GENERALIZED INTERVALS IN PARTIALLY ORDERED GROUPS

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1. *Introduction.* The present paper is chiefly concerned with a generalization, to be known as a 'D-interval', of the notion of interval or segment in an arbitrary partially ordered group. This idea is originally due to Duthie (2), but was developed by him only in a lattice. In analogy with the use of the interval in the normal sense, notions of 'D-distributivity' and 'D-modularity' are defined in terms of the D-interval, and analogues of known properties of lattice-groups or 'l-groups' can be formulated which might be valid when a lattice structure is no longer assumed to exist; in particular, an attempt is made to provide such a generalization of the result* of Freudenthal (3) that every l-group is a distributive lattice, but, for an arbitrary partially ordered group, it is shown that only an 'approximation' (in terms of non-Archimedean elements) to the desired result actually holds, although any Archimedean partially ordered group is necessarily D-distributive.

§ 2 introduces the concepts of upper and lower D-commensurability (in the sense of the Moore-Smith condition for a directed set), D-interval, D-convexity, D-modularity and D-distributivity, defined here first of all for an arbitrary partially ordered set, where it is not necessarily possible to define a compatible group operation. In § 3 such an operation is supposed to exist, and the concept of D-commensurability is seen to be definable in terms of the group operation alone; the notion of component is introduced, and some properties developed; also, further properties, expressed in terms of the group operation, are established for the D-interval. Finally, in § 4, the problem of the possible D-distributivity of an arbitrary partially ordered group is considered, with the results indicated above.

Another treatment of the problem of the generalization of l-group properties has been given by Fuchs (4), the generalization being made to any directed partially ordered group.

2. *Partially ordered sets.* Here, P denotes an arbitrary set of elements, partially ordered by a reflexive, antisymmetric, transitive relation $\leqslant$. Given any two elements x', x'' of P with $x' < x''$, the set of all elements x such that $x' \leqslant x \leqslant x''$ is called an interval, and is denoted by $[x', x'']$.

Definition 1. *Two elements a, b in P are said to be upper D-commensurable if there exists an element x of P such that $a \leqslant x$, $b \leqslant x$.*

Evidently P is a directed set (in the Moore-Smith sense) if and only if any two of its elements are upper D-commensurable. The notion of lower D-commensurability is defined dually. If two elements are both upper and lower D-commensurable, they are

* Also in Birkhoff (1), ch. xiv, p. 219, Th. 5.

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simply said to be D-commensurable, and we write $a \sim b$; evidently $a \sim b$ if and only if there exists an interval $[x', x'']$ containing both elements a, b.

In an *arbitrary* partially ordered set P, it is not necessarily true that (a) upper, lower, or (two-sided) D-commensurability is an equivalence relation, or that (b) if two elements are upper D-commensurable, then they must also be lower D-commensurable, and conversely.

Example. Let P_0 denote the set of ordered pairs (x, y) of real numbers such that $x+y \leqslant 2$, where P_0 is partially ordered according to the rule:

$$(x, y) \leqslant (x', y') \quad \text{if and only if} \quad x \leqslant x' \quad \text{and} \quad y \leqslant y'.$$

Then it is easily verified that any one of the statements in (a) or (b) above fails either in P_0 or in its dual.

However, in § 3 it will be seen that, for any partially ordered set with a compatible group operation, properties (a) and (b) hold.

DEFINITION 2. *If two elements a, b of P are D-commensurable, then the (evidently non-null) set-theoretic intersection of all the intervals containing both a and b is called the D-interval corresponding to the given pair of elements, and is denoted by $\langle a, b\rangle$ or $\langle b, a\rangle$.*

Evidently, if $a < b$, $\langle a, b\rangle$ coincides with the ordinary interval $[a, b]$; while, more generally, if the pair a, b possess a least upper bound $a \cup b$ and a greatest lower bound $a \cap b$, then $\langle a, b\rangle$ coincides with the interval: $[a \cap b, a \cup b]$. Clearly two elements are D-commensurable if and only if there exists some D-interval containing both.

DEFINITION 3. *A subset E of P is said to be D-convex if, for any two D-commensurable elements a, b contained in E, the corresponding D-interval $\langle a, b\rangle$ is contained in E.*

Evidently, any interval is D-convex, and similarly for the whole set P. Also, if E is D-convex, it is convex in the ordinary sense that, together with any two elements a, b with $a < b$, E contains the corresponding interval $[a, b]$.

