}(s)Z^{t_n,X^n}_s]\,ds. \tag{3.12}$$

Let $1 \le K_f \le N - n$. We approximate $\int_{t_n}^{t_{n+1}} \mathbb{E}^{X^n}_{t_n}[f^{t_n,X^n}_s\Delta\tilde{W}^*_{t_n,s}]\,ds$ in (3.12) by

$$\int_{t_n}^{t_{n+1}} \mathbb{E}^{X^n}_{t_n}[f^{t_n,X^n}_s\Delta\tilde{W}^*_{t_n,s}]\,ds = \Delta t_n \sum_{i=1}^{K_f} b^n_{K_f,i}\mathbb{E}^{X^n}_{t_n}[f^{t_n,X^n}_{t_{n+i}}\Delta\tilde{W}^*_{n,i}] + R^n_{z1}, \tag{3.13}$$

where $R^n_{z1} = \int_{t_n}^{t_{n+1}} (\mathbb{E}^{X^n}_{t_n}[f^{t_n,X^n}_s\Delta\tilde{W}^*_{t_n,s}] - \sum_{i=1}^{K_f} b^n_{K_f,i}\mathbb{E}^{X^n}_{t_n}[f^{t_n,X^n}_{t_{n+i}}\Delta\tilde{W}^*_{n,i}])ds$

About the second integral term on the right-hand side of (3.12), we have the identity

$$-\int_{t_n}^{t_{n+1}} \mathbb{E}^{X^n}_{t_n}[P_{K_z}(s)Z^{t_n,X^n}_s]\,ds = -\int_{t_n}^{t_{n+1}} P_{K_z}(s)\mathbb{E}^{X^n}_{t_n}[Z^{t_n,X^n}_{t_n}]\,ds + R^n_{z2}, \tag{3.14}$$

where $R^n_{z2} = -\int_{t_n}^{t_{n+1}} \left(\mathbb{E}^{X^n}_{t_n}[P_{K_z}(s)Z^{t_n,X^n}_s] - P_{K_z}(s)\mathbb{E}^{X^n}_{t_n}[Z^{t_n,X^n}_{t_n}]\right) ds$.

Now, combining (3.12), (3.13) and (3.14), and using the facts $\int_{t_n}^{t_{n+1}} P_{K_z}(s)\,ds = \Delta t_n$, $P_{K_z}(s)\mathbb{E}^{X^n}_{t_n}[Z^{t_n,X^n}_{t_n}] = P_{K_z}(s)Z^{t_n,X^n}_{t_n}$, we get the following reference equation for solving $Z^{t_n,X^n}_{t_n}$:

$$\Delta t_n Z^{t_n,X^n}_{t_n} = \mathbb{E}^{X^n}_{t_n}[Y^{t_n,X^n}_{t_{n+1}}\Delta\tilde{W}^*_{n,1}] + \Delta t_n \sum_{i=1}^{K_f} b^n_{K_f,i}\mathbb{E}^{X^n}_{t_n}[f^{t_n,X^n}_{t_{n+i}}\Delta\tilde{W}^*_{n,i}] + R^n_z, \tag{3.15}$$

where $R^n_z = R^n_{z1} + R^n_{z2}$.

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3.3. The semi-discrete multi-step scheme

Let (X^n, Y^n, Z^n) denote an approximation to the analytic solution (X_t, Y_t, Z_t) of (1.1) and (1.2) at the time $t = t_n$, $n = N, N-1, \ldots, 0$, where the values of Y^n and Z^n are the approximate values of Y_t and Z_t at the time space point (t_n, X^n). For simple presentation we denote $f(t_n, X^n, Y^n, Z^n)$ by f^n. Now based on (3.10) and (3.15), we propose the multi-step semi-discrete scheme for solving the decoupled FBSDEs (1.1) and (1.2) as follow:

Scheme 1. *Let $K = \max\{K_y, K_z, K_f\}$. Assume random variables X^0, Y^{N-i} and Z^{N-i}, $i = 0, 1, \ldots, K-1$, are known. Let $\{X^n\}_{n=0}^N$ be the numerical solution of the forward SDE in the decoupled FBSDEs by a numerical method for solving the SDE. For $n = N-K, \ldots, 0$,*

1. *Solve the random variable Z^n by*

$$Z^n = \frac{1}{\Delta t_n}\mathbb{E}_{t_n}^{X^n}[Y^{n+1}\Delta\tilde{W}_{n,1}^*] + \sum_{i=1}^{K_f} b_{K_f,i}^n \mathbb{E}_{t_n}^{X^n}[f^{n+i}\Delta\tilde{W}_{n,i}^*], \tag{3.16}$$

2. *Solve Y^n by*

$$Y^n = \mathbb{E}_{t_n}^{X^n}[Y^{n+1}] + \Delta t_n \sum_{i=0}^{K_y} b_{K_y,i}^n \mathbb{E}_{t_n}^{X^n}[f^{n+i}]. \tag{3.17}$$

Remark 3.1. The accuracy of the multi-step Scheme 1 depends not only on the accuracy of the discretizations of Y_t and Z_t in solving the BSDE (1.2) but also on the accuracy of the discretization of X_t in solving the SDE (1.1). And it is worth to note that the scheme is implicit for solving Y^n (see (3.17)), but it is always explicit for solving Z^n (see (3.16)).

4. General error estimate

In this section, we will give a general error estimate of the multi-step scheme (3.16) and (3.17). Before presenting the main theorem, we first introduce some backward discrete Gronwall inequalities in the following lemmas.

Lemma 4.1. *(Discrete Gronwall lemma) Suppose that N and K are two nonnegative integers with $N \geq K$ and Δt any positive number. Let $\{\eta_n\}$, $n = N-K, N-K-1, \ldots, 0$, satisfy*

$$|\eta_n| \leq \beta + \alpha\Delta t \sum_{j=n+1}^{N} |\eta_j|,$$

where α and β are two positive constants. Let $M_\eta = \max_{N-K+1\leq j\leq N} |\eta_j|$ and $\hat{T} = N\Delta t$. Then for $n = N-K, N-K-1, \ldots, 0$,

$$|\eta_n| \leq e^{\alpha\hat{T}}(\beta + \alpha K\Delta t M_\eta). \tag{4.1}$$

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Lemma 4.2. *Suppose that N and K are two nonnegative integers with $N \geq K$ and Δt is any positive number. Assume $\{a_n\}$, $\{b_n\}$, and $\{R^n\}$, $n = 0, 1, \ldots, N$, are nonnegative, and satisfy the inequality*

$$a_n + C_1 \Delta t\, b_n \leq (1 + C_2 \Delta t) a_{n+1} + C_3 \Delta t \sum_{i=1}^{K} (a_{n+i} + b_{n+i}) + R^n \tag{4.2}$$

for $0 \leq n \leq N - K$, where C_1, C_2 and C_3 are positive constants. Let

$$M_a = \max_{N-K+1 \leq i \leq N} a_i, \quad M_b = \max_{N-K+1 \leq i \leq N} b_i, \quad \hat{T} = N \Delta t.$$

If $C_1 - C_3 K \geq C_4$ for some constant $C_4 > 0$, then for $n = N - K, N - K - 1, \ldots, 0$, it holds that

$$a_n + \Delta t \sum_{i=n}^{N-K} b_i \leq C \left[a_{N-K+1} + \Delta t K (M_a + M_b) + \sum_{i=0}^{N-K} R^i \right], \tag{4.3}$$

where C is a constant depending on C_1, C_2, C_3, K and $\hat{T}$.

Proof. Let n be an integer with $0 \leq n \leq N - K$. Replacing n by $n + s$ with $0 \leq s \leq N - K - n$ in (4.2) and multiplying both sides of the pruned inequality by $(1 + C_2 \Delta t)^s$, we obtain

$$\begin{aligned}
&(1 + C_2 \Delta t)^s a_{n+s} + (1 + C_2 \Delta t)^s C_1 \Delta t b_{n+s} \\
&\leq (1 + C_2 \Delta t)^{s+1} a_{n+s+1} + (1 + C_2 \Delta t)^s C_3 \Delta t \sum_{i=1}^{K} a_{n+s+i} \\
&\quad + (1 + C_2 \Delta t)^s C_3 \Delta t \sum_{i=1}^{K} b_{n+s+i} + (1 + C_2 \Delta t)^s R^{n+s}.
\end{aligned} \tag{4.4}$$

By adding up the above inequality over $s = 0, 1, \ldots, N - K - n$, we deduce

$$\begin{aligned}
&a_n + C_1 \Delta t \sum_{i=n}^{N-K} (1 + C_2 \Delta t)^{i-n}\, b_i \\
&\leq (1 + C_2 \Delta t)^{N-K-n+1} a_{N-K+1} + C_3 \Delta t \sum_{j=n}^{N-K} (1 + C_2 \Delta t)^{j-n} \sum_{i=1}^{K} a_{j+i} \\
&\quad + C_3 \Delta t \sum_{j=n}^{N-K} (1 + C_2 \Delta t)^{j-n} \sum_{i=1}^{K} b_{j+i} + \sum_{i=n}^{N-K} (1 + C_2 \Delta t)^{i-n} R^i.
\end{aligned} \tag{4.5}$$

