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The horizon problem for random surfaces

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1. *Introduction*

Computer simulation of landscapes and skylines has recently attracted a great deal of interest: see [6, 7]. Specification of a 'landscape' requires a function $f: D \to \mathbb{R}$ on a subset D of $\mathbb{R}^2$, selected so that the apparent irregularity and randomness of the surface $\{(\mathbf{t}, f(\mathbf{t})) : \mathbf{t} \in D\}$ corresponds to what might be observed in nature. It is natural to look to random fields (that is, stochastic processes in two variables), and in particular to Gaussian fields, for functions with such properties. Even when an appropriate random field has been selected, determination of a typical sample function is far from easy [7].

A parameter frequently used to measure the irregularity of a curve or a surface is the 'fractal dimension' or, more precisely, the Hausdorff dimension or the (lower or upper) box-counting dimensions: see [3]. A surface that provides a 'realistic'-looking landscape has Hausdorff dimension of about 2·15, and a 'realistic' skyline has dimension of about 1·15. It appears to be a 'rule of thumb' that if a surface has dimension s ($2 \leqslant s \leqslant 3$) then its skyline or 'horizon' should have dimension $s-1$; there does not seem to be any mathematical basis for this assertion, and in this paper we embark on a study of such relationships.

Here we shall be concerned with the Hausdorff dimension of surfaces and curves, though similar problems may be posed for box-counting and other definitions of dimension. For definitions and basic properties of dimensions see [3]. In particular, recall that the *s-dimensional Hausdorff (outer) measure* of a subset E of $\mathbb{R}^n$ is defined by

$$\mathscr{H}^s(E) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s : E \subset \bigcup_{i=1}^{\infty} U_i \text{ and } \operatorname{diam} U_i \leqslant \delta \right\}.$$

Then $\mathscr{H}^s(E)$ is a Borel measure on $\mathbb{R}^n$. The *Hausdorff dimension* of E is given by

$$\dim_H E = \inf\{s : \mathscr{H}^s(E) < \infty\} = \sup\{s : \mathscr{H}^s(E) > 0\}.$$

It is convenient to use the vector and coordinate forms $\mathbf{t} = (t, u)$, $\mathbf{t}_1 = (t_1, u_1)$, etc., for points in $\mathbb{R}^2$.

We consider functions $f: I \to \mathbb{R}$, where $I = [0,1] \times [0,1]$ is the unit square, and define $g: [0,1] \to \mathbb{R}$ by

$$g(t) = \sup_{0 \leqslant u \leqslant 1} f(t, u); \tag{1·1}$$

thus g may be thought of as representing a horizon of the surface defined by f. We are interested in the relationship between $\dim_H$ graph f and $\dim_H$ graph g, where graph f and graph g are regarded as subsets of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.

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It should be pointed out at the outset that there is no general direct relationship between the dimension of a surface and that of its horizon, even when viewed in a 'typical' direction. By taking a smooth surface with a depression and replacing the bottom of the depression with a portion of a highly irregular surface, it is possible to construct a surface of dimension s for any $2 \leqslant s \leqslant 3$ such that the horizon viewed from any direction is smooth, and therefore of dimension 1. On the other hand, by choosing $f(t, \frac{1}{2})$ (regarded as a function of one variable) to have a graph of dimension s $(1 \leqslant s \leqslant 2)$, and by extending f smoothly to I so that $f(t, u)$ is, say, twice differentiable for $u \neq \frac{1}{2}$, and with $f(t, u) < f(t, \frac{1}{2})$ for $u \neq \frac{1}{2}$, we get a surface of dimension 2 with horizon of dimension s.

Thus, for dimensional relationships between surfaces and their horizons, we must look at particular classes of surfaces, preferably surfaces that are 'typical' in some sense. In this paper we study sample surfaces of certain Gaussian fields, and, in particular, obtain estimates for the dimensions of the horizon of the Brownian sheet and of the index-α Brownian field.

