Math. Struct. in Comp. Science (2006), *vol.* 16, *pp.* 901–924. © 2006 Cambridge University Press
doi:10.1017/S0960129506005639 Printed in the United Kingdom

Uniform regular enumerations

IVAN SOSKOV† and BOGOMIL KOVACHEV

Faculty of Mathematics and Computer Science, Sofia University,
5 James Bourchier Blvd., 1164 Sofia, Bulgaria
Email: soskov@fmi.uni-sofia.bg

Received 17 September 2005

In the paper we introduce and study the uniform regular enumerations for arbitrary recursive ordinals. As an application of the technique, we obtain a uniform generalisation of a theorem of Ash and a characterisation of a class of uniform operators on transfinite sequences of sets of natural numbers.

1. Introduction

Let ζ be a recursive ordinal and $\{B_\alpha\}_{\alpha\leqslant\zeta}$ be an arbitrary sequence of sets of natural numbers. The regular with respect to the sequence $\{B_\alpha\}_{\alpha\leqslant\zeta}$ enumerations were introduced in Soskov and Baleva (2002) and used to characterise the sets A satisfying the following condition:

$$(\forall X)[(\forall\alpha\leqslant\zeta)(B_\alpha \text{ is r.e. in } X^{(\alpha)} \text{ uniformly in } \alpha)\Rightarrow A \text{ is r.e. in } X^{(\alpha)}], \tag{1}$$

where α is a recursive ordinal.

In the present paper we are concerned with the characterisation of all sequences $\{A_\alpha\}_{\alpha\leqslant\zeta}$ of sets satisfying a uniform generalisation of (1):

$$\begin{aligned}(\forall X)[&(\forall\alpha\leqslant\zeta)(B_\alpha \text{ is r.e. in } X^{(\alpha)} \text{ uniformly in } \alpha)\\ &\Rightarrow(\forall\alpha\leqslant\zeta)(A_\alpha \text{ is r.e. in } X^{(\alpha)} \text{ uniformly in } \alpha)].\end{aligned} \tag{2}$$

It turns out that the technique developed in Soskov and Baleva (2002) cannot be applied directly to obtain this characterisation, which led us to the uniform regular enumerations presented here.

Using the technique of the uniform regular enumerations, we also obtain a characterisation of a class of uniform operators mapping transfinite sequences of sets of natural numbers to sequences of sets of natural numbers, which generalise the operators studied in Soskov (2005), and show that two sequences $\{B_\alpha\}_{\alpha\leqslant\zeta}$ and $\{A_\alpha\}_{\alpha\leqslant\zeta}$ satisfy (2) if and only if there exists a uniform operator Γ such that

$$\Gamma(\{B_\alpha\}_{\alpha\leqslant\zeta})=\{A_\alpha\}_{\alpha\leqslant\zeta}.$$

† The first author's research was partially supported by the Sofia University Science Fund.

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

2. Preliminaries

2.1. *Ordinal notations*

In the following we shall consider only recursive ordinals α that are below a fixed recursive ordinal η. We shall suppose that a notation $e \in \mathcal{O}$ for η is fixed and the notations for the ordinals $\alpha < \eta$ are elements a of $\mathcal{O}$ such that $a <_o e$. For definitions of the set $\mathcal{O}$ and the relation '$<_o$', consult Rogers (1967) or Sacks (1990). We shall identify every ordinal with its notation and denote the ordinals by the letters α, β, γ and δ. In particular, we shall write $\alpha < \beta$ instead of $\alpha <_o \beta$. If α is a limit ordinal, we shall use $\{\alpha(p)\}_{p\in\mathbb{N}}$ to denote the unique strongly increasing sequence of ordinals with limit α, determined by the notation of α, and write $\alpha = \lim \alpha(p)$.

2.2. *The enumeration jump*

Given two sets of natural numbers A and B, we say that A is enumeration reducible to B ($A \leqslant_e B$) if $A = \Phi_z(B)$ for some enumeration operator Φ_z. In other words, using the notation D_v for the finite set having canonical code v and $W_0, \ldots, W_z, \ldots$ for the Gödel enumeration of the r.e. sets, we have

$$A \leqslant_e B \iff \exists z \forall x (x \in A \iff \exists v (\langle v, x\rangle \in W_z \ \&\ D_v \subseteq B)).$$

The relation $\leqslant_e$ is reflexive and transitive and induces an equivalence relation $\equiv_e$ on all subsets of $\mathbb{N}$. The respective equivalence classes are called enumeration degrees – see Cooper (1990) for an introduction to the enumeration degrees.

Given a set A, we use A^+ to denote the set $A \oplus (\mathbb{N} \setminus A)$. The set A is called *total* if and only if $A \equiv_e A^+$. Clearly, A is recursively enumerable in B if and only if $A \leqslant_e B^+$, and A is recursive in B if and only if $A^+ \leqslant_e B^+$.

Since $B \leqslant_e B^+$, if $A \leqslant_e B$, then A is r.e. in B. It is easy to see that there exists a recursive function t such that if $A \subseteq \mathbb{N}$, then for every z,

$$\Phi_z(A) = W^A_{t(z)},$$

where, as usual, W^A_z denotes the domain of the z-th Turing machine using as oracle the characteristic function of A, in other words, the z-th r.e. in A set.

In the reverse direction, there exist recursive functions e_1 and e_2 such that for all total functions f,

$$W^f_z = \Phi_{e_1(z)}(G_f),$$

where $G_f = \{\langle x, y\rangle : f(x) \simeq y\}$, and for all $A \subseteq \mathbb{N}$,

$$W^A_z = \Phi_{e_2(z)}(A^+).$$

Notice that the graph of every total function is a total set.

The enumeration jump operator is defined in Cooper (1984) and further studied in McEvoy (1985). Here we shall use the following definition of the e-jump, which is m-equivalent to the original one, see McEvoy (1985):

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

Definition 2.1. Given a set A, let $K_A^0 = \{\langle x, z\rangle : x \in \Phi_z(A)\}$. Define the *e-jump* A'_e of A to be the set $(K_A^0)^+$.

The following properties of the enumeration jump are proved in McEvoy (1985). Let A and B be sets of natural numbers. Set $B_e^{(0)} = B$ and $B_e^{(n+1)} = (B_e^{(n)})'_e$. Then:

(J1) If $A \leqslant_e B$, then $A'_e \leqslant_e B'_e$.
(J2) A is Σ^0_{n+1} relative to B iff $A \leqslant_e (B^+)_e^{(n)}$.