Notice that

$$\begin{aligned}
\sum_{j=n}^{N-K} (1 + C_2 \Delta t)^{j-n} \sum_{i=1}^{K} a_{j+i} &\leq K (1 + C_2 \Delta t)^{N-K-n} \sum_{i=n+1}^{N} a_i, \\
\sum_{j=n}^{N-K} (1 + C_2 \Delta t)^{j-n} \sum_{i=1}^{K} b_{j+i} &\leq K \sum_{i=n}^{N} (1 + C_2 \Delta t)^{i-n} b_i.
\end{aligned}$$

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Inserting the above two inequalities into (4.5) leads to

$$\begin{aligned}
&a_n + C_1\Delta t \sum_{i=n}^{N-K} (1+C_2\Delta t)^{i-n} b_i \\
&\le (1+C_2\Delta t)^{N-K-n+1} a_{N-K+1} + C_3\Delta t K (1+C_2\Delta t)^{N-K-n} \sum_{i=n+1}^{N} a_i \\
&\quad + C_3\Delta t K \sum_{i=n}^{N} (1+C_2\Delta t)^{i-n} b_i + \sum_{i=n}^{N-K} (1+C_2\Delta t)^{i-n} R^i \\
&\le e^{C_2 N\Delta t} a_{N-K+1} + C_3 e^{C_2 N\Delta t} K\Delta t \sum_{i=n+1}^{N} a_i \\
&\quad + C_3 K\Delta t \sum_{i=n}^{N} (1+C_2\Delta t)^{i-n} b_i + \sum_{i=n}^{N-K} (1+C_2\Delta t)^{i-n} R^i, \qquad (4.6)
\end{aligned}$$

which implies

$$\begin{aligned}
&a_n + (C_1 - C_3K)\Delta t \sum_{i=n}^{N-K} (1+C_2\Delta t)^{i-n}\, b_i \\
&\le e^{C_2\hat{T}} a_{N-K+1} + C_3 e^{C_2\hat{T}} K\Delta t \sum_{i=n+1}^{N} a_i \\
&\quad + C_3 K\Delta t \sum_{i=N-K+1}^{N} (1+C_2\Delta t)^i\, b_i + \sum_{i=0}^{N-K} (1+C_2\Delta t)^i\, R^i. \qquad (4.7)
\end{aligned}$$

Since $C_1 - C_3K \ge C_4 > 0$, the inequality (4.7) gives us

$$a_n \le \beta + \alpha\Delta t \sum_{i=n+1}^{N} a_i, \qquad (4.8)$$

where

$$\begin{aligned}
&\alpha = C_3 e^{C_2\hat{T}} K, \\
&\beta = e^{C_2\hat{T}} a_{N-K+1} + C_3 K\Delta t \sum_{i=N-K+1}^{N} (1+C_2\Delta t)^i\, b_i + \sum_{i=0}^{N-K} (1+C_2\Delta t)^i\, R^i.
\end{aligned}$$

Then, by Lemma 4.1, we have

$$\begin{aligned}
a_n &\le e^{\alpha\hat{T}} (\beta + \alpha K\Delta t M_a) \\
&\le C_5 \left[a_{N-K+1} + \Delta t \sum_{i=N-K+1}^{N} (1+C_2\Delta t)^i b_i + \sum_{i=0}^{N-K} (1+C_2\Delta t)^i R^i + \Delta t M_a \right], \qquad (4.9)
\end{aligned}$$

where $C_5 = e^{\alpha\hat{T}} \max\{e^{C_2\hat{T}},\ C_3K,\ 1,\ \alpha K\}$.

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Let $C_6 = e^{C_2\hat{T}} \max\{1,\ C_3K\}$. Now inserting (4.9) into the right hand side of (4.7) gives us

$$\begin{aligned}
&a_n + C_4\Delta t \sum_{i=n}^{N-K} b_i \\
&\le a_n + C_4\Delta t \sum_{i=n}^{N-K} (1+C_2\Delta t)^{i-n} b_i \\
&\le C_6\Big[a_{N-K+1} + C_5\Delta t \sum_{i=n+1}^{N}\Big(a_{N-K+1} + \Delta t \sum_{j=N-K+1}^{N} (1+C_2\Delta t)^j b_j \\
&\qquad + \sum_{j=0}^{N-K} (1+C_2\Delta t)^j R^j + \Delta t M_a\Big)\Big] \\
&\qquad + C_3K\Delta t \sum_{i=N-K+1}^{N} (1+C_2\Delta t)^i b_i + \sum_{i=0}^{N-K} (1+C_2\Delta t)^i R^i \\
&\le C_7\Big[a_{N-K+1} + \Delta t(M_a + Ke^{C_2\hat{T}} M_b) + e^{C_2\hat{T}} \sum_{i=0}^{N-K} R^i\Big] \\
&\le C_8\Big[a_{N-K+1} + \Delta t(M_a + M_b) + \sum_{i=0}^{N-K} R^i\Big], \qquad (4.10)
\end{aligned}$$

where $C_7 = C_6C_5\hat{T} + \max\{C_6,\ C_3K,\ 1\}$ and $C_8 = C_7Ke^{C_2\hat{T}}$. The proof is then completed by using (4.10) and setting $C = \frac{C_8}{\min\{1,C_4\}}$ in (4.3). □

Without loss of generality, in the following we only consider the case of one-dimensional decoupled FBSDEs (i.e., $q = m = d = 1$). However we would like to remark that all error estimates obtained in the sequel also hold for d-dimensional decoupled FBSDEs where $d > 1$.

We first need the regularity of the solution (Y_t, Z_t) of the decoupled FBSDEs (1.1) and (1.2) when the terminal condition $\xi = \varphi(X_T)$ in (1.2).

Lemma 4.3. (see ([10, 12, 23, 27])) *Suppose the functions b, σ, f and φ are deterministic and are uniformly Lipschitz continuous w.r.t. (X, Y, Z) and Hölder continuous of parameter $\frac{1}{2}$ w.r.t. t. In addition, assume φ is of class $C_b^{2+\alpha}$ for some $\alpha \in (0,1)$ and the matrix-valued function $a = \sigma\sigma^* = (a_{ij})$ is uniformly elliptic. Then the solution (Y_t, Z_t) of* (1.1) *can be represented as $(u(t, X_t), \nabla_x u(t, X_t)\sigma(t, X_t))$, where $u(t,x)$ is the smooth solution of the following PDEs:*

$$(\partial_t + \mathcal{L}_{t,x})u(t,x) + f(t,x,u(t,x),\nabla_x u(t,x)\sigma(t,x)) = 0$$

with the terminal condition $u(T,x) = \varphi(x)$, where $\mathcal{L}$ is the second order differential operator defined by

$$\mathcal{L}_{t,x} = \frac{1}{2}\sum_{i,j}[\sigma\sigma^*]_{ij}(t,x)\partial^2_{x_ix_j} + \sum_i b_i(t,x)\partial_{x_i}.$$

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Furthermore for $k = 0, 1, 2, \ldots$, *if* $b, \sigma \in C_b^{1+k,2+2k}$, $f \in C_b^{1+k,2+2k,2+2k,2+2k}$ *and* $\varphi \in C_b^{2+2k+\alpha}$ *for some* $\alpha \in (0,1)$, *then* $u \in C_b^{1+k,2+2k}$.

In order to give a useful theorem, we need the following notations.

Let us denote by $\tilde{Y}_{t_{n+1}}^{t_n,X^n}$ and $\tilde{Z}_{t_{n+1}}^{t_n,X^n}$ the approximate values of $Y_{t_{n+1}}^{t_n,X^n}$ and $Z_{t_{n+1}}^{t_n,X^n}$ at the time-space point (t_{n+1}, X^{n+1}), respectively, where X^{n+1} is the approximate solution of $X_{t_{n+1}}^{t_n,X^n}$ calculated from (3.4). It is easy to verify $\tilde{Y}_{t_{n+1}}^{t_n,X^n} = Y_{t_{n+1}}^{t_{n+1},X^{n+1}}$ and $\tilde{Z}_{t_{n+1}}^{t_n,X^n} = Z_{t_{n+1}}^{t_{n+1},X^{n+1}}$. Similarly, we denote by $\tilde{Y}_{t_{n+i}}^{t_n,X^n}$ and $\tilde{Z}_{t_{n+i}}^{t_n,X^n}$ the approximate values of $Y_{t_{n+i}}^{t_n,X^n}$ and $Z_{t_{n+i}}^{t_n,X^n}$ at the time-space point (t_{n+i}, X^{n+i}), respectively, where X^{n+i} is the approximate solution of $X_{t_{n+i}}^{t_n,X^n}$ calculated from (3.4). It is easy to verify

$$\tilde{Y}_{t_{n+i}}^{t_n,X^n} = Y_{t_{n+i}}^{t_{n+i},X^{n+i}}, \quad \tilde{Z}_{t_{n+i}}^{t_n,X^n} = Z_{t_{n+i}}^{t_{n+i},X^{n+i}}.$$