The following proposition gives a simple upper bound for the dimension of the horizon of surfaces satisfying a Hölder condition. Note that (1·2) implies that graph f has Hausdorff and box dimensions at most $3-\beta$.

PROPOSITION 1. *Suppose that* $f: I \to \mathbb{R}$ *satisfies*

$$|f(\mathbf{t}_1) - f(\mathbf{t}_2)| \leqslant c|\mathbf{t}_1 - \mathbf{t}_2|^\beta \tag{1·2}$$

for all $\mathbf{t}_1, \mathbf{t}_2 \in I$, *where* $0 \leqslant \beta \leqslant 1$ *and* $c > 0$. *With* g *as in* (1·1) *we have*

$$|g(t_1) - g(t_2)| \leqslant c|t_1 - t_2|^\beta, \tag{1·3}$$

so in particular, graph g *has Hausdorff (and also lower and upper box) dimension of at most* $2-\beta$.

Proof. Fix t_1 and t_2, and suppose that $g(t_1) = f(t_1, u)$ for some u (since f is continuous the supremum is attained). Then

$$g(t_2) \geqslant f(t_2, u) \geqslant f(t_1, u) - c|t_1 - t_2|^\beta = g(t_1) - c|t_1 - t_2|^\beta,$$

so $g(t_1) - g(t_2) \leqslant c|t_1 - t_2|^\beta$. Combining this with the symmetric inequality gives (1·3). The Hausdorff and box dimensions of the graph of a function satisfying (1·3) are at most $2-\beta$: see Falconer [3], corollary 11·2.

2. *Stochastic processes on an interval*

The horizon function of a random field on $I = [0, 1] \times [0, 1]$ may be thought of as a random function on $[0, 1]$. We first obtain a criterion concerning the almost sure dimension of the graph of a random function on a real interval. The following very convenient test does not, as far as I know, appear elsewhere, although the proof is of a fairly standard form.

PROPOSITION 2. *Let* $X(t)$ $(0 \leqslant t \leqslant 1)$ *be a stochastic process. Suppose that there are* $0 \leqslant \alpha < 1$ *and* $c > 0$ *such that, for* $0 \leqslant t_1 < t_2 < 1$ *and* $a \leqslant b$

$$P(a < X(t_1) - X(t_2) \leqslant b) \leqslant c(b-a)|t_1 - t_2|^{-\alpha}. \tag{2·1}$$

Then $\dim_H$ *graph* $X \geqslant 2-\alpha$ *almost surely.*

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Proof. Fix t_1, t_2 for the time being, and let

$$p(x) = P(X(t_1) - X(t_2) \leqslant x).$$

From (2·1), dp/dx exists for almost all x, with

$$0 \leqslant dp/dx \leqslant c|t_1 - t_2|^{-\alpha}.$$

Thus, for $s > 1$,

$$\begin{aligned} E((|t_1-t_2|^2 + |X(t_1)-X(t_2)|^2)^{-\frac{1}{2}s}) &= \int_{-\infty}^{\infty} (|t_1-t_2|^2 + x^2)^{-\frac{1}{2}s} \frac{dp}{dx} dx \\ &\leqslant c|t_1-t_2|^{-\alpha} \int_{-\infty}^{\infty} (|t_1-t_2|^2 + x^2)^{-\frac{1}{2}s} dx \\ &\leqslant c_1 |t_1-t_2|^{1-s-\alpha}, \end{aligned} \tag{2·2}$$

for some constant c_1, by estimating the integral separately over $|x| \leqslant |t-t_2|$ and its complement.