Let α be a recursive ordinal. To define the α-th enumeration jump of a set A, we are going to use a construction very similar to that used in the definition of the α-th Turing jump. The idea is to modify the definition of the sets H_α^A (see Rogers (1967) or Sacks (1990)) by taking the enumeration jump instead of the Turing jump.

Definition 2.2.

(i) $E_0^A = A$.
(ii) $E_{\beta+1}^A = (E_\beta^A)'_e$.
(iii) If $\alpha = \lim \alpha(p)$, then $E_\alpha^A = \{\langle p, x\rangle : x \in E_{\alpha(p)}^A\}$.

From now on, $A_e^{(\alpha)}$ will stand for E_α^A.

Of course, the definition of the set $A_e^{(\alpha)}$ depends on the fixed notation of the ordinal α. On the other hand, it is easy to see by a minor modification of the proof of the corresponding theorem of Spector for the sets H_α^A (see Rogers (1967) or Sacks (1990)) that if α_1 and α_2 are two notations of the same recursive ordinal, then $A_e^{(\alpha_1)} \equiv_e A_e^{(\alpha_2)}$.

The following properties of the transfinite iteration of the enumeration jump follow easily from the definition:

(E1) If $\beta \leqslant \alpha$ are recursive ordinals, then $A_e^{(\beta)} \leqslant_e A_e^{(\alpha)}$ uniformly in β and α.
(E2) If $A \leqslant_e B$, then for every recursive ordinal α, we have $A_e^{(\alpha)} \leqslant_e B_e^{(\alpha)}$.
(E3) If $\alpha > 0$, then $A_e^{(\alpha)}$ is a total set.

Finally, for total sets, the α-th enumeration jump and the α-th Turing jump are equivalent. In other words, the following proposition is true.

Proposition 2.3. Let A be a total set of natural numbers. Then for every recursive ordinal α, we have $E_\alpha^A \equiv_e (H_\alpha^A)^+$ uniformly in α.

Since we are going to consider e-jumps only here, from now on we shall omit the subscript e in the notation of the enumeration jump. So, for every recursive ordinal α, we use $A^{(\alpha)}$ to denote the α-th enumeration jump of A.

For every function f and every recursive ordinal α, we use f^α to denote the α-th jump of the graph G_f of f. It is easy to see that there exists a recursive function $e(z, \alpha)$ such that if f is a total function and α is a recursive ordinal, then

$$W_z^{f^{(\alpha)}} = \Phi_{e(z,\alpha)}(f^{(\alpha)}).$$

Hence, for jumps of total functions, the relations 'r.e.' in and '$\leqslant_e$' are uniformly equivalent.

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

2.3. *The jump set of a sequence of sets*

Let ζ be a recursive ordinal and $\{B_\alpha\}_{\alpha\leqslant\zeta}$ be a sequence of sets of natural numbers. For every recursive ordinal α we define the *jump set* $\mathscr{P}_\alpha$ *of the sequence* $\{B_\alpha\}$ as follows using transfinite recursion on α.

Definition 2.4.

— $\mathscr{P}_0 = B_0$.

— Let $\alpha = \beta + 1$.
Then
$$\mathscr{P}_\alpha = \begin{cases} \mathscr{P}'_\beta \oplus B_\alpha & \text{if } \alpha \leqslant \zeta \\ \mathscr{P}'_\beta & \text{otherwise.} \end{cases}$$

— Let $\alpha = \lim \alpha(p)$ and set $\mathscr{P}_{<\alpha} = \{\langle p, x\rangle : x \in \mathscr{P}_{\alpha(p)}\}$.
Then
$$\mathscr{P}_\alpha = \begin{cases} \mathscr{P}_{<\alpha} \oplus B_\alpha & \text{if } \alpha \leqslant \zeta \\ \mathscr{P}_{<\alpha} & \text{otherwise.} \end{cases}$$

Note that if the sequence $\{B_\alpha\}$ contains only one member, that is, $\zeta = 0$, then for every recursive α, we have $\mathscr{P}_\alpha = B_0^{(\alpha)}$.

The properties of the jump sets $\mathscr{P}_\alpha$ are similar to the properties of the enumeration jumps. Again we have that if α_1 and α_2 are two notations of the same recursive ordinal, then $\mathscr{P}_{\alpha_1} \equiv_e \mathscr{P}_{\alpha_2}$. We shall omit the proof since it is very close to the proof of the corresponding result for the H_α^A sets mentioned above.

We shall use the following properties of the jump sets, which follow easily from the definition:

($\mathscr{P}$1) If $\beta \leqslant \alpha$, then $\mathscr{P}_\beta \leqslant_e \mathscr{P}_\alpha$ uniformly in β and α.
($\mathscr{P}$2) If $\alpha \leqslant \zeta$, then $B_\alpha \leqslant_e \mathscr{P}_\alpha$ uniformly in α.
($\mathscr{P}$3) Let $(\forall\alpha \leqslant \zeta)(B_\alpha \leqslant_e A^{(\alpha)}$ uniformly in $\alpha)$. Then $\mathscr{P}_\zeta \leqslant_e A^{(\zeta)}$.
($\mathscr{P}$4) If α is a limit ordinal, then the set $\mathscr{P}_{<\alpha}$ is total.
($\mathscr{P}$5) If $\zeta < \alpha$, then the set $\mathscr{P}_\alpha$ is total.

2.4. *The results*

Let us fix a recursive ordinal ζ and a sequence $\{B_\alpha\}_{\alpha\leqslant\zeta}$ of subsets of $\mathbb{N}$. For every recursive ordinal α we use $\mathscr{P}_\alpha$ to denote the α-th jump set of the sequence $\{B_\alpha\}_{\alpha\leqslant\zeta}$.

The following version of Ash's Theorem (Ash 1992) is proved in Soskov and Baleva (2002).

Theorem 2.5. Let α be a recursive ordinal and $A \subseteq \mathbb{N}$. Suppose that for all total sets X such that $(\forall\gamma \leqslant \zeta)(B_\gamma \leqslant_e X^{(\gamma)})$ uniformly in γ we have $A \leqslant_e X^{(\alpha)}$. Then $A \leqslant_e \mathscr{P}_\alpha$.

In particular, if we take $\zeta = \alpha = 0$, Theorem 2.5 gives Selman's Theorem (Selman 1971), which describes the enumeration reducibility in terms of the relation 'r.e. in'.

Theorem 2.6 (Selman). Let A and B be sets of natural numbers. Then $A \leqslant_e B$ if and only if $(\forall X)(B$ is r.e. in $X \Rightarrow A$ is r.e. in $X)$.

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

Another direct consequence of Theorem 2.5 is the following theorem due to Case (Case 1974).