Further, given any subset E of P, there exists a least D convex set $\hat{E}$ containing E, namely the set intersection of all D-convex sets containing E; $\hat{E}$ is called the D-convex hull of E, which is itself D-convex if and only if E coincides with $\hat{E}$.

It is evident that any D-interval is D-convex, i.e. if $x, y \in \langle a, b\rangle$, then $\langle x, y\rangle \subset \langle a, b\rangle$.

It is easy to prove

THEOREM 1. *For any set E in P,*

$$E \subset \bigcup_{x, y \in E} \langle x, y\rangle \subset \hat{E};$$

in particular, if E is D-convex, $\quad E = \bigcup_{x, y \in E} \langle x, y\rangle$.

In Birkhoff (1), ch. IX, it is shown that a lattice L is distributive in the usual sense if and only if, for any three elements a, x, y, the equality of $x \cup a$, $y \cup a$ and that of $x \cap a$, $y \cap a$ together imply the equality of x and y; i.e. the equality of the intervals $[x \cap a, x \cup a]$ and $[y \cap a, y \cup a]$ implies that of x and y. We could generalize this notion to any partially ordered set as follows:

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DEFINITION 4. *A partially ordered set P is said to be D-distributive if, for any three elements a, x, y of P such that $x \sim a$, $y \sim a$, the equality of the D-intervals $\langle x, a\rangle$ and $\langle y, a\rangle$ implies that $x = y$.*

Similarly (cf. (1) ch. v, p. 66), the notion of modularity could be generalized:

DEFINITION 5. *A partially ordered set P is said to be D-modular if, for any three elements a, x, y of P such that $x \sim a$, $y \sim a$ and $x \geqslant y$, the equality of $\langle x, a\rangle$ and $\langle y, a\rangle$ implies that $x = y$.*

Although largely logically prior to the notion of D-interval, the following notes are of little interest in the sequel, and so have been deferred to the end of the section:

NOTE 1. The notion of the D-interval corresponding to a suitable pair of elements of P is evidently a particular case of that of the D-interval $\langle E\rangle$ corresponding to any set E which is bounded (in the sense that there exists an interval containing E), when $\langle E\rangle$ is defined to be the set intersection of all such intervals.

NOTE 2. Where two elements a, b of P are given to be D-commensurable in a one-sided sense only, the notion of D-interval could be replaced by that of the left, or right D-principal ideal corresponding to the pair a, b. In fact, given any element $x \in P$, let the corresponding principal ideal (the set of all elements z of P such that $z \leqslant x$) be denoted by $I(x)$; then if the two elements a, b are upper D-commensurable, then the set-theoretic intersection of all principal ideals $I(x)$ containing both a, b is called the left D-principal ideal corresponding to the pair a, b, and could be denoted by $a, b\rangle$ or $b, a\rangle$, while if two elements a, b are given to be lower D-commensurable, the corresponding right D-principal ideal $\langle a, b$ or $\langle b, a$ could be analogously defined. Evidently, when a, b are (two-sidedly) D-commensurable, the D-interval $\langle a, b\rangle$ is the set-theoretic intersection of $a, b\rangle$ and $\langle a, b$.

THEOREM 2. *If a, b are upper D-commensurable, then a, b possess a least upper bound $a \cup b$ if and only if the corresponding left D-principal ideal $a, b\rangle$ contains some element $x \geqslant a, b$; further, if such an element x exists, it is unique, equalling $a \cup b$, and $a, b\rangle$ reduces to the principal ideal $I(a \cup b)$.*

This result (and its dual) follows at once from the definitions of $a, b\rangle$ (or $\langle a, b$) and of the least upper bound $a \cup b$ (or $a \cap b$). We may clearly replace $a, b\rangle$, in its first occurrence in Theorem 2, by $\langle a, b\rangle$ if this exists.

3. *Partially ordered groups.* In the following two sections, G denotes an additively written (though not necessarily commutative) group which has a partial ordering relation $\leqslant$ such that $x \leqslant y$, implies $a+x+b \leqslant a+y+b$, for all $x, y, a, b \in G$ (cf. (1), pp. 214–15).