Now let μ_X be the random probability measure supported by graph X defined by

$$\mu_X(A) = m\{t \in [0,1] : (t, X(t)) \in A\}$$

for Borel sets A, where m is Lebesgue measure. We estimate the energy integral

$$\begin{aligned} I_s(\mu_X) &= \int_0^1 \int_0^1 |(t_1, X(t_1)) - (t_2, X(t_2))|^{-s} \, d\mu_X(t_1) \, d\mu_X(t_2) \\ &= \int_0^1 \int_0^1 |(t_1-t_2)^2 + (X(t_1)-X(t_2))^2|^{-\frac{1}{2}s} \, dt_1 \, dt_2. \end{aligned}$$

From (2·2)

$$E(I_s(\mu_X)) \leqslant c_1 \int_0^1 \int_0^1 |t_1 - t_2|^{1-s-\alpha} \, dt_1 \, dt_2 < \infty$$

if $1 < s < 2-\alpha$. Thus, if $1 < s < 2-\alpha$, the energy integral $I_s(\mu_X)$ is almost surely finite, implying that $\dim_H$ (support μ_X) $\geqslant s$. (See Falconer [3], §4·2.) The result follows.

To apply the preceding result to Gaussian processes, it is useful to note that, if W is a normally distributed random variable with mean μ and variance σ^2, then

$$\begin{aligned} P(a < W \leqslant b) &= (2\pi)^{-\frac{1}{2}} \sigma^{-1} \int_a^b \exp(-(x-\mu)^2/2\sigma^2) \, dx \\ &\leqslant (2\pi)^{-\frac{1}{2}} \sigma^{-1} |b-a|. \end{aligned} \tag{2·3}$$

We point out, in passing, that the following known result (see Adler [1]) is now immediate; in particular it gives the lower estimate for the almost sure dimension of the graph of Brownian motion.

COROLLARY 3. *Let $X(t)$ be a Gaussian process on $[0,1]$ such that for some $0 < \alpha < 1$ and $c > 0$*

$$\operatorname{var}(X(t_1) - X(t_2)) \geqslant c|t_1 - t_2|^{2\alpha} \tag{2·4}$$

for all $t_1, t_2 \in [0,1]$. Then $\dim_H$ graph $X \geqslant 2-\alpha$ *almost surely.*

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Proof. Using (2·3)

$$P(a < X(t_1) - X(t_2) \leqslant b) \leqslant c_1 |b-a| \, |t_1 - t_2|^{-\alpha}$$

for some constant c_1. The result follows from Proposition 2.

3. *Gaussian fields*

For the basic definitions and properties of random fields, see Adler[1], and for further theory see German & Horowitz[4], Pitt[8] or Kahane[5]. We shall be concerned with a real-valued random field X on the unit square $I = [0,1] \times [0,1]$, so that $X(\mathbf{t})$ is a random variable for each $\mathbf{t} \in I$. We write $X(\mathbf{t}, \omega)$, $X(\mathbf{t})$ (where ω is a point of the sample space Ω) or just X, depending on the emphasis required. Any random field X has a random horizon Y given by

$$Y(t) = \sup_{0 \leqslant u \leqslant 1} X(t,u); \tag{3·1}$$

thus Y is a stochastic process on $[0,1]$.

Of particular interest to us are the Gaussian fields. A random field $X(\mathbf{t})$ is *Gaussian* if for any finite set of points $\mathbf{t}_1, \ldots, \mathbf{t}_n \in I$ and scalars $\lambda_1, \ldots, \lambda_n$ the random variable $\sum_{j=1}^n \lambda_j X(\mathbf{t}_j)$ has Gaussian distribution.

In what follows, X will be a Gaussian field with zero mean, i.e. $E(X(\mathbf{t})) = 0$ for all $\mathbf{t} \in I = [0,1] \times [0,1]$. Such a Gaussian field is completely determined (in the sense that the joint distribution of any finite set of $X(\mathbf{t}_j)$ is determined) by the covariance functions

$$E(X(\mathbf{t}_1) X(\mathbf{t}_2)) \quad (\mathbf{t}_1, \mathbf{t}_2 \in I).$$

A random field has a *continuous version* if there is a field with the same finite-dimensional distributions with sample paths that are almost surely everywhere continuous. It may be shown [1] that the processes that we are particularly interested in have continuous versions, and in what follows we shall refer to such continuous versions when this is the case.