Theorem 2.7 (Case). Let A and B be sets of natural numbers and $n < \omega$. Then $A \leqslant_e B \oplus \varnothing^{(n)}$ if and only if $(\forall X)(B \text{ is } \Sigma_n^X \Rightarrow A \text{ is } \Sigma_n^X)$.

So, while the theorems of Selman and Case describe a kind of positive reducibility between sets of natural numbers in terms of the classical relation 'Σ_n^0 in', Theorem 2.5 gives a description of a positive reducibility of a transfinite sequence of sets to a set. A natural further step is to generalise Theorem 2.5 in order to obtain a reducibility of transfinite sequences of sets to transfinite sequences of sets.

Definition 2.8. Let ζ be a recursive ordinal, and $\mathscr{A} = \{A_\alpha\}_{\alpha \leqslant \zeta}$ and $\mathscr{B} = \{B_\alpha\}_{\alpha \leqslant \zeta}$ be two sequences of sets of natural numbers. Then $\mathscr{A}$ is *uniformly reducible* to $\mathscr{B}$ ($\mathscr{A} \leqslant_u \mathscr{B}$) if there exists a recursive function g such that

$$(\forall \alpha \leqslant \zeta)(A_\alpha = \Phi_{g(\alpha)}(\mathscr{P}_\alpha(\mathscr{B}))),$$

where $\mathscr{P}_\alpha(\mathscr{B})$ denotes the α-th jump set of the sequence $\mathscr{B}$.

The following theorem is our first result.

Theorem 2.9. Let ζ be a recursive ordinal, and $\mathscr{A} = \{A_\alpha\}_{\alpha \leqslant \zeta}$ and $\mathscr{B} = \{B_\alpha\}_{\alpha \leqslant \zeta}$ be two sequences of sets of natural numbers. Then

$$\begin{aligned}\mathscr{A} \leqslant_u \mathscr{B} \iff (\forall \text{ total } X)[&(\forall \alpha \leqslant \zeta)(B_\alpha \leqslant_e X^{(\alpha)} \text{ uniformly in } \alpha)\\ &\Rightarrow (\forall \alpha \leqslant \zeta)(A_\alpha \leqslant_e X^{(\alpha)} \text{ uniformly in } \alpha)].\end{aligned}$$

The uniform operators are introduced in Soskov (2005). We fix natural numbers $k_0, \ldots, k_r$ and k.

Definition 2.10. A mapping $\Gamma : \mathscr{P}(\mathbb{N})^r \to \mathscr{P}(\mathbb{N})$ is defined to be a *uniform operator* of type $(k_0, \ldots, k_r \to k)$ if there exists a function g on the natural numbers such that for all $X \subseteq \mathbb{N}$ and for all $b_0, \ldots, b_r \in \mathbb{N}$,

$$\Gamma(W_{b_0}^{X^{(k_0)}}, \ldots, W_{b_r}^{X^{(k_r)}}) = W_{g(b_0,\ldots,b_r)}^{X^{(k)}}.$$

Given sets $B_0, \ldots, B_r$ of natural numbers, we use $\mathscr{P}^k_{k_0,\ldots,k_r}(B_0, \ldots, B_r)$ to denote the k-th jump set of the sequence $\{A_l\}_{l \leqslant k}$, where

$$A_l = \begin{cases} \varnothing, & \text{if } l \notin \{k_0, \ldots, k_r\} \\ B_m, & \text{if } l = k_m,\ 0 \leqslant m \leqslant r. \end{cases}$$

Theorem 2.11 (Soskov 2005). A mapping $\Gamma : \mathscr{P}(\mathbb{N})^r \to \mathscr{P}(\mathbb{N})$ is a uniform operator of type $(k_0, \ldots, k_r \to k)$ if and only if there exists an enumeration operator Φ such that for all sets $B_0, \ldots, B_r$,

$$\Gamma(B_0, \ldots, B_r) = \Phi(\mathscr{P}^k_{k_0,\ldots,k_r}(B_0, \ldots, B_r)).$$

Combining this result with the theorems of Selman and Case, we obtain the following corollary.

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

Corollary 2.12. Let $n \geqslant 1$ and A and B be sets of natural numbers. Then

$$(\forall X)(B \text{ is } \Sigma_n^X \Rightarrow A \text{ is } \Sigma_n^X)$$

if and only if there exists a uniform operator Γ of type $(n \to n)$ such that $A = \Gamma(B)$.

Here we are going to study uniform operators on transfinite sequences of sets. Let ζ be a recursive ordinal. We use $\mathscr{S}_\zeta$ to denote the set of all sequences $\{A_\alpha\}_{\alpha \leqslant \zeta}$ of sets of natural numbers.

Definition 2.13. A mapping $\Gamma : \mathscr{S}_\zeta \to \mathscr{S}_\zeta$ is a *uniform operator* if there exists a function g on the natural numbers such that if a is an index of a recursive function φ, then $g(a)$ is an index of a recursive function ψ such that for all $X \subseteq \mathbb{N}$,

$$\Gamma(\{W_{\varphi(\alpha)}^{X^{(\alpha)}}\}_{\alpha \leqslant \zeta}) = \{W_{\psi(\alpha)}^{X^{(\alpha)}}\}_{\alpha \leqslant \zeta}.$$

Our second result is the following generalisation of Theorem 2.11. Given a sequence $\mathscr{B} \in \mathscr{S}_\zeta$, we use $\mathscr{P}_\alpha(\mathscr{B})$ to denote the α-th jump set of $\mathscr{B}$.

Theorem 2.14. A mapping $\Gamma : \mathscr{S}_\zeta \to \mathscr{S}_\zeta$ is a uniform operator if and only if there exists a recursive function h such that for every sequence $\mathscr{B} \in \mathscr{S}_\zeta$,

$$\Gamma(\mathscr{B}) = \{\Phi_{h(\alpha)}(\mathscr{P}_\alpha(\mathscr{B}))\}.$$

Combining Theorems 2.9 and 2.14, we obtain the following corollary.

Corollary 2.15. Let $\mathscr{A} = \{A_\alpha\}$ and $\mathscr{B} = \{B_\alpha\}$ be elements of $\mathscr{S}_\zeta$. Then the following assertions are equivalent:

(i) $\mathscr{A} \leqslant_{\mathrm{u}} \mathscr{B}$.
(ii) For all total X, we have the following implication: if $(\forall \alpha \leqslant \zeta)(B_\alpha \leqslant_{\mathrm{e}} X^{(\alpha)}$ uniformly in $\alpha)$, then $(\forall \alpha \leqslant \zeta)(A_\alpha \leqslant_{\mathrm{e}} X^{(\alpha)}$ uniformly in $\alpha)$.
(iii) There exists a uniform operator $\Gamma : \mathscr{S}_\zeta \to \mathscr{S}_\zeta$ such that $\mathscr{A} = \Gamma(\mathscr{B})$.