For simple presentation, in the sequel, we let

$$\tilde{f}_{t_{n+1}}^{t_n,X^n} = f\Big(t_{n+1}, X^{n+1}, \tilde{Y}_{t_{n+1}}^{t_n,X^n}, \tilde{Z}_{t_{n+1}}^{t_n,X^n}\Big),$$

and similarly,

$$\tilde{f}_{t_{n+i}}^{t_n,X^n} = f\Big(t_{n+i}, X^{n+i}, \tilde{Y}_{t_{n+i}}^{t_n,X^n}, \tilde{Z}_{t_{n+i}}^{t_n,X^n}\Big).$$

We denote $Y_{t_n}^{t_n,X^n} - Y^n$ by e_y^n, $Z_{t_n}^{t_n,X^n} - Z^n$ by e_z^n, and $f(t_n, X^n, Y_{t_n}^{t_n,X^n}, Z_{t_n}^{t_n,X^n}) - f^n$ by e_f^n. Then it can be verified that

$$e_y^{n+1} = \tilde{Y}_{t_{n+1}}^{t_n,X^n} - Y^{n+1}, \quad e_z^{n+1} = \tilde{Z}_{t_{n+1}}^{t_n,X^n} - Z^{n+1}, \quad \tilde{f}_{t_{n+1}}^{t_n,X^n} - f^{n+1} = e_f^{n+1}.$$

Similarly,

$$e_y^{n+i} = \tilde{Y}_{t_{n+i}}^{t_n,X^n} - Y^{n+i}, \quad e_z^{n+i} = \tilde{Z}_{t_{n+i}}^{t_n,X^n} - Z^{n+i}, \quad \tilde{f}_{t_{n+i}}^{t_n,X^n} - f^{n+i} = e_f^{n+i}.$$

By Lemma 4.3, solutions (Y_t, Z_t) of (1.1)-(1.2) and $(Y_t^{t_n,X^n}, Z_t^{t_n,X^n})$ of (3.3) can be represented as: $(u(t,X_t), \nabla_x u(t,X_t)\sigma(t,X_t))$, $(u(t,X_t^{t_n,X^n}), \nabla_x u(t,X_t^{t_n,X^n})\sigma(t,X_t^{t_n,X^n}))$, respectively, for $t \in [t_n, T]$, where u is the solution of the PDE in Lemma 4.3.

From the definition of our scheme, Y^n and Z^n are functions of X^n, i.e $Y^n = Y^n(X^n)$, $Z^n = Z^n(X^n)$. Then

$$\begin{aligned} e_y^n &= Y_{t_n}^{t_n,X^n} - Y^n = u(t_n, X^n) - Y^n(X^n), \\ e_z^n &= Z_{t_n}^{t_n,X^n} - Z^n = \nabla_x u(t_n, X^n)\sigma(t_n, X^n) - Z^n(X^n). \end{aligned}$$

According to the previous definition of e_y^n and e_z^n, we have

$$\begin{aligned} e_y^{n+1} &= Y_{t_{n+1}}^{t_{n+1},X^{n+1}} - Y^{n+1} = u(t_{n+1}, X^{n+1}) - Y^{n+1}(X^{n+1}), \\ e_z^{n+1} &= Z_{t_{n+1}}^{t_{n+1},X^{n+1}} - Z^{n+1} = \nabla_x u(t_{n+1}, X^{n+1})\sigma(t_{n+1}, X^{n+1}) - Z^{n+1}(X^{n+1}). \end{aligned}$$

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So the definitions about e_y^i and e_z^i are consistent for different i, $0 \le i \le N$.

Before we state our main result on the general error estimate of the multi-step scheme for solving the system of decoupled FBSDEs (3.16) and (3.17) in the following theorem.

Theorem 4.1. *Let (X_t, Y_t, Z_t), $t \in [0,T]$ and (X^n, Y^n, Z^n), $n = 0, 1, \dots, N$ be the exact solutions of the decoupled FBSDEs* (1.1) *and* (1.2) *and the approximate solution obtained by the multi-step semi-discrete Scheme 1, respectively. Assume that the function $f(t, X_t, Y_t, Z_t)$ is Lipschitz continuous with respect to X_t, Y_t and Z_t and the Lipschitz constant is L. Let c_0 be the time partition regularity parameter defined in* (3.2). *Let $K = \max\{K_y, K_z, K_f\}$, $B = \max\{B_{K_y}, B_{K_f}\}$ and $\bar{P}_{K_z} = \max\limits_{0 \le s \le Kc_0} Q_{K_z}(s)$. Let $M_{ey} = \max\limits_{N-K+1 \le i \le N} \mathbb{E}[|e_y^i|^2]$ and $M_{ez} = \max\limits_{N-K+1 \le i \le N} \mathbb{E}[|e_z^i|^2]$. Then if the time step Δt is sufficiently small, it holds that for $n = N-K, N-K-1, \dots, 0$,*

$$\begin{aligned}
&\mathbb{E}[|e_y^n|^2] + \Delta t \sum_{i=n}^{N-K} \mathbb{E}[|e_z^i|^2] \\
&\le C\Big[\mathbb{E}[|e_y^{N-K+1}|^2] + \Delta t(M_{ey} + M_{ez}) + \sum_{i=0}^{N-K} \Big\{ \frac{1}{\Delta t}\Big(\mathbb{E}[|R_{ey}^i|^2] \\
&\quad + (\Delta t)^2 \sum_{j=0}^{K} \mathbb{E}[|R_{ef}^{i+j}|^2] + \mathbb{E}[|R_y^i|^2]\Big) + \frac{1}{\Delta t}\Big(\mathbb{E}[|R_{wy}^i|^2] + (\Delta t)^2 \sum_{j=1}^{K} \mathbb{E}[|R_{wf}^{i+j}|^2] + \mathbb{E}[|R_z^i|^2]\Big)\Big\}\Big],
\end{aligned} \tag{4.11}$$

where C is a constant depending on c_0, T, L, B, K, $Q_{K_z}(0)$ and $\bar{P}_{K_z}$, R_y^i and R_z^i are defined in (3.10) *and* (3.15)*, respectively, and*

$$\begin{aligned}
R_{ey}^i &= \mathbb{E}_{t_i}^{X^i}[Y_{t_{i+1}}^{t_i,X^i} - \tilde{Y}_{t_{i+1}}^{t_i,X^i}], & R_{ef}^{i+j} &= \mathbb{E}_{t_i}^{X^i}[f_{t_{i+j}}^{t_i,X^i} - \tilde{f}_{t_{i+j}}^{t_i,X^i}], \\
R_{wy}^i &= \mathbb{E}_{t_i}^{X^i}[(Y_{t_{i+1}}^{t_i,X^i} - \tilde{Y}_{t_{i+1}}^{t_i,X^i})\Delta\tilde{W}_{i,1}^*], & R_{wf}^{i+j} &= \mathbb{E}_{t_i}^{X^i}[(f_{t_{i+j}}^{t_i,X^i} - \tilde{f}_{t_{i+j}}^{t_i,X^i})\Delta\tilde{W}_{i,j}^*].
\end{aligned} \tag{4.12}$$

Proof. There are mainly three parts in the proof of the theorem: (i) the estimate of e_y^n; (ii) the estimate of e_z^n; (iii) the general error estimate in the theorem.

(i) The estimate of e_y^n. Let n be an integer with $0 \le n \le N-K$. Subtracting (3.17) from (3.10), we get

$$\begin{aligned}
e_y^n &= \mathbb{E}_{t_n}^{X^n}[Y_{t_{n+1}}^{t_n,X^n} - Y^{n+1}] + \Delta t_n \sum_{i=0}^{K_y} b_{K_y,i}^n \mathbb{E}_{t_n}^{X^n}[f_{t_{n+i}}^{t_n,X^n} - f^{n+i}] + R_y^n \\
&= \mathbb{E}_{t_n}^{X^n}[Y_{t_{n+1}}^{t_n,X^n} - \tilde{Y}_{t_{n+1}}^{t_n,X^n} + \tilde{Y}_{t_{n+1}}^{t_n,X^n} - Y^{n+1}] \\
&\quad + \Delta t_n \sum_{i=0}^{K_y} b_{K_y,i}^n \mathbb{E}_{t_n}^{X^n}[f_{t_{n+i}}^{t_n,X^n} - \tilde{f}_{t_{n+i}}^{t_n,X^n} + \tilde{f}_{t_{n+i}}^{t_n,X^n} - f^{n+i}] + R_y^n \\
&= \mathbb{E}_{t_n}^{X^n}[e_y^{n+1}] + \Delta t_n \sum_{i=0}^{K_y} b_{K_y,i}^n \mathbb{E}_{t_n}^{X^n}[e_f^{n+i}] + R_{ey}^n + \Delta t_n \sum_{i=0}^{K_y} b_{K_y,i}^n R_{ef}^{n+i} + R_y^n.
\end{aligned} \tag{4.13}$$