These covariance functions lead to the variances

$$E(|X(\mathbf{t}_1) - X(\mathbf{t}_2)|^2) = E(X(\mathbf{t}_1)^2) + E(X(\mathbf{t}_2)^2) - 2E(X(\mathbf{t}_1) X(\mathbf{t}_2)).$$

The field X has *stationary increments* if $E(|(\mathbf{t}_2) - X(\mathbf{t}_2)|^2)$ depends only on $\mathbf{t}_1 - \mathbf{t}_2$, and has *isotropic stationary increments* if it depends only on $|\mathbf{t}_2 - \mathbf{t}_2|$. Examples are the Brownian sheet, and the index-α Brownian field respectively; see below. Note that the increments of random fields are not usually independent – this leads to many of the difficulties of this subject.

Let Ω be the sample space underlying a Gaussian field on I and write $L^2(\Omega)$ for the space of zero mean random variables X on Ω satisfying $E(X^2) < \infty$. Then $L^2(\Omega)$ is a (real) Hilbert space under the inner product given by $E(X_1 X_2)$, with $H_0 = \operatorname{span}\{X(\mathbf{t}) : \mathbf{t} \in I\}$ a subspace of $L^2(\Omega)$. We write H for the completion of H_0; equivalently H is the closure of H_0 in $L^2(\Omega)$. With this Hilbert space approach, the inner product $E(X(\mathbf{t}_1) X(\mathbf{t}_2))$ is just one of the given covariances. Moreover this viewpoint provides a very convenient way of thinking of conditional variances of the $X(\mathbf{t})$. For example, if $S \subset I$, then $\operatorname{var}(X(\mathbf{t}) \mid X(\mathbf{s}) : \mathbf{s} \in S)$ is just the Hilbert space distance from $X(\mathbf{t})$ to the subspace K of H spanned by $(X(\mathbf{s}) : x \in S)$. In fact we

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have a decomposition $X(\mathbf{t}) = X_1(\mathbf{t}) + X_2(\mathbf{t})$, where $X_1(\mathbf{t}) \in \bar{K}$ and $X_2(\mathbf{t}) \in \bar{K}^\perp$, with the Gaussian random variables $X_1(\mathbf{t})$ and $X_2(\mathbf{t})$ orthogonal and therefore independent.

PROPOSITION 4. *Let $X(t,u)$ be a Gaussian field on I. Suppose that there are numbers $0 \leqslant \alpha \leqslant 1$ and $c > 0$ such that*

$$\operatorname{var}(X(t_2,0)-X(t_1,0) \mid X(t_1,u_1)-X(t_1,u_2), X(t_2,u_1)-X(t_2,u_2): 0 \leqslant u_1, u_2 \leqslant 1) \geqslant c|t_2-t_1|^{2\alpha} \quad (3{\cdot}2)$$

for $0 \leqslant t_1 \leqslant t_2 \leqslant 1$. Then $\dim_H$ graph $Y(t) \geqslant 2-\alpha$ *almost surely, where Y is given by* (3·1).

Proof. Write $\mathscr{F}_t$ for the σ-field generated by the increments of $X(t,u)$ in the u direction, i.e. $\mathscr{F}_t = \sigma(X(t,u_1)-X(t,u_2): 0 \leqslant u_1, u_2 \leqslant 1)$. Then $Y(t)-X(t,0)$ is $\mathscr{F}_t$-measurable for each t. We have

$$Y(t_2)-Y(t_1) = (Y(t_2)-X(t_2,0))-(Y(t_1)-X(t_1,0))+(X(t_2,0)-X(t_1,0)),$$

so, writing $\sigma(\mathscr{F}_{t_1}, \mathscr{F}_{t_2})$ for the σ-field generated by $\mathscr{F}_{t_1}$ and $\mathscr{F}_{t_2}$,