In the rest of the paper we shall introduce the technique of uniform regular enumerations, which will be used in the proofs of Theorems 2.9 and 2.14.

3. Ordinal approximations

Definition 3.1. Given an ordinal $\alpha > 0$, an *ordinal approximation of* α is a finite sequence $\overline{\alpha} = \alpha_1 < \alpha_2 < \ldots \alpha_n < \alpha$ of ordinals, where $n \geqslant 1$ and $\alpha_1 = 0$. The only ordinal approximation of 0 is 0.

For every ordinal approximation $\overline{\alpha} = \alpha_1, \alpha_2, \ldots, \alpha_n, \alpha$ and every $\beta < \alpha$, we define the *β-predecessor* $\overline{\beta}$ of $\overline{\alpha}$ by means of the following inductive definition.

Definition 3.2.

1. Let $\beta \leqslant \alpha_n$. Then:
   1.1. If $\beta = \alpha_i$ for some $i \in [1, n]$, then $\overline{\beta} = \alpha_1, \ldots, \alpha_i$.

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

1.2. Otherwise, if α_i is the least element of the sequence $\alpha_1, \ldots, \alpha_n$ such that $\beta < \alpha_i$, then $\overline{\beta}$ is the β-predecessor of $\alpha_1, \ldots, \alpha_i$.

2. Let $\alpha_n < \beta < \alpha$. Then:

2.1. If $\alpha = \delta + 1$ and $\beta = \delta$, then $\overline{\beta} = \alpha_1, \ldots, \alpha_n, \delta$.

2.2. If $\alpha = \delta + 1$ and $\beta < \delta$, then $\overline{\beta}$ is the β-predecessor of $\alpha_1, \ldots, \alpha_n, \delta$.

2.3. If $\alpha = \lim \alpha(p)$, then $\overline{\beta}$ is the β-predecessor of

$$\alpha_1, \ldots, \alpha_n, \alpha(p_0), \alpha(p_0 + 1), \ldots, \alpha(p_1),$$

where

$$p_0 = \mu p[\alpha_n < \alpha(p)] \text{ and } p_1 = \mu p[\beta < \alpha(p)].$$

The following simple lemma can be proved by means of transfinite induction on α.

Lemma 3.3. For every ordinal approximation $\overline{\alpha}$ and every $\beta < \alpha$, there exists exactly one β-predecessor $\overline{\beta}$ of $\overline{\alpha}$.

From the definition it follows immediately that there exists a recursive function π such that if $\overline{\alpha}$ is an ordinal approximation and $\beta < \alpha$, then $\pi(\overline{\alpha}, \beta)$ yields the β-predecessor of $\overline{\alpha}$.

We use $\overline{\beta} \prec \overline{\alpha}$ to denote the fact that $\overline{\beta}$ is the β-predecessor of $\overline{\alpha}$. As usual, $\overline{\beta} \preceq \overline{\alpha}$ will stand for $\overline{\beta} \prec \overline{\alpha}$ or $\overline{\beta} = \overline{\alpha}$.

We now point out some useful properties of the predecessor relation, which follow directly from the definition.

Lemma 3.4. Let $\overline{\alpha} = \alpha_1, \ldots, \alpha_n, \alpha$ be an ordinal approximation of α. Then the following assertions hold:

(1) If $\beta \leqslant \alpha_k$, $1 \leqslant k \leqslant n$, then $\overline{\beta} \prec \overline{\alpha} \iff \overline{\beta} \preceq \alpha_1, \ldots, \alpha_k$.

(2) If for some $k \in [1, n]$, $\alpha_k \leqslant \beta < \alpha$ and $\beta_1, \ldots, \beta_l$ is the β-predecessor of α, then $k \leqslant l$ and $\alpha_i = \beta_i, i = 1, \ldots, k$.

(3) Let $\alpha = \delta + 1$, $\alpha_n < \delta$ and $\beta \leqslant \delta$. Then $\overline{\beta} \prec \overline{\alpha} \iff \overline{\beta} \preceq \alpha_1, \ldots, \alpha_n, \delta$.

(4) Let $\alpha = \lim \alpha(p)$ be a limit ordinal and $p_0 = \mu p[\alpha_n < \alpha(p)]$. Let $\beta < \alpha$, $p_1 \geqslant p_0$ and $\alpha(p_1) \geqslant \beta$. Then

$$\overline{\beta} \prec \overline{\alpha} \iff \overline{\beta} \preceq \alpha_1, \ldots, \alpha_n, \alpha(p_0), \alpha(p_0 + 1), \ldots, \alpha(p_1).$$

Lemma 3.5. Let $\gamma < \beta < \alpha$ be ordinals, $\overline{\gamma} \prec \overline{\beta}$ and $\overline{\beta} \prec \overline{\alpha}$. Then $\overline{\gamma} \prec \overline{\alpha}$.

Proof. We use transfinite induction on α. Suppose $\overline{\alpha} = \alpha_1, \ldots, \alpha_n, \alpha$.

Let $\beta \leqslant \alpha_n$. Then $\overline{\beta} \preceq \alpha_1, \ldots, \alpha_n$. By the induction hypothesis, $\overline{\gamma} \prec \alpha_1, \ldots, \alpha_n$. So $\overline{\gamma} \prec \overline{\alpha}$ by Lemma 3.4.

Suppose now that $\alpha_n < \beta$. Let $\alpha = \delta + 1$. Set $\overline{\delta} = \alpha_1, \ldots, \alpha_n, \delta$. Since $\beta \leqslant \delta$, we have $\alpha_n < \delta$. By Lemma 3.4, $\overline{\beta} \preceq \overline{\delta}$. By the induction hypothesis, $\overline{\gamma} \prec \overline{\delta}$. From here, and again by Lemma 3.4, it follows that $\overline{\gamma} \prec \overline{\alpha}$.