THEOREM 3. *Two elements a, b of G are upper D-commensurable if and only if there exist two elements $x \geqslant 0$, $y \geqslant 0$ of G such that $a-b = x-y$.*

Proof. If $z \geqslant a, b$, we can write

$$a-b = (z-b)-\{-(a-b)+(z-b)\},$$

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where both of the elements $(z-b)$ and $\{-(a-b)+(z-b)\}$ are $\geqslant 0$. Conversely, if $a-b=x-y$, with $x, y \geqslant 0$, then

$$x+b \geqslant (x-y)+b = (a-b)+b = a$$

and $x+b \geqslant b$. Taking $b=0$ in Theorem 3, we have:

COROLLARY. *An element $a \in G$ can be expressed as the difference of two positive elements if and only if there exists $z \geqslant 0$ such that $z \geqslant a$.*

If H denotes the set of all elements $a \in G$ expressible as the difference of two positive elements, then (Shimbireva (7), proof of Th. 2, § 2) we can prove:

THEOREM 4. *H is a normal (or invariant) subgroup of G and coincides with the subgroup $[G^+]$ of G which is generated by the semi-group G^+ of all positive elements of G.*

Thus, from the last two theorems, two elements a, b of G are upper D-commensurable if and only if $a-b \in [G^+] = [-G^+]$ and so, from the duals of these two theorems, it follows immediately that, in a partially ordered group G, the notions of upper and lower D-commensurability are equivalent to one another, and hence to (two-sided) D-commensurability. Theorem 4 furthermore shows that this latter relation is, in a partially ordered group, also an equivalence relation, and a congruence relation with respect to the group operation, where $a \sim b$ if and only if a, b belong to the same coset with respect to H, that is, to the same member of the quotient group G/H. That member of G/H which contains a given element a of G is called the *component* containing a, and is denoted by H_a; as is noted in (7), H_a is the set of all elements $x \in G$ such that, for some $y \in G$, $y \geqslant a$ and $y \geqslant x$, — and dually.

THEOREM 5. *Each component H_a is the union of a set of intervals.*

Proof. We show that H_a coincides with the union of all intervals having an element in common with it.

If I denotes any such interval, let $x \in H_a \cap I$. Then if $u \in I$, $u \sim x$ and so, also, $u \sim a$; that is, $u \in H_a$. Hence $I \subset H_a$.

The converse, that H_a is contained in the union of all such intervals, is immediate.

COROLLARY. *Each component H_a is D-convex.*

For if $x, y \in H_a$, then, since $x \sim y$, there exists an interval I such that $x, y \in I$. But, by the theorem, $I \subset H_a$; hence $\langle x, y \rangle \subset I \subset H_a$.

By this corollary, H, in particular, is convex in the ordinary sense, and the next theorem shows how it can be characterized among the normal convex subgroups of G.

In fact, if C is any normal convex subgroup of G, the quotient group G/C can be made into a partially ordered group by ordering its elements according to the rule:*

$C_x \geqslant C_y$ if and only if there exist elements $u \in C_x$ and $v \in C_y$ such that $u \geqslant v$, where C_x denotes that element of G/C containing the element x of G. This result was established by Fuchs (5) for the case where G itself is a directed set (i.e. where H coincides with G), but can easily be seen to hold for an arbitrary partially ordered group G.

THEOREM 6. *H is the least normal convex subgroup C of G such that the induced ordering in G/C is null; that is, such that the set $(G/C)^+$ of positive elements of G/C contains the zero element C only.*

* An equivalent condition is: corresponding to *every* $u \in C_x$, there exists $v \in C_y$ with $u \geqslant v$.

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Proof. To show that the induced ordering in G/H is null, we need only note that, if the coset $H_x \geqslant H$ then there must exist $u \in H_x$ and $v \in H$ such that $u \geqslant v$; hence, *a fortiori*, $u \sim v$, which is impossible unless H_x coincides with H.

On the other hand, let C be any normal convex subgroup such that the induced ordering in G/C is null. If $x \in G^+$, then $C_x \geqslant C_0 = C$. But, since this ordering is null, $C_x = C$, and so $x \in C$. Hence $G^+ \subset C$, and since, by Theorem 4, $H = [G^+]$, it follows that H also is contained in the subgroup C.

Evidently each component H_a is a directed set; in particular, since H itself is furthermore a normal convex subgroup, it is an *ideal* in the sense of Fuchs's definition ((5), Th. 6).