$$\begin{aligned}\operatorname{var}(Y(t_2)-Y(t_1) \mid \sigma(\mathscr{F}_{t_1}, \mathscr{F}_{t_2})) &= \operatorname{var}(X(t_2,0)-X(t_1,0) \mid \sigma(\mathscr{F}_{t_2}, \mathscr{F}_{t_1})) \\ &\geqslant c|t_2-t_1|^{2\alpha}\end{aligned}$$

using (3·2). Thus, conditioned on $\sigma(\mathscr{F}_{t_1}, \mathscr{F}_{t_2})$, $Y(t_2)-Y(t_1)$ has Gaussian distribution with variance at least $c|t_2-t_1|^{2\alpha}$. It follows from (2·3) that the conditional probabilities, and thus the unconditional probability, satisfy

$$P(a < Y(t_2)-Y(t_1) \leqslant b) \leqslant c_1|b-a|\,|t_2-t_1|^{-\alpha}.$$

The result follows by Proposition 2.

4. *The Brownian sheet*

The *Brownian sheet* or *2-parameter Wiener process* is the Gaussian field $X(t,u)$ with zero mean and with covariance function

$$E(X(t_1,u_1)X(t_2,u_2)) = (t_1 \wedge t_2)(u_1 \wedge u_2),$$

where '$\wedge$' denotes the minimum of the two arguments. It follows that

$$E(|X(t_1,u_1)-X(t_2,u_2)|^2) = t_1u_1 + t_2u_2 - 2(t_1 \wedge t_2)(u_1 \wedge u_2).$$

Note that $X(t,u) = 0$ almost surely if $t = 0$ or $u = 0$. It is well known that, almost surely, the sample surface defined by the continuous version of X has Hausdorff and box dimensions of $2\frac{1}{2}$ and satisfies a Hölder condition of order β for every $\beta < \frac{1}{2}$: see Adler [1], §8·9.

THEOREM 5. *Let $X(t,u)$ be the Brownian sheet on the unit square I, and let Y be the horizon function* (3·1). *Then* $\dim_H$ graph $Y = 1\frac{1}{2}$, *almost surely.*

Proof. Since X almost surely satisfies a Hölder condition of order β for every $\beta < \frac{1}{2}$, Proposition 1 gives that $\dim_H$ graph $Y \leqslant 1\frac{1}{2}$ almost surely.

Choose $u_0 > 0$ small. Then, if $t_1 \leqslant t_2$ and $0 \leqslant u \leqslant 1$,

$$E((X(t_2,u_0)-X(t_1,u_0))X(t_1,u)) = t_1(u_0 \wedge u) - t_1(u_0 \wedge u) = 0,$$

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and if, in addition, $u_0 \leqslant u$,

$$E((X(t_2,u_0)-X(t_1,u_0))(X(t_2,u)-X(t_2,u_0))) = t_2u_0-t_2u_0-t_1u_0+t_1u_0 = 0.$$

Hence $X(t_2,u_0)-X(t_1,u_0)$ is independent of $X(t_1,u_1)-X(t_1,u_2)$ and also of $X(t_2,u_1)-X(t_2,u_2)$ for $u_1,u_2 \geqslant u_0$ and for $0 \leqslant t_1,t_2 \leqslant 1$. Thus

$$\operatorname{var}(X(t_2,u_0)-X(t_1,u_0)\,|\,X(t_1,u_1)-X(t_1,u_2),X(t_2,u_1)-X(t_2,u_2):$$
$$u_0 \leqslant u_1,u_2 \leqslant 1) = \operatorname{var}(X(t_2,u_0)-X(t_1,u_0)) = |t_2-t_1|\,u_0.$$

It follows by a very minor variant of Proposition 4 (working on $[0,1]\times[u_0,1]$ instead of the unit square) that $\dim_H \text{graph } Y \geqslant 1\frac{1}{2}$.