This leaves the case $\alpha_n < \beta$ and $\alpha = \lim \alpha(p)$. Let

$$p_0 = \mu p[\alpha_n < \alpha(p)] \text{ and } p_\beta = \mu p[\beta < \alpha(p)].$$

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

Set $\overline{\alpha(p_\beta)} = \alpha_1, \ldots, \alpha_n, \alpha(p_0), \alpha(p_0+1), \ldots, \alpha(p_\beta)$. Now we have that $\overline{\beta} \prec \overline{\alpha(p_\beta)}$. By induction, $\overline{\gamma} \prec \overline{\alpha(p_\beta)}$, so by Lemma 3.4, $\overline{\gamma} \prec \overline{\alpha}$. □

From the last lemma it follows that if we fix an ordinal approximation $\overline{\alpha}$ and consider the set of all ordinal approximations $\overline{\beta} \prec \overline{\alpha}$, this set is well ordered by the relation $\prec$ and its order type is α.

4. Regular finite parts

We fix a sequence $\{B_\alpha\}_{\alpha \leqslant \zeta}$, of subsets of $\mathbb{N}$. For every $\alpha \leqslant \zeta$ set $B^*_\alpha = \mathbb{N} \oplus B_\alpha$.

In the following we shall use the term *finite part* for finite mappings of $\mathbb{N}$ into $\mathbb{N}$ defined on finite segments $[0, q-1]$ of $\mathbb{N}$. Finite parts will be denoted by the letters τ, ρ. If $\mathrm{dom}(\tau) = [0, q-1]$, let $\mathrm{lh}(\tau) = q$.

We shall suppose that an effective coding of all finite sequences, and hence of all finite parts, is fixed. Given two finite parts τ and ρ, we shall say that τ is less than or equal to ρ if the code of τ is less than or equal to the code of ρ. We shall use $\tau \subseteq \rho$ to denote the fact that the partial mapping ρ extends τ, and say that ρ is an extension of τ. For any τ, we shall use $\tau \restriction n$ to denote the restriction of τ on $[0, n-1]$.

Below, we define for every $\alpha \leqslant \zeta$ and every ordinal approximation $\overline{\alpha}$ of α the $\overline{\alpha}$-regular finite parts. The definition is by transfinite recursion on α.

Let $\alpha \leqslant \zeta$. Suppose that for all $\beta < \alpha$ we have defined the $\overline{\beta}$-regular finite parts, and for every $\overline{\beta}$-regular τ we have defined the $\overline{\beta}$-rank $|\tau|_{\overline{\beta}}$ of τ. Suppose also that for all finite parts ρ and for all $e, x \in \mathbb{N}$ we have defined the forcing relations $\rho \Vdash_{\overline{\beta}} F_e(x)$ and $\rho \Vdash_{\overline{\beta}} \neg F_e(x)$.

Definition 4.1. Given a $\overline{\beta}$-regular finite part τ, we say that $\rho \supseteq \tau$ is a *normal* $\overline{\beta}$-regular extension of τ if $|\rho|_{\overline{\beta}} = |\tau|_{\overline{\beta}} + 1$.

We fix an ordinal approximation $\overline{\alpha}$ of α.

(1) $\alpha = 0$.

Then $\overline{\alpha} = 0$. The 0-regular finite parts are finite parts τ such that $\mathrm{dom}(\tau) = [0, 2q+1]$ and for all odd $z \in \mathrm{dom}(\tau), \tau(z) \in B^*_0$.

If $\mathrm{dom}(\tau) = [0, 2q+1]$, the 0-rank $|\tau|_0$ of τ is equal to the number $q+1$ of the odd elements of $\mathrm{dom}(\tau)$.

Given a finite part ρ, let:

$$\rho \Vdash_0 F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(\rho((u)_0) \simeq (u)_1))$$
$$\rho \Vdash_0 \neg F_e(x) \iff \forall(0\text{-regular } \tau)(\rho \subseteq \tau \Rightarrow \tau \nVdash_0 F_e(x)).$$

(2) $\alpha = \beta + 1$.

Let $\overline{\beta}$ be the β-predecessor of $\overline{\alpha}$.

Set $X^{\overline{\beta}}_p = \{\rho : \rho \text{ is } \overline{\beta}\text{-regular } \&\ \rho \Vdash_{\overline{\beta}} F_{(p)_0}((p)_1)\}$.

Given a finite part τ and a set X of $\overline{\beta}$-regular finite parts, let $\mu_{\overline{\beta}}(\tau, X)$ be the least extension of τ belonging to X if any, and $\mu_{\overline{\beta}}(\tau, X)$ be the least $\overline{\beta}$-regular extension of τ

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

otherwise. We shall assume that $\mu_{\overline{\beta}}(\tau, X)$ is undefined if there is no $\overline{\beta}$-regular extension of τ.

Let τ be a finite part defined on $[0, q-1]$ and $r \geqslant 0$. Then τ is *$\overline{\alpha}$-regular* of *$\overline{\alpha}$-rank* $r+1$ if there exist natural numbers

$$0 < n_0 < \ell_0 < m_0 < b_0 < n_1 < \ell_1 < m_1 < b_1 \ldots < n_r < \ell_r < m_r < b_r < n_{r+1} = q$$

such that $\tau \restriction n_0$ is a $\overline{\beta}$-regular finite part of rank 1 and for all j, $0 \leqslant j \leqslant r$, the following conditions are satisfied:

(s_a) $\tau \restriction \ell_j$ is a normal $\overline{\beta}$-regular extension of $\tau \restriction n_j$.

(s_b) $\tau \restriction m_j = \begin{cases} \mu_{\overline{\beta}}(\tau \restriction (\ell_j + 1), X^{\overline{\beta}}_{\langle e, \ell_j \rangle}), & \text{if } \tau(n_j) \simeq \langle 0, \beta, e \rangle + 1 \\ \mu_{\overline{\beta}}(\tau \restriction (\ell_j + 1), X^{\overline{\beta}}_{p}), & \text{if } \tau(n_j) \simeq \langle 1, \beta, p \rangle + 1 \\ \text{a normal } \overline{\beta}\text{-regular extension of } \tau \restriction \ell_j, & \text{otherwise.} \end{cases}$

(s_c) $\tau \restriction b_j$ is a normal $\overline{\beta}$-regular extension of $\tau \restriction m_j$ and $\tau(b_j) \in B^*_\alpha$.

(s_d) $\tau \restriction n_{j+1}$ is a normal $\overline{\beta}$-regular extension of $\tau \restriction b_j$.