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Recall that $B = \max\{B_{K_y}, B_{K_f}\}$, we have

$$\begin{aligned}|e_y^n| &\le |\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]| + \Delta t_n \sum_{i=0}^{K_y} |b_{K_y,i}^n||\mathbb{E}_{t_n}^{X^n}[e_f^{n+i}]| + |R_{ey}^n| + \Delta t_n \sum_{i=0}^{K_y} |b_{K_y,i}^n||R_{ef}^{n+i}| + |R_y^n| \\ &\le |\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]| + LB\Delta t_n \sum_{i=0}^{K} \mathbb{E}_{t_n}^{X^n}[|e_y^{n+i}| + |e_z^{n+i}|] + |R_{ey}^n| \ + B\Delta t_n \sum_{i=0}^{K} |R_{ef}^{n+i}| + |R_y^n|.\end{aligned} \tag{4.14}$$

Let $\gamma > 0$ be a real number. By the inequalities

$$(a+b)^2 \le (1+\gamma\Delta t)a^2 + \left(1 + \frac{1}{\gamma\Delta t}\right)b^2, \quad \left(\sum_{n=1}^{m} a_n\right)^2 \le m\sum_{n=1}^{m} a_n^2,$$

and taking square on both sides of (4.14), we obtain

$$\begin{aligned}|e_y^n|^2 \le& (1+\gamma\Delta t)|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]|^2 + (1+\tfrac{1}{\gamma\Delta t})\bigg(LB\Delta t_n \sum_{i=0}^{K} \mathbb{E}_{t_n}^{X^n}[|e_y^{n+i}| + |e_z^{n+i}|] \\ &+ |R_{ey}^n| + B\Delta t_n \sum_{i=0}^{K} |R_{ef}^{n+i}| + |R_y^n|\bigg)^2 \\ \le& (1+\gamma\Delta t)|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]|^2 \\ &+ 8L^2B^2(K+1)(1+\tfrac{1}{\gamma\Delta t})(\Delta t_n)^2 \sum_{i=0}^{K} (\mathbb{E}_{t_n}^{X^n}[|e_y^{n+i}|]^2 + \mathbb{E}_{t_n}^{X^n}[|e_z^{n+i}|]^2) \\ &+ 4(1+\tfrac{1}{\gamma\Delta t})\bigg(|R_{ey}^n|^2 + (K+1)B^2(\Delta t_n)^2 \sum_{i=0}^{K} |R_{ef}^{n+i}|^2 + |R_y^n|^2\bigg).\end{aligned} \tag{4.15}$$

(ii) The estimate of e_z^n. Subtracting (3.16) from (3.15) results

$$\begin{aligned}\Delta t_n e_z^n =& \mathbb{E}_{t_n}^{X^n}[(Y_{t_{n+1}}^{t_n,X^n} - Y^{n+1})\Delta\tilde{W}_{n,1}^*] \\ &+ \Delta t_n \sum_{i=1}^{K_f} b_{K_f,i}^n \mathbb{E}_{t_n}^{X^n}[(f_{t_{n+i}}^{t_n,X^n} - f^{n+i})\Delta\tilde{W}_{n,i}^*] + R_z^n.\end{aligned} \tag{4.16}$$

Notice that

$$\begin{aligned}&\mathbb{E}_{t_n}^{X^n}[(Y_{t_{n+1}}^{t_n,X^n} - Y^{n+1})\Delta\tilde{W}_{n,1}^*] \\ =& \mathbb{E}_{t_n}^{X^n}[(Y_{t_{n+1}}^{t_n,X^n} - \tilde{Y}_{t_{n+1}}^{t_n,X^n} + e_y^{n+1})\Delta\tilde{W}_{n,1}^*] = R_{wy}^n + \mathbb{E}_{t_n}^{X^n}[e_y^{n+1}\Delta\tilde{W}_{n,1}^*] \\ &\mathbb{E}_{t_n}^{X^n}[(f_{t_{n+i}}^{t_n,X^n} - f^{n+i})\Delta\tilde{W}_{n,i}^*] \\ =& \mathbb{E}_{t_n}^{X^n}[(f_{t_{n+i}}^{t_n,X^n} - \tilde{f}_{t_{n+i}}^{t_n,X^n} + e_f^{n+i})\Delta\tilde{W}_{n,i}^*] = R_{wf}^{n+i} + \mathbb{E}_{t_n}^{X^n}[e_f^{n+i}\Delta\tilde{W}_{n,i}^*],\end{aligned}$$

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we easily get from (4.16) that

$$\begin{aligned}|e_z^n| \leq & \tfrac{1}{\Delta t_n}\big|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}\Delta\tilde{W}_{n,1}^*]\big| + \sum_{i=1}^{K_f}\big|b_{K_f,i}^n\mathbb{E}_{t_n}^{X^n}[e_f^{n+i}\Delta\tilde{W}_{n,i}^*]\big| \\ & + \tfrac{1}{\Delta t_n}\big|R_{wy}^n\big| + \sum_{i=1}^{K_f}\big|b_{K_f,i}^n R_{wf}^{n+i}\big| + \tfrac{1}{\Delta t_n}\big|R_z^n\big|.\end{aligned} \tag{4.17}$$

Then taking square on both sides of (4.17) and using the Hölder inequality, we get

$$\begin{aligned}|e_z^n|^2 \leq & \tfrac{2}{(\Delta t_n)^2}|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}\Delta\tilde{W}_{n,1}^*]|^2 + 2\bigg(\sum_{i=1}^{K_f}|b_{K_f,i}^n\mathbb{E}_{t_n}^{X^n}[e_f^{n+i}\Delta\tilde{W}_{n,i}^*]| \\ & + \tfrac{1}{\Delta t_n}|R_{wy}^n| + \sum_{i=1}^{K_f}|b_{K_f,i}^n||R_{wf}^{n+i}| + \tfrac{1}{\Delta t_n}|R_z^n|\bigg)^2 \\ \leq & \tfrac{2}{(\Delta t_n)^2}|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}\Delta\tilde{W}_{n,1}^*]|^2 + 8KB^2\sum_{i=1}^{K}\mathbb{E}_{t_n}^{X^n}[|e_f^{n+i}|^2]\mathbb{E}_{t_n}^{X^n}[|\Delta\tilde{W}_{n,i}^*|^2] \\ & + 8\bigg(\tfrac{1}{(\Delta t_n)^2}|R_{wy}^n|^2 + KB^2\sum_{i=1}^{K}|R_{wf}^{n+i}|^2 + \tfrac{1}{(\Delta t_n)^2}|R_z^n|^2\bigg).\end{aligned} \tag{4.18}$$

From the definition of $\Delta\tilde{W}_{t_n,s}$ and the properties of $P_{K_z}(s)$ on $[t_n, t_{n+1}]$, we have

$$\mathbb{E}_{t_n}^{X^n}[|\Delta\tilde{W}_{n,1}|^2] = \int_{t_n}^{t_{n+1}} P_{K_z}^2(s)ds = \ Q_{K_z}(0)\Delta t_n.$$

Thus we deduce

$$\begin{aligned}|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}\Delta\tilde{W}_{n,1}^*]|^2 &= |\mathbb{E}_{t_n}^{X^n}[(e_y^{n+1} - \mathbb{E}_{t_n}^{X^n}[e_y^{n+1}])\Delta\tilde{W}_{n,1}^*]|^2 \\ &\leq \mathbb{E}_{t_n}^{X^n}[|\Delta\tilde{W}_{n,1}|^2]\mathbb{E}_{t_n}^{X^n}[(e_y^{n+1} - \mathbb{E}_{t_n}^{X^n}[e_y^{n+1}])^2] \\ &= Q_{K_z}(0)\Delta t_n(\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|^2] - |\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]|^2).\end{aligned}$$

From the Lipschitz condition of f, we have the estimate

$$\mathbb{E}_{t_n}^{X^n}[|e_f^{n+i}|^2] \leq \mathbb{E}_{t_n}^{X^n}[|L(|e_y^{n+i}| + |e_z^{n+i}|)|^2] \leq 2L^2\mathbb{E}_{t_n}^{X^n}[|e_y^{n+i}|^2 + |e_z^{n+i}|^2].$$

In addition, we have

$$\mathbb{E}_{t_n}^{X^n}[|\Delta\tilde{W}_{n,i}|^2] = \int_{t_n}^{t_{n+i}} P_{K_z}^2(s)ds \leq i\ \Delta t\bar{P}_{K_z}.$$