Note that since the box counting dimensions are always greater than or equal to the Hausdorff dimensions, it follows using Proposition 1 that the (lower and upper) box dimensions of the horizon of the Brownian sheet are also $1\frac{1}{2}$.

5. *The index-α Brownian field*

The *index-α Brownian field* ($0<\alpha<1$) on the unit square I is the Gaussian process $X(t,u)$ with zero mean and with covariance function

$$E(X(\mathbf{t}_1)X(\mathbf{t}_2)) = \tfrac{1}{2}(|\mathbf{t}_1|^{2\alpha}+|\mathbf{t}_2|^{2\alpha}-|\mathbf{t}_1-\mathbf{t}_2|^{2\alpha}) \tag{5·1}$$

(where $\mathbf{t}_i = (t_i,u_i)\in I$). It is easy to check that

$$E(|X(\mathbf{t}_1)-X(\mathbf{t}_2)|^2) = |\mathbf{t}_1-\mathbf{t}_2|^{2\alpha},$$

and that $E(X(\mathbf{t})^2) = |\mathbf{t}|^{2\alpha}$, so that $X(\mathbf{0}) = 0$ almost surely.

The index-$\frac{1}{2}$ Brownian field is called *isotropic* or *Lévy Brownian motion*, but it should be noted that it is a very different process from the Brownian sheet.

The index-α Brownian field has stationary isotropic increments, and is scale-invariant, in the sense that for a non-zero real number h, the Gaussian process $|h|^{-\alpha}X(h\mathbf{t})$ has the same covariance functions, and therefore the same distributions, as $X(\mathbf{t})$.

A simple calculation shows that we may write the covariance function (5·1) as

$$E(X(\mathbf{t})X(\mathbf{s})) = b\int_{\mathbf{R}^2} |\boldsymbol{\lambda}|^{-2-2\alpha}(e^{i\mathbf{t}\cdot\boldsymbol{\lambda}}-1)(e^{-i\mathbf{s}\cdot\boldsymbol{\lambda}}-1)\,d\boldsymbol{\lambda}, \tag{5·2}$$

where b is a positive number depending only on α. In particular, this leads to

$$E((X(\mathbf{t}_1)-X(\mathbf{t}_2))(X(\mathbf{s}_1)-X(\mathbf{s}_2))) = b\int_{\mathbf{R}^2} |\boldsymbol{\lambda}|^{-2-2\alpha}(e^{i\mathbf{t}_1\cdot\boldsymbol{\lambda}}-e^{i\mathbf{t}_2\cdot\boldsymbol{\lambda}})(e^{-i\mathbf{s}_1\cdot\boldsymbol{\lambda}}-e^{-i\mathbf{s}_2\cdot\boldsymbol{\lambda}})\,d\boldsymbol{\lambda}. \tag{5·3}$$

The following lemma on exponential sums will be needed in our calculations on the horizon of the index-σ Brownian field.

LEMMA 6. *Given $\alpha \geqslant 0$, there exists $c>0$ such that, for every positive integer N and $u_1,\dots,u_N\in[0,1]$ and $a_1,\dots,a_N\in\mathbb{R}$ such that $\sum_{j=1}^N a_j = 1$, we have*

$$\int_{-\infty}^{\infty} \frac{|\sum_1^N a_j e^{i\lambda u_j}|^2}{|a^2+\lambda^2|^{1+\alpha}}\,d\lambda \geqslant c|a|^{-2-2\alpha}$$

for $a\in\mathbb{R}$.