To conclude the definition of the $\overline{\alpha}$-regular finite parts in this case, for every ρ, e and x let

$$\begin{aligned} \rho \Vdash_{\overline{\alpha}} F_e(x) &\iff \exists v(\langle v, x \rangle \in W_e \ \&\ (\forall u \in D_v)((u = \langle e_u, x_u, 0 \rangle \ \&\ \rho \Vdash_{\overline{\beta}} F_{e_u}(x_u)) \\ &\qquad \vee (u = \langle e_u, x_u, 1 \rangle \ \&\ \rho \Vdash_{\overline{\beta}} \neg F_{e_u}(x_u)))) \\ \rho \Vdash_{\overline{\alpha}} \neg F_e(x) &\iff \forall(\overline{\alpha}\text{-regular } \tau)(\rho \subseteq \tau \Rightarrow \tau \nVdash_{\overline{\alpha}} F_e(x)). \end{aligned}$$

(3) $\alpha = \lim \alpha(p)$ is a limit ordinal.

Let $\overline{\alpha} = \alpha_1, \ldots, \alpha_n, \alpha$. Set $p_0 = \mu p[\alpha(p) > \alpha_n]$. For every p we use $\overline{\alpha(p)}$ to denote the $\alpha(p)$-predecessor of $\overline{\alpha}$. Note that for every $p \geqslant p_0$

$$\overline{\alpha(p)} = \alpha_1, \ldots, \alpha_n, \alpha(p_0), \alpha(p_0 + 1), \ldots, \alpha(p).$$

A finite part τ defined on $[0, q-1]$ is $\overline{\alpha}$-regular with $\overline{\alpha}$-rank $r+1$ if there exist natural numbers

$$0 < n_0 < b_0 < m_0 < n_1 < b_1 < m_1 \ldots < n_r < b_r < m_r < n_{r+1} = q,$$

such that $\tau \restriction n_0$ is an $\alpha_1, \ldots, \alpha_n$-regular finite part of rank 1 and for all $j, 0 \leqslant j \leqslant r$, the following conditions are satisfied:

(l_a) $\tau \restriction b_j$ is an $\overline{\alpha(p_0 + 3j)}$-regular finite part of rank 1 and $\tau(b_j) \in B^*_\alpha$.

(l_b) $\tau \restriction m_j$ is an $\overline{\alpha(p_0 + 3j + 1)}$-regular finite part of rank 1.

(l_c) $\tau \restriction n_{j+1}$ is an $\overline{\alpha(p_0 + 3j + 2)}$-regular finite part of rank 1.

For every finite part ρ and every $e, x \in \mathbb{N}$ set:

$$\begin{aligned} \rho \Vdash_{\overline{\alpha}} F_e(x) &\iff \exists v(\langle v, x \rangle \in W_e \ \& \\ &\qquad (\forall u \in D_v)(u = \langle p_u, e_u, x_u \rangle \ \&\ \rho \Vdash_{\overline{\alpha(p_u)}} F_{e_u}(x_u))) \\ \rho \Vdash_{\overline{\alpha}} \neg F_e(x) &\iff \forall(\overline{\alpha}\text{-regular } \tau)(\rho \subseteq \tau \Rightarrow \tau \nVdash_{\overline{\alpha}} F_e(x)). \end{aligned}$$

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

The following lemma shows that the $\overline{\alpha}$-rank is well defined.

Lemma 4.2. Let $\alpha \leqslant \zeta$ and τ be an $\overline{\alpha}$-regular finite part. Then the following assertions hold:

(1) Suppose that $\alpha = \beta + 1$. Let $n'_0, \ell'_0, m'_0, b'_0, \ldots, n'_p, \ell'_p, m'_p, n'_{p+1}$ and $n_0, \ell_0, m_0, b_0, \ldots, n_r, \ell_r, m_r, b_r, n_{r+1}$ be two sequences of natural numbers satisfying **s_a–s_d** from the definition above. Then $r = p, n_{p+1} = n'_{p+1}$, and for all $j \leqslant r$ we have $n_j = n'_j, \ell_j = \ell'_j, m_j = m'_j$ and $b_j = a_j$.

(2) Suppose that $\alpha = \lim \alpha(p)$ is a limit ordinal and let $n'_0, b'_0, m'_0 \ldots, n'_p, b'_p, m'_p, n'_{p+1}$ and $n_0, b_0, m_0 \ldots, n_r, b_r, m_r, n_{r+1}$ be two sequences of natural numbers satisfying the conditions **l_a–l_c**. Then $r = p, n_{p+1} = n'_{p+1}$, and for all $j \leqslant r$ we have $n'_j = n_j$, $b'_j = b_j$ and $m'_j = m_j$.

(3) If ρ is $\overline{\alpha}$-regular, $\tau \subseteq \rho$ and $|\tau|_{\overline{\alpha}} = |\rho|_{\overline{\alpha}}$, then $\tau = \rho$.

Proof. The proof follows easily from the definition of the $\overline{\alpha}$-regular finite parts by transfinite induction on α. □

Corollary 4.3. Let $\alpha = \beta + 1$, $\overline{\alpha}$ be an ordinal approximation of α, and $\overline{\beta}$ be the β-predecessor of $\overline{\alpha}$. Then every $\overline{\alpha}$-regular finite part τ is $\overline{\beta}$-regular and $|\tau|_{\overline{\beta}} > |\tau|_{\overline{\alpha}}$.

We shall state several properties of the regular finite parts omitting the proofs which can be found in Soskov and Baleva (2002).

Lemma 4.4. Let $1 \leqslant \alpha \leqslant \zeta$ and $\overline{\alpha} = \alpha_1, \ldots, \alpha_n, \alpha$, $1 \leqslant k \leqslant n$. Then every $\overline{\alpha}$-regular finite part τ is $\alpha_1, \ldots, \alpha_k$-regular and the $\alpha_1, \ldots, \alpha_k$-rank of τ is strictly greater than $|\tau|_{\overline{\alpha}}$.

Lemma 4.5. Let $\alpha = \lim \alpha(p)$ be a limit ordinal. Let $\overline{\alpha} = \alpha_1, \ldots, \alpha_n, \alpha$ and $p_0 = \mu p[\alpha(p) > \alpha_n]$. Suppose that $p_1 \geqslant p_0$ and τ is an $\alpha_1, \ldots, \alpha_n, \alpha(p_0), \ldots, \alpha(p_1)$-regular finite part of rank 1. Then for every $\overline{\beta} \prec \overline{\alpha}$ if τ is $\overline{\beta}$-regular, then $\beta \leqslant \alpha(p_1)$.