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Now, inserting the above inequalities into (4.18), we obtain

$$\begin{aligned}
|e_z^n|^2 &\le \tfrac{2Q_{K_z}(0)}{(\Delta t_n)^2}\Delta t_n(\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|^2]-|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]|^2)\\
&\quad+8KB^2\sum_{i=1}^{K}2L^2\ i\ \Delta t\ \bar{P}_{K_z}\mathbb{E}_{t_n}^{X^n}[|e_y^{n+i}|^2+|e_z^{n+i}|^2]\\
&\quad+8\Big(\tfrac{1}{(\Delta t_n)^2}\big|R_{wy}^n\big|^2+KB^2\sum_{i=1}^{K}\big|R_{wf}^{n+i}\big|^2+\tfrac{1}{(\Delta t_n)^2}\big|R_z^n\big|^2\Big)\\
&\le \tfrac{2Q_{K_z}(0)}{\Delta t_n}(\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|^2]-|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]|^2)\\
&\quad+16L^2B^2K^2\Delta t\bar{P}_{K_z}\sum_{i=1}^{K}\mathbb{E}_{t_n}^{X^n}[|e_y^{n+i}|^2+|e_z^{n+i}|^2]\\
&\quad+8\Big(\tfrac{1}{(\Delta t_n)^2}\big|R_{wy}^n\big|^2+KB^2\sum_{i=1}^{K}\big|R_{wf}^{n+i}\big|^2+\tfrac{1}{(\Delta t_n)^2}\big|R_z^n\big|^2\Big).
\end{aligned}\tag{4.19}$$

Dividing both sides of (4.19) by $\frac{2Q_{K_z}(0)}{\Delta t}$ and using the inequality (3.2), it is easy to get

$$\begin{aligned}
\tfrac{\Delta t}{2Q_{K_z}(0)}|e_z^n|^2 &\le c_0\big(\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|^2]-|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]|^2\big)\\
&\quad+\tfrac{8L^2B^2K^2\bar{P}_{K_z}}{Q_{K_z}(0)}(\Delta t)^2\sum_{i=1}^{K}\mathbb{E}_{t_n}^{X^n}[|e_y^{n+i}|^2+|e_z^{n+i}|^2]\\
&\quad+\tfrac{4\Delta t}{Q_{K_z}(0)}\Big(\tfrac{1}{(\Delta t_n)^2}\big|R_{wy}^n\big|^2+KB^2\sum_{i=1}^{K}\big|R_{wf}^{n+i}\big|^2+\tfrac{1}{(\Delta t_n)^2}\big|R_z^n\big|^2\Big).
\end{aligned}\tag{4.20}$$

(iii) The estimate of (4.11).

Now multiplying (4.15) by c_0 and adding the derived inequality to (4.20), we obtain

$$\begin{aligned}
&c_0\big|e_y^n\big|^2+\tfrac{\Delta t}{2Q_{K_z}(0)}\big|e_z^n\big|^2\\
&\le c_0(1+\gamma\Delta t)|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]|^2+8c_0L^2B^2(K+1)(1+\frac{1}{\gamma\Delta t})(\Delta t)^2\\
&\quad\sum_{i=0}^{K}\mathbb{E}_{t_n}^{X^n}[|e_y^{n+i}|^2+|e_z^{n+i}|^2]+c_0\Big(\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|^2]-|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]|^2\Big)\\
&\quad+\tfrac{8L^2B^2K^2\bar{P}_{K_z}}{Q_{K_z}(0)}(\Delta t)^2\sum_{i=1}^{K}\mathbb{E}_{t_n}^{X^n}[|e_y^{n+i}|^2+|e_z^{n+i}|^2]\\
&\quad+4c_0(1+\tfrac{1}{\gamma\Delta t})\Big(\big|R_{ey}^n\big|^2+(K+1)B^2(\Delta t)^2\sum_{i=0}^{K}\big|R_{ef}^{n+i}\big|^2+\big|R_y^n\big|^2\Big)\\
&\quad+\frac{4\Delta t}{Q_{K_z}(0)}\Big(\frac{1}{(\Delta t_n)^2}\big|R_{wy}^n\big|^2+KB^2\sum_{i=1}^{K}\big|R_{wf}^{n+i}\big|^2+\frac{1}{(\Delta t_n)^2}\big|R_z^n\big|^2\Big),
\end{aligned}\tag{4.21}$$

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or in the equivalent form

$$\begin{aligned}
&c_0(1-D_1\Delta t)|e_y^n|^2+\Big(\tfrac{1}{2Q_{K_z}(0)}-c_0D_1\Big)\Delta t|e_z^n|^2\\
&\le c_0\big(1+\gamma\Delta t\big)\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|^2]+\Big(c_0D_1+\frac{8L^2B^2K^2\bar{P}_{K_z}}{Q_{K_z}(0)}\Delta t\Big)\Delta t\sum_{i=1}^{K}\mathbb{E}_{t_n}^{X^n}[|e_y^{n+i}|^2]\\
&\quad+\Big(c_0D_1+\tfrac{8L^2B^2K^2\bar{P}_{K_z}}{Q_{K_z}(0)}\Delta t\Big)\Delta t\sum_{i=1}^{K}\mathbb{E}_{t_n}^{X^n}[|e_z^{n+i}|^2] \qquad (4.22)\\
&\quad+4c_0\big(1+\tfrac{1}{\gamma\Delta t}\big)\Big(\big|R_{ey}^n\big|^2+(K+1)B^2(\Delta t)^2\sum_{i=0}^{K}\big|R_{ef}^{n+i}\big|^2+\big|R_y^n\big|^2\Big)\\
&\quad+\tfrac{4\Delta t}{Q_{K_z}(0)}\Big(\tfrac{1}{(\Delta t_n)^2}\big|R_{wy}^n\big|^2+KB^2\sum_{i=1}^{K}\big|R_{wf}^{n+i}\big|^2+\tfrac{1}{(\Delta t_n)^2}\big|R_z^n\big|^2\Big),
\end{aligned}$$

where $D_1=\frac{8L^2B^2(K+1)}{\gamma}+8L^2B^2(K+1)\Delta t$.

If Δt is sufficiently small, by letting γ be a big enough number, we can have $1-D_1\Delta t>0$. Furthermore, we can choose the constants C_1,C_2,C_3 such that

$$\begin{aligned}
&C_1\le\frac{1}{1-D_1\Delta t}\Big(\frac{1}{2c_0Q_{K_z}(0)}-D_1\Big), && \frac{1+\gamma\Delta t}{1-D_1\Delta t}\le 1+C_2\Delta t,\\
&\frac{1}{1-D_1\Delta t}\Big(D_1+\frac{8L^2B^2K^2\bar{P}_{K_z}}{c_0Q_{K_z}(0)}\Delta t\Big)\le C_3, && C_1-C_3K\ge C_4>0.
\end{aligned}$$

Then take the expectation on both sides of the inequality (4.22), we deduce

$$\begin{aligned}
&\mathbb{E}[|e_y^n|^2]+C_1\Delta t\mathbb{E}[|e_z^n|^2]\\
&\le(1+C_2\Delta t)\mathbb{E}[|e_y^{n+1}|^2]+C_3\Delta t\sum_{i=1}^{K}\Big(\mathbb{E}[|e_y^{n+i}|^2]+\mathbb{E}[|e_z^{n+i}|^2]\Big)+R^n, \qquad (4.23)
\end{aligned}$$

where

$$\begin{aligned}
R^n=&\frac{1}{1-D_1\Delta t}\Big\{4(1+\frac{1}{\gamma\Delta t})\Big(\mathbb{E}[|R_{ey}^n|^2]+(K+1)B^2(\Delta t)^2\sum_{i=0}^{K}\mathbb{E}[|R_{ef}^{n+i}|^2]+\mathbb{E}[|R_y^n|^2]\Big)\\
&+\frac{4c_0}{Q_{K_z}(0)}\frac{1}{\Delta t}\Big(\mathbb{E}[|R_{wy}^n|^2]+\mathbb{E}[|R_z^n|^2]\Big)+\frac{4KB^2}{c_0Q_{K_z}(0)}\Delta t\sum_{i=1}^{K}\mathbb{E}[|R_{wf}^{n+i}|^2]\Big\}.
\end{aligned}$$

By (4.23) and Lemma 4.2, we finally obtain (4.11) with some suitable constant C which completes the proof. □

Remark 4.1. Theorem 4.1 implies that Scheme 1 is stable, and its solution continuously depends on terminal conditions, that is, for any given positive number ϵ, there exists a

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positive integer δ, for different terminal conditions $(\overline{Y}^i, \overline{Z}^i)$ and (Y^i, Z^i), $i = N-K-1, \ldots, N$, if

$$\max_{N-K+1\le i\le N} \mathbb{E}[|\overline{Y}^i - Y^i|^2] < \delta, \quad \max_{N-K+1\le i\le N} \mathbb{E}[|\overline{Z}^i - Z^i|^2] < \delta,$$

then for $0 \le n \le N-K$, we have

$$\mathbb{E}[|\overline{Y}^n - Y^n|^2] + \Delta t \sum_{i=n}^{N-K} \mathbb{E}[|\overline{Z}^i - Z^i|^2] < \epsilon.$$

Remark 4.2. The terms R_y^n and R_z^n defined in (3.10) and (3.15) are truncated error terms for solving Y_t and Z_t in the BSDE in (1.2) by the discretizations (3.17) and (3.16) in Scheme 1, respectively. The four terms R_{ey}^n, R_{ef}^n, R_{wy}^n and R_{wf}^n are determined by the numerical scheme used for solving the SDE (1.1), which reflect the weak errors of the scheme for solving the SDE. Under certain regularity conditions on the coefficients b, σ, f and φ, as long as the estimates of R_y^n, R_z^n, R_{ey}^n, R_{ef}^n, R_{wy}^n and R_{wf}^n are obtained, then it is easy to get error estimates by Theorem 4.1 for Scheme 1.