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Proof. Defining the Fourier transform $\hat{g}$ of a 'well-behaved' function g by

$$\hat{g}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda u} g(u)\, du$$

in the usual way, the inversion formula gives

$$g(-u) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\lambda u} \hat{g}(\lambda)\, \mathrm{d}\lambda.$$

Let g be a C^∞ 'bump function' of compact support with $g(u) = 1$ for $-1 \leqslant u \leqslant 1$. Then

$$1 = \sum_1^N a_j g(-u_j) = (2\pi)^{-1} \int_{-\infty}^{\infty} \left(\sum_1^N a_j e^{i\lambda u_j} \right) \hat{g}(\lambda)\, d\lambda$$

$$\leqslant (2\pi)^{-1} \left[\int_{-\infty}^{\infty} \frac{|\sum_1^N a_j e^{i\lambda u_j}|^2}{|a^2+\lambda^2|^{1+\alpha}} d\lambda \right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} (a^2+\lambda^2)^{1+\alpha} |\hat{g}(\lambda)|^2\, d\lambda \right]^{\frac{1}{2}} \tag{5·4}$$

using the Cauchy–Schwarz inequality. But

$$\int_{-\infty}^{\infty} (a^2+\lambda^2)^{1+\alpha} |\hat{g}(\lambda)|^2\, d\lambda \leqslant \int_{|\lambda| \leqslant a} (2a^2)^{1+\alpha} |\hat{g}(\lambda)|^2\, d\lambda$$

$$+ \int_{|\lambda| > a} (2\lambda^2)^{1+\alpha} |\hat{g}(\lambda)|^2\, d\lambda \leqslant c_1 |a|^{2+2\alpha}$$

for some constant c_1, since $\hat{g}(\lambda)$ is bounded and rapidly decreasing. Combining this with (5·4) gives the result.

Note in passing that a variation of this proof gives that

$$\int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 (a^2+\lambda^2)^{-1-\alpha}\, d\lambda \geqslant c|a|^{-2-2\alpha}$$

if $f \in L^2(\mathbb{R})$ with support $f \subset [0,1]$ and

$$\int_0^1 f(u)\, du = 1.$$

These results are reminiscent of Heisenberg's uncertainty principle; that is, impossible for both a function and its Fourier transform to be too concentrated (see, for example, [2], §2·8).

THEOREM 7. *Let $X(t,u)$ be index-α Brownian field on the unit square I, and let Y be the horizon function* (3·1). *Then, almost surely*

$$\max\{1, 1\tfrac{1}{2} - \alpha\} \leqslant \dim_H \text{graph } Y \leqslant 2 - \alpha. \tag{5·5}$$

Proof. We work in the Hilbert space H with inner product on the subspace spanned by $\{X(\mathbf{s}) : \mathbf{s} \in I\}$ given by $E(X(\mathbf{t}_1) X(\mathbf{t}_2))$ as in (5·1) or (5·2). In order to apply Proposition 4, we must estimate

$$v(t_1,t_2) \equiv \text{var}\,(X(t_2,0) - X(t_1,0) \,|\, X(t_1,u_1) - X(t_1,u_2),$$

$$X(t_2,u_1) - X(t_2,u_2) : 0 \leqslant u_1, u_2 \leqslant 1). \tag{5·6}$$

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In Hilbert space parlance, this is the distance from $X(t_2,0)-X(t_1,0)$ to the subspace spanned by

$$\{X(t_1,u_1)-X(t_1,u_1), X(t_2,u_1)-X(t_2,u_2): 0 \leqslant u_1, u_2 \leqslant 1\}.$$

This distance is given by

$$v(t_1,t_2) = \inf E(X(t_2,0)-X(t_1,0)-\sum_1^N a_i(X(t_2,u_j)-X(t_2,u_j))+\sum_1^M b_j(X(t_1,v_j)-X(t_1,v_j)))^2$$

where the infimum is taken over $N, M \in \mathbb{Z}^+$, $0 \leqslant u_j, v_j \leqslant 1$ and $a_i, b_i \in \mathbb{R}$. But the set of expressions of the form $X(t_i,0)-\sum_1^N a_j(X(t_i,u_j)-X(t_i,u_j))$ are the same as those of the form $\sum_1^N a_j X(t_i,u_j)$ for $i = 1, 2$, subject to the condition $\sum_1^N a_j = 1$ (for different N, a_j, u_j, of course). Thus