The next theorem is an analogue of the distributive law ((1), p. 214, (2)) for an l-group:

THEOREM 7. *If $x \sim y$, then*

(i) $\langle -x, -y\rangle = -\langle x, y\rangle$;

(ii) *for $a, b \in G$, $\langle b+x+a, b+y+a\rangle = b+\langle x, y\rangle + a$.*

Proof. Since D-commensurability is a congruence relation in G, $\langle -x, -y\rangle$ and $\langle b+x+a, b+y+a\rangle$ exist. Then

$$\begin{aligned}\langle -x, -y\rangle &= \bigcap_{-x,-y\in I} I = -\bigcap_{-x,-y\in I}(-I)\\ &= -\bigcap_{x,y\in -I}(-I) = -\langle x, y\rangle,\end{aligned}$$

while

$$\begin{aligned}\langle b+x+a, b+y+a\rangle &= \bigcap_{\substack{b+x+a\\ b+y+a\}\in I}} I = \bigcap_{x,y\in -b+I-a} I\\ &= \bigcap_{x,y\in -b+I-a}\{b+(-b+I-a)+a\}\\ &= b+\bigcap_{x,y\in -b+I-a}\{-b+I-a\}+a\\ &= b+\langle x, y\rangle + a.\end{aligned}$$

From Theorem 7, there follow two theorems on D-convexity:

THEOREM 8. *If C is a D-convex set in G, then so are $-C$, and $b+C+a$ for any $a, b \in G$.*

THEOREM 9. *If E is a subset of G, the D-convex hulls of $-E$ and $b+E+a$ are $-\hat{E}$ and $b+\hat{E}+a$ respectively.*

The final theorem of this section is an analogue of Theorem 4 of (1):

THEOREM 10. *If $a \sim b$, then*

$$a-\langle a, b\rangle + b = \langle a, b\rangle \tag{1}$$

Proof. If $x \sim y$, then, by Theorem 7,

$$\begin{aligned}a-\langle x, y\rangle + b &= a-(-\langle -x, -y\rangle)+b\\ &= a+\langle -x, -y\rangle + b\\ &= \langle a-x+b, a-y+b\rangle.\end{aligned}$$

Then this result follows on setting $x = a, y = b$.

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COROLLARY. *If* $a \sim 0$, *then*

(i) $a - \langle a, 0\rangle = \langle a, 0\rangle = -\langle a, 0\rangle + a$;

(ii) $\langle a, 0\rangle + \langle -a, 0\rangle = \langle -a, 0\rangle + \langle a, 0\rangle$.

Proof. (i) By symmetry from Theorem 10,

$$b - \langle a, b\rangle + a = \langle a, b\rangle. \tag{2}$$

Now substitute $b = 0$ in (1) and (2).

(ii) By Theorem 7,

$$\begin{aligned}\langle a, 0\rangle + \langle -a, 0\rangle &= \langle a, 0\rangle - \langle a, 0\rangle \\ &= (-\langle a, 0\rangle + a) - (-\langle a, 0\rangle + a), \quad \text{by (i)} \\ &= \langle -a, 0\rangle + a - a + \langle a, 0\rangle.\end{aligned}$$

4. *D-distributivity in a partially ordered group.* Regarding the question of whether an arbitrary partially ordered group might be D-distributive in the sense of Def. 4 (just as an arbitrary l-group is distributive in the normal sense), it is only possible to obtain what could be called an approximation to that result in terms of elements which are not Archimedean in the sense of the following definition:

DEFINITION 6. *A non-zero element* x *of a partially ordered group* G *is said to be Archimedean if given any element* $y \in G$, *there exists a positive or negative integer* n *such that* $nx \geqslant y$.

The group G is itself said to be Archimedean if all its non-zero elements possess this property.

THEOREM 11. *If* a, x, y *are any three elements of* G *such that* $x \sim y \sim a$ *and such that* $\langle x, a\rangle$ *equals* $\langle y, a\rangle$, *then* $x - y$ *is either zero or non-Archimedean.*

Proof. First of all, since the relation

$$\langle x, a\rangle = \langle y, a\rangle$$

is equivalent to

$$\langle x - a, 0\rangle + a = \langle y - a, 0\rangle + a,$$

or

$$\langle x - a, 0\rangle = \langle y - a, 0\rangle,$$

we can, without loss of generality, take $a = 0$.