5. Error estimate for Markovian decoupled FBSDEs with $\xi = \varphi(X_T)$

Under some regular conditions on the coefficient functions b, σ, f and φ, we will first estimate the truncation error terms R_y^n, R_z^n, R_{ey}^n, R_{ef}^n, R_{wy}^n and R_{wf}^n, and then present a sharp estimation of $Y_{t_n} - Y^n$ and $Z_{t_n} - Z^n$ with explicitly given convergence rate of the multi-step Scheme 1. For simplicity, we make the following assumption.

Hypothesis 1. *The functions $b(t, X_t)$, $\sigma(t, X_t)$, $f(t, X_t, Y_t, Z_t)$ and $\varphi(X_T)$ are bounded and smooth enough with bounded derivatives.*

Note that under Hypothesis 1, by Lemma 4.3, we have

$$(Y_t, Z_t) = (u(t, X_t), \nabla_x u(t, X_t)\sigma(t, X_t)),$$

where $u(t, x)$ and its derivatives are bounded. This means that (Y_t, Z_t) is bounded and has bounded derivatives with respect to t and X_t.

To estimate the errors of Scheme 1, we make the following hypothesis on the numerical solution X^n of the forward SDE in FBSDEs.

Hypothesis 2. *Let $\{X^n\}_{n=0}^N$ be the approximate solutions of the forward SDE. We assume that the approximation solutions $\{X^n\}_{n=0}^N$ have the following properties: there exist positive numbers r, r_1, r_2, β, γ and a constant $C > 0$, such that for any $g \in C_p^{2\beta+2}$,*

$$\max_{0\le n\le N} \mathbb{E}[|X^n|^r] \le C(1 + \mathbb{E}[|X_0|^r]), \tag{5.1}$$

$$|\mathbb{E}_{t_n}^{X^n}[g(X_{t_{n+1}}^{t_n, X^n}) - g(X^{n+1})]| \le C(1 + |X^n|^{2r_1})(\Delta t)^{\beta+1}, \tag{5.2}$$

$$|\mathbb{E}_{t_n}^{X^n}[\Big(g(X_{t_{n+1}}^{t_n, X^n}) - g(X^{n+1})\Big)\, \Delta\tilde{W}_{n,1}]| \le C(1 + |X^n|^{2r_2})(\Delta t)^{\gamma+1}, \tag{5.3}$$

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$n = 0, 1, \ldots, N-1$ where $C > 0$ is a constant which does not depend on Δt. The number $\beta + 1$ will be called the local weak order of the approximation.

Remark 5.1. In fact, many numerical schemes for solving forward stochastic differential equations, such as the Euler scheme, the Milstein scheme, and the Itô-Taylor type weak or strong schemes (see [14]) have the approximation properties (5.1) -(5.3) with $\beta = \gamma$, which can be proved by using Lemma 5.7.2, Lemma 5.7.5 and Lemma 5.11.4 in [14].

By using the standard Itô-Taylor expansion formula [14], it is easy to prove the following Lemma (similar proof to that of Lemma 3.1 in [36]).

Lemma 5.1. *Suppose that the functions $g = g(s,x)$ and $v = v(s,x)$ are bounded and smooth with bounded derivatives. Define $G(s) = \mathbb{E}_t^x[g(s, v(s, X_s))]$ and $\hat{G} = \mathbb{E}_t^x[g(s, v(s, X_s))(X_s - X_t)]$, $t \le s$. Then $G(s)$ and $\hat{G}(s)$ have bounded derivatives.*

And about the terms R_y^n and R_z^n defined in (3.10) and (3.15), we have the following lemma.

Lemma 5.2. *Let R_y^n and R_z^n be the local truncation errors defined in the reference Eqs.* (3.10) *and* (3.15). *Then under Hypothesis* 1*, we have the following local estimates:*

$$|R_y^n| \le C(\Delta t)^{K_y+2}, \quad |R_z^n| \le C\Big((\Delta t)^{K_f+2-K_z} + (\Delta t)^{K_z+2}\Big), \tag{5.4}$$

where $C \ge 0$ is a generic constant depending only on T, the upper bounds of derivatives of b, σ, φ and f.

Proof. Under Hypothesis 1, we know that the solution $u(s, X_s)$ and its derivatives are bounded [15], and Lemma 4.3 tells us that the solution of (1.1) and (1.2) can be represented by $Y_s = u(s, X_s)$ and $Z_s = \nabla_x u(s, X_s)\sigma(s, X_s)$ for $s \in [0, T]$. Then by Lemma 5.1 the derivatives of $\mathbb{E}_{t_n}^{X^n}[f_s^{t_n, X^n}]$ with respect to time s up to order $K_y + 1$ is bounded and so is $\mathbb{E}_{t_n}^{X^n}[Z_s^{t_n, X^n}]$ with respect to time s up to order $K_z + 1$. From (3.10), we have

$$\begin{aligned} R_y^n &= \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n}[f_s^{t_n, X^n}]\, ds - \Delta t_n \sum_{i=0}^{K_y} b_{K_y,i}^n \mathbb{E}_{t_n}^{X^n}[f_{t_{n+i}}^{t_n, X^n}] \\ &= \int_{t_n}^{t_{n+1}} \Big[\mathbb{E}_{t_n}^{X^n}[f_s^{t_n, X^n}]\, ds - p_{K_y}^{t_n, X^n}(s)\Big]\, ds, \end{aligned} \tag{5.5}$$

where $p_{K_y}^{t_n, X^n}(s)$ is the K_y-th order Lagrange interpolation polynomial through the points $(t_{n+i},\ \mathbb{E}_{t_n}^{X^n}[f_{t_{n+i}}^{t_n, X^n}])$, $i = 0, 1, \ldots, K_y$. Thus by the theory of numerical integrals, we easily obtain the estimate

$$|R_y^n| \le C(\Delta t)^{K_y+2}. \tag{5.6}$$

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And by Lemma 2.3 and the properties of the polynomial $P_{K_z}(s)$, we obtain

$$\begin{aligned}|R_{z2}^n| &= \left| \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n}[P_{K_z}(s) Z_s^{t_n,X^n}]ds - Z_{t_n}^{t_n,X^n}\Delta t_n \right| \\ &= \left| \int_{t_n}^{t_{n+1}} P_{K_z}(s)\mathbb{E}_{t_n}^{X^n}[Z_s^{t_n,X^n}]ds - Z_{t_n}^{t_n,X^n}\Delta t_n \right| \\ &\le C(\Delta t_n)^{K_z+2}.\end{aligned} \tag{5.7}$$

By the definition of R_{z1}^n, using the interpolation theory [31], we get

$$\begin{aligned}|R_{z1}^n| &= \left| \int_{t_n}^{t_{n+1}} \Big(\mathbb{E}_{t_n}^{X^n}[f_s^{t_n,X^n}\Delta\tilde{W}_{t_n,s}^*] - \sum_{i=1}^{K_f} b_{K_f,i}^n \mathbb{E}_{t_n}^{X^n}[f_{t_{n+i}}^{t_n,X^n}\Delta\tilde{W}_{n,i}^*] \Big) ds \right| \\ &\le \frac{C}{(K_f+1)!}(\Delta t)^{K_f+1} \int_{t_n}^{t_{n+1}} |F^{(K_f+1)}(\xi_s)| ds,\end{aligned} \tag{5.8}$$

where $F(s) = \mathbb{E}_{t_n}^{X^n}[f_s^{t_n,X^n}\Delta\tilde{W}_{t_n,s}^*]$, and $\xi_s \in (t_n, s)$. By Itô's formula we have

$$f_{\xi_s}^{t_n,X^n} = f_{t_n}^{t_n,X^n} + \int_{t_n}^{\xi_s} L^0 f_r^{t_n,X^n} dr + \int_{t_n}^{\xi_s} L^1 f_r^{t_n,X^n} dW_r,$$

where L^0 and L^1 be two differential operators defined by

$$L^1 = \sigma\partial_x, \qquad L^0 = \partial_t + b\partial_x + \frac{1}{2}\sigma^2\partial_{xx}^2.$$