Definition 4.6. For every finite part τ and every ordinal approximation $\overline{\alpha}$ let

$$\mathrm{Reg}(\tau, \overline{\alpha}) = \{\overline{\beta} : \overline{\beta} \preceq \overline{\alpha} \text{ and } \tau \text{ is } \overline{\beta}\text{-regular}\}.$$

Lemma 4.7. Let $\alpha \leqslant \zeta$, and let $\overline{\alpha} = \alpha_1, \ldots, \alpha_n, \alpha$ be an ordinal approximation of α and τ be an $\overline{\alpha}$-regular finite part. Then the following assertions are true:

(1) If $\alpha = \delta + 1$ and $\overline{\delta}$ is the δ-predecessor of $\overline{\alpha}$, then

$$\overline{\beta} \in \mathrm{Reg}(\tau, \overline{\alpha}) \iff \overline{\beta} = \overline{\alpha} \vee \overline{\beta} \in \mathrm{Reg}(\tau, \overline{\delta}).$$

(2) Let $\alpha = \lim \alpha(p)$. Set $p_0 = \mu p[\alpha(p) > \alpha_n]$, and for every $p \geqslant p_0$, let $\overline{\alpha(p)}$ be the $\alpha(p)$-predecessor of $\overline{\alpha}$. Suppose that $p_1 \geqslant p_0$ and τ is an $\overline{\alpha(p_1)}$-regular finite part of rank 1. Then

$$\overline{\beta} \in \mathrm{Reg}(\tau, \overline{\alpha}) \iff \overline{\beta} = \overline{\alpha} \vee \overline{\beta} \in \mathrm{Reg}(\tau, \overline{\alpha(p_1)}).$$

Lemma 4.8. Let $\alpha \leqslant \zeta$ and $\overline{\alpha}$ be an approximation of α. Suppose that $\overline{\beta} \preceq \overline{\alpha}$. Then there exists a natural number $k(\overline{\alpha}, \overline{\beta})$ such that every $\overline{\alpha}$-regular finite part of rank greater than or equal to $k(\overline{\alpha}, \overline{\beta})$ is $\overline{\beta}$-regular.

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

Proof. We use transfinite induction on α. The assertion is obviously true for $\alpha = 0$.

Suppose that $\alpha = \delta + 1$ and $\overline{\delta}$ is the δ-predecessor of $\overline{\alpha}$. Let $\overline{\beta} \prec \overline{\alpha}$. Then $\overline{\beta} \preceq \overline{\delta}$. By induction, every $\overline{\delta}$-regular finite part of rank at least $k(\overline{\delta}, \overline{\beta})$ is $\overline{\beta}$-regular. Set $k(\overline{\alpha}, \overline{\beta}) = k(\overline{\delta}, \overline{\beta}) = k$. Consider an $\overline{\alpha}$-regular finite part τ of rank at least k. Then τ is $\overline{\delta}$-regular of rank greater than k, so τ is $\overline{\beta}$-regular.

Let $\alpha = \lim \alpha(p)$, $\overline{\alpha} = \alpha_1, \ldots, \alpha_n, \alpha$ and $\overline{\beta} \prec \overline{\alpha}$. Set $p_0 = \mu p[\alpha(p) > \alpha_n]$. By definition, an $\overline{\alpha}$-regular finite part τ is of rank $r+1$ if and only if τ is $\overline{\alpha(p_0 + 3r + 2)}$-regular of rank 1. Let $p_1 = \mu p[p > p_0 \wedge \alpha(p) > \beta]$.

By Lemma 3.4, $\overline{\beta} \preceq \overline{\alpha(p_1)}$. Hence, by induction, every $\overline{\alpha(p_1)}$-regular finite part of rank at least $k = k(\overline{\alpha(p_1)}, \overline{\beta})$ is $\overline{\beta}$-regular.

Now let r be the least natural number such that $p_0 + 3r + 2 > p_1 + k$. Consider an $\overline{\alpha}$-regular finite part τ of rank greater than or equal to $r+1$. Then, by Lemma 4.4, τ is $\overline{\alpha(p_1 + k)}$-regular of rank at least 2, and hence, again by Lemma 4.4, τ is $\overline{\alpha(p_1)}$-regular of rank greater than k. So τ is $\overline{\beta}$-regular. □

Remark 4.9. The proof above means that we can assume that the function k is recursive.

Corollary 4.10. Let $\alpha \leqslant \zeta$, $\overline{\alpha}$ be an ordinal approximation of α, and $\overline{\beta} \preceq \overline{\alpha}$. Suppose that τ is an $\overline{\alpha}$-regular finite part of rank greater than $k(\overline{\alpha}, \overline{\beta}) + s$. Then $|\tau|_{\overline{\beta}} > s$.

Let $\alpha \leqslant \zeta$ and $\overline{\alpha}$ be an ordinal approximation of α. We use $\mathscr{R}_{\overline{\alpha}}$ to denote the set of all $\overline{\alpha}$-regular finite parts. For every $p \in \mathbb{N}$, let

$$Y_p^{\overline{\alpha}} = \{\tau : \tau \in \mathscr{R}_{\overline{\alpha}} \ \& \ (\exists \rho \supseteq \tau)(\rho \in \mathscr{R}_{\overline{\alpha}} \ \& \ \rho \Vdash_{\overline{\alpha}} F_{(p)_0}((p)_1))\},$$
$$Z_p^{\overline{\alpha}} = \{\tau : \tau \in \mathscr{R}_{\overline{\alpha}} \& \ \tau \Vdash_{\overline{\alpha}} \neg F_{(p)_0}((p)_1)\}.$$

Let $\mu_{\overline{\alpha}}^X(\tau, p) \simeq \mu_{\overline{\alpha}}(\tau, X_p^{\overline{\alpha}})$.

Proposition 4.11. There exist recursive functions $h_1, \ldots, h_5$ such that for every sequence $\mathscr{B} \in \mathscr{S}_\zeta$ and for every ordinal approximation $\overline{\alpha}$, $\alpha \leqslant \zeta$, the following assertions are true:

(1) $\mathscr{R}_{\overline{\alpha}} = \Phi_{h_1(\overline{\alpha})}(\mathscr{P}_\alpha(\mathscr{B}))$.
(2) For every $p \in \mathbb{N}$, $X_p^{\overline{\alpha}} = \Phi_{h_2(\overline{\alpha}, p)}(\mathscr{P}_\alpha(\mathscr{B}))$.
(3) For every $p \in \mathbb{N}$, $Y_p^{\overline{\alpha}} = \Phi_{h_3(\overline{\alpha}, p)}(\mathscr{P}_\alpha(\mathscr{B}))$.
(4) For every $p \in \mathbb{N}$, the characteristic function of $\{Z_p^{\overline{\alpha}}\}$ is equal to $\{h_4(\overline{\alpha})\}^{\mathscr{P}'_\alpha(\mathscr{B})}$.
(5) $\mu_{\overline{\alpha}}^X = \{h_5(\overline{\alpha})\}^{\mathscr{P}'_\alpha(\mathscr{B})}$.