Then, by Th. 10, Cor. (i), the equality of $\langle x, 0\rangle$ and $\langle y, 0\rangle$ implies that

$$-\langle x, 0\rangle + x = -\langle y, 0\rangle + y = -\langle x, 0\rangle + y.$$

Thus

$$\begin{aligned}-(-\langle x, 0\rangle + x) &= -(-\langle x, 0\rangle + y), \\ -x + \langle x, 0\rangle &= -y + \langle x, 0\rangle, \\ \langle x, 0\rangle &= z + \langle x, 0\rangle, \quad \text{where} \quad z = x - y. \end{aligned} \tag{1}$$

Now if all the intervals, each of which contains both elements x, 0 are indexed as I_α, $\alpha \in A$, then, by definition,

$$\langle x, 0\rangle = \bigcap_{\alpha \in A} I_\alpha;$$

and so, from (1),

$$\bigcap_{\alpha \in A} I_\alpha = z + \bigcap_{\alpha \in A} I_\alpha = \bigcap_{\alpha \in A} (z + I_\alpha).$$

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Hence the two elements $x, 0$ are contained also in every interval of the form $z+I_\alpha$, and so, if ω is any element with $\omega \leqslant x, \omega \leqslant 0$, then also $z+\omega \leqslant x, z+\omega \leqslant 0$. Since it is similarly true that

$$\bigcap_{\alpha \in A} I_\alpha = -z+\bigcap_{\alpha \in A} I_\alpha = \bigcap_{\alpha \in A} (-z+I_\alpha)$$

it follows too that $-z+\omega \leqslant x$, $-z+\omega \leqslant 0$. Consequently for any positive or negative integer n,

$$nz+\omega \leqslant x, 0 \quad \text{or} \quad nz \leqslant x-\omega, -\omega.$$

COROLLARY. *Any Archimedean partially ordered group is D-distributive.*

We can verify that the elements x, y appearing in Theorem 11 need not be equal, by considering the following example (which appeared in a different context in Loonstra (6)) of a partially ordered group which is neither a lattice nor Archimedean.

Example. Let G_0 denote the group whose elements are triples $(m, n; p)$ of integers, where addition is defined by letting

$$(m_1, n_1; p_1)+(m_2, n_2; p_2) = (m_1+m_2, n_1+n_2; p_1+p_2)$$

and where the partial ordering is defined as follows: $(m_1, n_1; p_1) < (m_2, n_2; p_2)$ if and only if

(i) $m_1 \leqslant m_2$ and $n_1 \leqslant n_2$ (without equality in both cases), or

(ii) $m_1 = m_2$, $n_1 = n_2$ and $p_1 < p_2$.

G_0 is a partially ordered group which is neither a lattice (the elements $(0, 1; 0)$ and $(1, 0; 0)$ have no least upper bound) nor Archimedean (if $z = (0, 0: 1)$ and n is any integer then $nz = (0, 0; n) < (1, 0; 0)$). If $x = (1, -1; 1)$ and $y = (1, -1; 0)$, both pairs $x, 0$ and $y, 0$ are evidently contained in every interval $I_{p,q}$ with end-points of the form $(0, -1; p)$ and $(1, 0; q)$ respectively, where p, q denote positive and negative integers respectively; on the other hand, it can easily be seen that *any* interval containing both $x, 0$ or both $y, 0$ must contain some interval $I_{p,q}$. Hence the D-intervals $\langle x, 0\rangle$ and $\langle y, 0\rangle$ exist and both coincide with the set-theoretic intersection of the intervals $I_{p,q}$, although it is true that $x \neq y$. Actually, $x-y = z$, the element noted above to be non-Archimedean; further, since $z > 0$, this example also shows that a partially ordered group G need not be D-modular in the sense of Def. 5.

REFERENCES

(1) BIRKHOFF, G. *Lattice theory*, 2nd ed. (New York, 1948).
(2) DUTHIE, W. D. Segments of ordered sets. *Trans. Amer. Math. Soc.* 51 (1942), 1–14.
(3) FREUDENTHAL, H. Teilweise geordnete Moduln. *Proc. K. Akad. Wet. Amst.* 39 (1936), 641–51.
(4) FUCHS, L. Absolutes in partially ordered groups. *Nederl. Akad. Wetensch. Proc.* 52, 251–55; = *Indagationes Math.* 11 (1949), 66–70.
(5) FUCHS, L. On partially ordered groups. *Nederl. Akad. Wetensch. Proc.* 53, 828–34; = *Indagationes Math.* 12 (1950), 272–78.
(6) LOONSTRA, F. The class of partially ordered groups. *Compositio Math.* 9 (1951), 130–40.
(7) SHIMBIREVA. On the theory of partially ordered groups. *Rec. Math.* [*Mat. Sbornik*], N.S., 20 (1947), 145–78.

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