Thus about the integrand $F^{(K_f+1)}(\xi_s)$ in inequality (5.8), we have

$$\begin{aligned}F^{(K_f+1)}(\xi_s) &= \Big(\mathbb{E}_{t_n}^{X^n}[f_{\xi_s}^{t_n,X^n}\Delta\tilde{W}_{t_n,\xi_s}^*] \Big)_{\xi_s}^{(K_f+1)} \\ &= \left(\mathbb{E}_{t_n}^{X^n}\Big[\Big(f_{t_n}^{t_n,X^n} + \int_{t_n}^{\xi_s} L^0 f_r^{t_n,X^n} dr + \int_{t_n}^{\xi_s} L^1 f_r^{t_n,X^n} dW_r \Big) \int_{t_n}^{\xi_s} P_{K_z}(r) dW_r \Big] \right)_{\xi_s}^{(K_f+1)} \\ &= \left(\int_{t_n}^{\xi_s} \mathbb{E}_{t_n}^{X^n}[(L^1 f_r^{t_n,X^n}) P_{K_z}(r)] dr \right)_{\xi_s}^{(K_f+1)} \\ &= \left(\Big(\int_{t_n}^{\xi_s} \mathbb{E}_{t_n}^{X^n}[(L^1 f_r^{t_n,X^n}) P_{K_z}(r)] dr \Big)'_{\xi_s} \right)^{(K_f)} = \Big(\mathbb{E}_{t_n}^{X^n}[(L^1 f_{\xi_s}^{t_n,X^n})] P_{K_z}(\xi_s) \Big)_{\xi_s}^{(K_f)}.\end{aligned}$$

Then under the conditions of the lemma and by the identity $P_i^{(j)}(r) = \frac{1}{(b-a)^j} Q_i^{(j)}(\frac{r-a}{b-a})$, we easily obtain the estimate

$$|R_{z1}^n| \le \int_{t_n}^{t_{n+1}} \frac{C}{(\Delta t)^{K_z}} \frac{(\Delta t)^{K_f+1}}{(K_f+1)!}\, ds \le C(\Delta t)^{K_f+2-K_z}. \tag{5.9}$$

Now by (5.6), (5.7) and (5.9), and $R_z^n = R_{z1}^n + R_{z2}^n$, we complete the proof. □

Now, in the following theorem we present the error estimates of $Y_{t_n} - Y^n$ and $Z_{t_n} - Z^n$ with explicit convergence orders.

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Theorem 5.1. *Assume Hypothesis* 1 *and Hypothesis* 2 *hold, and suppose the initial values satisfy* $M_{ey} = O((\Delta t)^{K_y+1})$ *and* $M_{ez} = O((\Delta t)^{K_f+1-K_z} + (\Delta t)^{K_z+1})$. *Let* $K = \max(K_y, K_z, K_f)$, $B = \max\{B_{K_y}, B_{K_f}\}$ *and* $\bar{P}_{K_z} = \max_{0\le s\le Kc_0} Q_{K_z}(s)$. *If the order-*$\beta$ *weak Taylor Scheme is used to solve the SDE* (1.1) *with* $\beta = \gamma = \hat{K} + 1$ *in Scheme* 1, *then for* $0 \le n \le N - K$, *we have the following error estimate:*

$$\begin{aligned}&\mathbb{E}[|e_y^n|^2] + \Delta t \sum_{i=n}^{N-K} \mathbb{E}[|e_z^i|^2]\\ &\le C\Big((\Delta t)^{2\hat{K}+2} + (\Delta t)^{2K_y+2} + (\Delta t)^{2K_f+2-2K_z} + (\Delta t)^{2K_z+2}\Big),\end{aligned} \tag{5.10}$$

where C *is a constant depending on* c_0, T, L, K, B, $Q_{K_z}(0)$, $\bar{P}_{K_z}$, *the initial value of* X_t *in* (1.1), *and the upper bounds of the derivatives of* b, σ, f *and* φ.

Proof. From the definitions of R_{ey}^i, R_{ef}^i, R_{wy}^i and R_{wf}^i in Theorem 4.1, under the conditions of the theorem, we have the estimates

$$\mathbb{E}[|X^i|^2] \le C\Big(1 + \mathbb{E}[|X_0|^2]\Big), \tag{5.11}$$

$$\mathbb{E}[|R_{ey}^i|^2] \le C\Big(1 + \mathbb{E}[|X^i|^{4r_1}]\Big)(\Delta t)^{2\beta+2} \le C\Big(1 + \mathbb{E}[|X_0|^{4r_1}]\Big)(\Delta t)^{2\beta+2}, \tag{5.12}$$

$$\mathbb{E}[|R_{ef}^i|^2] \le C\Big(1 + \mathbb{E}[|X^i|^{4r_1}]\Big)(\Delta t)^{2\beta+2} \le C\Big(1 + \mathbb{E}[|X_0|^{4r_1}]\Big)(\Delta t)^{2\beta+2}, \tag{5.13}$$

$$\mathbb{E}[|R_{wy}^i|^2] \le C\Big(1 + \mathbb{E}[|X^i|^{4r_2}\Big)(\Delta t)^{2\gamma+2} \le C\Big(1 + \mathbb{E}[|X_0|^{4r_2}\Big)(\Delta t)^{2\gamma+2}, \tag{5.14}$$

$$\mathbb{E}[|R_{wf}^i|^2] \le C\Big(1 + \mathbb{E}[|X^i|^{4r_2}]\Big)(\Delta t)^{2\gamma+2} \le C\Big(1 + \mathbb{E}[|X_0|^{4r_2}]\Big)(\Delta t)^{2\gamma+2}, \tag{5.15}$$

for $i = 0, 1, \ldots, N-1$. We note that in this proof C is a generic constant which could change its value from place to place.

According to Lemma 5.2, we know that for $0 \le i \le N - K$,

$$|R_y^i| \le C(\Delta t)^{K_y+2}, \quad |R_z^i| \le C\Big((\Delta t)^{K_f+2-K_z} + (\Delta t)^{K_z+2}\Big).$$

By the above two inequalities and (5.11), we deduce

$$\begin{aligned}&\sum_{i=0}^{N-K} \frac{1}{\Delta t}\Big(\mathbb{E}[|R_{ey}^i|^2] + (\Delta t)^2 \sum_{j=0}^{K} \mathbb{E}[|R_{ef}^{i+j}|^2] + \mathbb{E}[|R_y^i|^2]\Big)\\ &\quad\le C\Big((\Delta t)^{2\beta} + (\Delta t)^{2K_y+2}\Big),\end{aligned} \tag{5.16}$$

$$\begin{aligned}&\sum_{i=0}^{N-K} \frac{1}{\Delta t}\Big(\mathbb{E}[|R_{wy}^i|^2] + (\Delta t)^2 \sum_{j=1}^{K} \mathbb{E}[|R_{wf}^{i+j}|^2] + \mathbb{E}[|R_z^i|^2]\Big)\\ &\quad\le C\Big((\Delta t)^{2\gamma} + (\Delta t)^{2K_f+2-2K_z} + (\Delta t)^{2K_z+2}\Big).\end{aligned} \tag{5.17}$$

Then we immediately obtain the estimate result (5.10) by (5.16), (5.17) and Theorem 4.1. □

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6. Numerical experiments

In this section, some numerical tests will be performed by using the proposed multi-step scheme – Scheme 1 to solve the system of decoupled FBSDEs (1.1) and (1.2). Here we consider one-dimensional problems only since our purpose is just to demonstrate effectiveness and accuracy of the multi-step scheme and verify the above theoretical results.

For our numerical experiments, space partition and approximation of $\mathbb{E}_{t_n}^{X^n}[\cdot]$ at discrete space grid point x_i are needed. With the spatial step size h, the discrete grid points are set to be $x_i = ih$, $i = 0, \pm 1, \pm 2, \ldots$. In the calculations of the conditional mathematical expectation $\mathbb{E}_{t_n}^{X^n}[\cdot]$, the Gauss-Hermite quadrature rule is used, and the values of the integrands of the conditional mathematical expectations at non-grid points are approximated by local cubic interpolations. Since our goal is to test the accuracy of the scheme with respect to the time step size, we set the number of the Gauss-Hermite quadrature points to be big enough so that the errors contributed by spacial approximation are very small and almost have no effect on the spatial convergence of the method. For simplicity, we also take a uniform partition with a time step size Δt. Then the time partition number N is given by $N = \frac{T}{\Delta t}$ where T is the terminal time. The time step sizes used in our experiments are $N = \frac{T}{2^i}$ $(i = 4, \ldots, 7)$. In the following numerical examples we choose $T = 1$.

Let $|Y_{t_0} - Y^0|$ and $|Z_{t_0} - Z^0|$ represent the errors between the exact solution (Y_t, Z_t) at time t_0 and the approximate solution (Y^n, Z^n) of Scheme 1 at $n = 0$. The convergence rate (CR) with respect to time step Δt is obtained by using linear least square fitting to the errors.

Example 6.1. The aim of this example is to numerically verify the high convergence rates of the proposed multi-step scheme (Scheme 1) with different step numbers K_f, K_y and K_z for solving the pure BSDE (the Eq. (1.1) becomes $X_t = W_t$). Without loss of generality, we choose the BSDE (1.2) as

$$-dY_t = (Z_t Y_t - \tfrac{3}{4} Z_t)dt - Z_t dW_t \tag{6.1}$$

with the terminal condition $Y_T = \dfrac{\exp(X_T + \frac{T}{4})}{1 + \exp(X_T + \frac{T}{4})}$. The analytic solution of (6.1) is

$$Y_t = \frac{\exp(X_t + \frac{t}{4})}{1 + \exp(X_t + \frac{t}{4})}, \qquad Z_t = \frac{\exp(X_t + \frac{t}{4})}{(1 + \exp(X_t + \frac{t}{4}))^2}.$$

We let $X_{t_0} = 0$. Then the exact solution (Y_t, Z_t) at the time $t_0 = 0$ is $(Y_{t_0}, Z_{t_0}) = (1/2, 1/4)$. We report the errors $|Y_{t_0} - Y^0|$ and $|Z_{t_0} - Z^0|$ and their convergence rates for different step numbers K_f, K_y and K_z in Table 1.