$$v(t_1,t_2) = \inf\left\{E\left(\sum_1^N a_j X(t_2,u_j)-\sum_1^M b_j X(t_1,v_j)\right)^2 : \sum_1^N a_j = \sum_1^M b_j = 1\right\},$$

and, by renumbering, and making some of the terms zero,

$$v(t_1,t_2) = \inf\left\{E\left(\sum_1^N (a_j X(t_2,u_j)-b_j X(t_1,u_j))\right)^2 : \sum_1^N a_j = \sum_1^N b_j = 1\right\}.$$

It is now easy to see (by appealing to symmetry or by minimization under constraint) that, for given N and u_j, this infimum is attained when $a_j = b_j$ $(1 \leqslant j \leqslant N)$. Thus, using (5·3),

$$\begin{aligned} v(t_1,t_2) &= \inf\left\{E\left(\sum_1^N a_j(X(t_2,u_j)-X(t_1,u_j))\right)^2\right\} \\ &= b\inf\int\int_{\mathbf{R}^2} \frac{|\sum_1^N a_j(e^{i(\lambda_1 t_2+\lambda_2 u_j)}-e^{i(\lambda_1 t_1+\lambda_2 u_j)})|^2}{|\lambda_1^2+\lambda_2^2|^{1+\alpha}}\,\mathrm{d}\lambda_1\,d\lambda_2 \\ &= b\inf\int\int_{\mathbf{R}^2} \frac{|e^{i\lambda_1 t_2}-e^{i\lambda_1 t_1}|^2|\sum_1^N a_j e^{i\lambda_2 u_j}|^2}{|\lambda_1^2+\lambda_2^2|^{1+\alpha}}\,d\lambda_1\,d\lambda_2 \end{aligned}$$

with the infimum taken over N, u_j and a_j such that $0 \leqslant u_j \leqslant 1$ and $\sum_1^N a_j = 1$. By Lemma 6,

$$v(t_1,t_2) \geqslant bc\int_{-\infty}^{\infty} |e^{i\lambda_1 t_2}-e^{i\lambda_1 t_1}|^2|\lambda_1|^{-2-2\alpha}\,d\lambda_1 = 4bc\int_{-\infty}^{\infty} \sin^2(\tfrac{1}{2}\lambda_1(t_2-t_1))\,|\lambda_1|^{-2-2\alpha}\,d\lambda_1.$$

Using that $\sin t \geqslant t\sin\frac{1}{2}$ for $0 \leqslant t \leqslant \frac{1}{2}$, we get that, for appropriate constants c_1 and c_2,

$$v(t_1,t_2) \geqslant c_1\int_0^{1/|t_2-t_1|} \lambda_1^2|t_2-t_1|^2\lambda_1^{-2-2\alpha}\,d\lambda_1 \geqslant c_2|t_2-t_1|^{2\alpha+1}$$

provided that $0 < \alpha < \frac{1}{2}$. It follows from Proposition 4 that $\dim_H \operatorname{graph} Y(t) \geqslant 1\frac{1}{2}-\alpha$ almost surely.

The index-α Brownian field almost surely satisfies a Hölder condition

$$|X(\mathbf{t}_1)-X(\mathbf{t}_2)| \leqslant C|t_1-t_2|^\beta$$

for every $\beta < \alpha$, so the upper bound follows from Proposition 1.

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6. *Further developments*

Our analysis raises many open questions. The most obvious is to show that the horizon of the index-α Brownian field has Hausdorff dimension (and therefore box dimensions) $(2-\alpha)$ almost surely. But we have not even been able to prove a zero-one law for the dimension of the horizon of a Gaussian surface. What general properties of a surface ensure that the horizon has dimension one less than that of the surface?

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