Table 1 shows that Scheme 1 is a highly accurate and effective numerical method for solving BSDE; its accuracy depends on the step numbers K_f, K_y and K_z. Increasing

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Table 1: Errors and convergence rates of the multi-step scheme in Example 6.1.

$\lvert Y_{t_0} - Y^0\rvert$					
Step Number	$N = 16$	$N = 32$	$N = 64$	$N = 128$	CR
$K_z = 0, K_f = 0, K_y = 1$	5.451E-04	2.753E-04	1.382E-04	6.926E-05	0.992
$K_z = 0, K_f = 1, K_y = 0$	4.445E-04	2.227E-04	1.114E-04	5.575E-05	0.998
$K_z = 0, K_f = 1, K_y = 1$	2.697E-04	1.368E-04	6.890E-05	3.457E-05	0.988
$K_z = 2, K_f = 2, K_y = 2$	1.362E-03	6.920E-04	3.471E-04	1.736E-04	0.991
$K_z = 1, K_f = 2, K_y = 1$	3.036E-06	7.717E-07	1.958E-07	4.945E-08	1.979
$K_z = 1, K_f = 2, K_y = 3$	1.031E-05	2.610E-06	6.572E-07	1.649E-07	1.988
$K_z = 2, K_f = 4, K_y = 2$	5.543E-06	7.541E-07	9.937E-08	1.282E-08	2.918
$K_z = 2, K_f = 4, K_y = 4$	5.897E-06	8.065E-07	1.059E-07	1.366E-08	2.919
$\lvert Z_{t_0} - Z^0\rvert$					
Step Number	$N = 16$	$N = 32$	$N = 64$	$N = 128$	CR
$K_z = 0, K_f = 0, K_y = 1$	3.754E-03	1.891E-03	9.492E-04	4.754E-04	0.993
$K_z = 0, K_f = 1, K_y = 0$	1.083E-03	5.224E-04	2.563E-04	1.269E-04	1.030
$K_z = 0, K_f = 1, K_y = 1$	1.907E-03	9.531E-04	4.764E-04	2.381E-04	1.000
$K_z = 2, K_f = 2, K_y = 2$	8.995E-03	4.636E-03	2.350E-03	1.183E-03	0.975
$K_z = 1, K_f = 2, K_y = 1$	1.227E-04	3.208E-05	8.228E-06	2.083E-06	1.960
$K_z = 1, K_f = 2, K_y = 3$	1.331E-04	3.486E-05	8.950E-06	2.267E-06	1.958
$K_z = 2, K_f = 4, K_y = 2$	1.866E-04	2.117E-05	3.185E-06	4.285E-07	2.903
$K_z = 2, K_f = 4, K_y = 4$	1.838E-04	2.134E-05	3.197E-06	4.301E-07	2.895

K_y does not improve the convergence rate of Scheme 1 when we fix K_z and set $K_f = 2K_z$. It is easy to verify that all the convergence rates presented in Table 1 match the theoretical result in Theorem 5.1 very well, i.e., the convergence rate is $\min(K_y + 1, K_z + 1, K_f + 1 - K_z)$. Thus, in order to obtain the best accuracy and efficiency at the same time, we should choose $K_y = K_f = 2K_z$ in Scheme 1.

Example 6.2. In this example, we solve a system of decoupled FBSDEs to demonstrate the convergence rates of Scheme 1 with optimal setting of the step numbers K_f, K_y and K_z ($K_y = K_f = 2K_z$ as demonstrated by Example 6.1) and different numerical methods for SDEs (Euler, Milstein and Order-2.0 Weak Taylor). We choose the decouple FBSDE as

$$\begin{cases} dX_t = \sin(t + X_t)dt + \frac{3}{10}\cos(t + X_t)dW_t, \\ -dY_t = \left(\frac{3}{20}Y_tZ_t - \cos(t + X_t)(1 + Y_t)\right)dt - Z_tdW_t. \end{cases} \tag{6.2}$$

With the terminal condition $Y_T = \sin(T + X_T)$, the analytic solution of (6.2) is given by

$$Y_t = \sin(t + X_t), \qquad Z_t = \tfrac{3}{10}\cos^2(t + X_t).$$

In this example, we still let $X_{t_0} = 0$, then the exact solution (Y_t, Z_t) at the time $t_0 = 0$ is $(Y_{t_0}, Z_{t_0}) = (0, \frac{3}{10})$. The errors $|Y_{t_0} - Y^0|$ and $|Z_{t_0} - Z^0|$ and their convergence rates are shown in Table 2.

The errors and convergence rates listed in Table 2 demonstrate that Scheme 1 is a stable, effective and accurate numerical method for solving decoupled FBSDEs. The

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Table 2: Errors and convergence rates of the multi-step scheme in Example 6.2.

Step Number	$K_y=0,\quad K_f=1,\quad K_z=0$					
SDE Scheme	Euler		Milstein		Order-2.0 Weak Taylor	
$\hat{K}$	$\hat{K}=0$		$\hat{K}=0$		$\hat{K}=1$	
N	$\lvert Y_{t_0}-Y^0\rvert$	$\lvert Z_{t_0}-Z^0\rvert$	$\lvert Y_{t_0}-Y^0\rvert$	$\lvert Z_{t_0}-Z^0\rvert$	$\lvert Y_{t_0}-Y^0\rvert$	$\lvert Z_{t_0}-Z^0\rvert$
16	3.077E-02	3.349E-02	3.076E-02	3.348E-02	1.512E-02	2.873E-02
32	1.617E-02	1.738E-02	1.616E-02	1.737E-02	7.836E-03	1.467E-02
64	8.287E-03	8.844E-03	8.281E-03	8.841E-03	3.990E-03	7.408E-03
128	4.195E-03	4.461E-03	4.191E-03	4.459E-03	2.013E-03	3.722E-03
CR	0.959	0.969	0.959	0.970	0.970	0.983
Step Number	$K_y=1,\quad K_f=2,\quad K_z=1$					
SDE Scheme	Euler		Milstein		Order-2.0 Weak Taylor	
$\hat{K}$	$\hat{K}=0$		$\hat{K}=0$		$\hat{K}=1$	
N	$\lvert Y_{t_0}-Y^0\rvert$	$\lvert Z_{t_0}-Z^0\rvert$	$\lvert Y_{t_0}-Y^0\rvert$	$\lvert Z_{t_0}-Z^0\rvert$	$\lvert Y_{t_0}-Y^0\rvert$	$\lvert Z_{t_0}-Z^0\rvert$
16	1.673E-02	1.425E-02	1.670E-02	1.424E-02	8.417E-04	1.218E-03
32	8.704E-03	7.579E-03	8.690E-03	7.573E-03	2.129E-04	2.537E-04
64	4.438E-03	3.898E-03	4.431E-03	3.895E-03	5.338E-05	5.708E-05
128	2.240E-03	1.976E-03	2.237E-03	1.974E-03	1.336E-05	1.348E-05
CR	0.967	0.951	0.967	0.950	1.992	2.164

accuracy of the scheme depends on not only the step numbers K_f, K_y and K_z but also the numerical methods used for solving the forward SDE ($\hat{K}+1$ represent the SDE solver accuracy). The numerical convergence rates are again very consistent with the result of Theorem 5.1, i.e., the convergence rate is $\min(\hat{K}+1, K_y+1, K_z+1, K_f+1-K_z)$.

7. Conclusions

In this paper, by introducing a kind of novel backward orthogonal polynomials and a Gaussian process, we propose a multi-step numerical scheme for solving the decoupled forward-backward stochastic differential equations. Under Lipschtiz conditions on the coefficients of decoupled FBSDEs, we obtain a general error estimate result which implies the stability of the proposed multistep-scheme. For the decoupled FBSDEs of Markovian type, we further prove that the convergence rate of the scheme could be of high order when the coefficients b, σ, f and φ are smooth enough. Numerical experiment results are consistent with our theoretical results. All these results show that the multistep scheme is stable, effective and highly accurate, and has the capacity of using it to solve decoupled FBSDEs and other related problems.

Acknowledgments This work is partially supported by the National Natural Science Foundations of China under grant numbers 11571206 and 11571024; by Beijing Postdoctoral Research Foundation under grant number 2015ZZ-27 and by Chaoyang District Postdoctoral Research Foundation under grant number 2015ZZ-9. The authors

https://doi.org/10.4208/nmtma.2016.m1421 Published online by Cambridge University Press

would like to thank the referees for their valuable comments, which improve the paper significantly.

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