5. Regular enumerations

For every $\overline{\alpha}$-regular finite part τ of rank $r+1$, we define the subsets $N_{\overline{\alpha}}^\tau$ and $B_{\overline{\alpha}}^\tau$ of $\mathrm{dom}(\tau)$ as follows.

Definition 5.1.

(a) If $\alpha = 0$, let

$$N_{\overline{\alpha}}^\tau = \{n : n \in \mathrm{dom}(\tau) \ \& \ n \text{ is even}\}$$
$$B_{\overline{\alpha}}^\tau = \{b : b \in \mathrm{dom}(\tau) \ \& \ b \text{ is odd}\}.$$

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

(b) Let $\alpha = \beta + 1$ and $n_0, \ell_0, m_0, b_0, \ldots, n_r, \ell_r, m_r, b_r, n_{r+1}$ satisfy the conditions **s_a–s_d** from the definition of the regular finite parts. Set $N^{\tau}_{\overline{\alpha}} = \{n_0, \ldots, n_r\}$ and $B^{\tau}_{\overline{\alpha}} = \{b_0, \ldots, b_r\}$.
(c) Let $\alpha = \lim \alpha(p)$ and $n_0, b_0, m_0 \ldots, n_r, b_r, m_r, n_{r+1}$ satisfy the conditions **l_a–l_c** from the definition of the regular finite parts. Set $N^{\tau}_{\overline{\alpha}} = \{n_0, \ldots, n_r\}$ and $B^{\tau}_{\overline{\alpha}} = \{b_0, \ldots, b_r\}$.

Definition 5.2. Let $\overline{\zeta}$ be an ordinal approximation of ζ. A total mapping f of $\mathbb{N}$ in $\mathbb{N}$ is called a *regular enumeration* (with respect to $\overline{\zeta}$) if the following conditions hold:

(i) For every finite part $\rho \subseteq f$, there exists a $\overline{\zeta}$-regular extension τ of ρ such that $\tau \subseteq f$.
(ii) If $\overline{\alpha} \preceq \overline{\zeta}$ and $z \in B^*_{\alpha}$, then there exists an $\overline{\alpha}$-regular $\tau \subseteq f$ such that $z \in \tau(B^{\tau}_{\overline{\alpha}})$.
(iii) If $\overline{\alpha + 1} \preceq \overline{\zeta}$ and $p \in \mathbb{N}$, then there exists an $\overline{\alpha + 1}$-regular $\tau \subseteq f$ such that $\langle 1, \alpha, p\rangle + 1 \in \tau(N^{\tau}_{\overline{\alpha+1}})$.

Clearly, if f is a regular enumeration and $\overline{\alpha} \preceq \overline{\zeta}$, then for every $\rho \subseteq f$, there exists an $\overline{\alpha}$-regular $\tau \subseteq f$ such that $\rho \subseteq \tau$. Moreover, there exist $\overline{\alpha}$-regular finite parts of f of arbitrary large rank.

Given a regular f and $\overline{\alpha} \preceq \overline{\zeta}$, let $B^f_{\overline{\alpha}} = \{b : (\exists \tau \subseteq f)(\tau \text{ is } \overline{\alpha}\text{-regular } \& \ b \in B^{\tau}_{\overline{\alpha}})\}$. Evidently, $f(B^f_{\overline{\alpha}}) = B^*_{\alpha}$.

Proposition 5.3. Suppose that f is a regular enumeration. Then:

(1) $B_0 \leqslant_e f$.
(2) If $\alpha = \beta + 1 \leqslant \zeta$, then $B_{\alpha} \leqslant_e f \oplus \mathscr{P}'_{\beta}$ uniformly in α.
(3) If $\alpha \leqslant \zeta$ is a limit ordinal, then $B_{\alpha} \leqslant_e f \oplus \mathscr{P}_{<\alpha}$ uniformly in α.
(4) $\mathscr{P}_{\alpha} \leqslant_e f^{(\alpha)}$ uniformly in α.
(5) $B_{\alpha} \leqslant_e f^{(\alpha)}$ uniformly in α.

Proof. Note that since $B^*_{\alpha} = \mathbb{N} \oplus B_{\alpha}$, we have for every α that

$$B_{\alpha} = \{x : 2x + 1 \in B^*_{\alpha}\}.$$

Since f is regular, $B^*_0 = f(B^f_0)$. Clearly, B^f_0 is equal to the set of all odd natural numbers.

We now turn to the proof of (2) and (3). We shall describe an effective procedure satisfying the requirements of (2) and (3) by means of effective transfinite recursion on α.

Let $\alpha = \beta + 1$. Suppose that $\overline{\alpha}$ is the α-predecessor of $\overline{\zeta}$ and $\overline{\beta}$ is the β-predecessor of $\overline{\alpha}$.

Since f is regular, for every finite part ρ of f there exists an $\overline{\alpha}$-regular $\tau \subseteq f$ such that $\rho \subseteq \tau$. Hence there exist natural numbers

$$0 < n_0 < \ell_0 < m_0 < b_0 < n_1 < \ell_1 < m_1 < b_1 < \ldots < n_r < \ell_r < m_r < b_r < \ldots,$$

such that for every $r \geqslant 0$, the finite part $\tau_r = f \restriction n_{r+1}$ is $\overline{\alpha}$-regular, and $n_0, \ell_0, m_0, b_0, \ldots, n_r, \ell_r, m_r, b_r, n_{r+1}$ are the numbers satisfying the conditions **s_a–s_d** from the definition of the $\overline{\alpha}$-regular finite part τ_r. Clearly, $B^f_{\overline{\alpha}} = \{b_0, b_1 \ldots\}$. We shall show that there exists a recursive in $f \oplus \mathscr{P}'_{\beta}$ way to list $n_0, \ell_0, m_0, b_0, \ldots$ in an increasing order.

Clearly, $f \restriction n_0$ is $\overline{\beta}$-regular and $|f \restriction n_0|_{\overline{\beta}} = 1$. By Proposition 4.11, $\mathscr{R}_{\overline{\beta}}$ is uniformly recursive in $\mathscr{P}'_{\beta}$. Using f, we can generate consecutively the finite parts $f \restriction q$ for $q = 1, 2 \ldots$. By Lemma 4.2, $f \restriction n_0$ is the first element of this sequence that belongs to $\mathscr{R}_{\overline{\beta}}$. Clearly, $n_0 = \mathrm{lh}(f \restriction n_0)$.

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

Suppose that $r \geqslant -1$ and $n_0, \ell_0, m_0, b_0, \ldots, n_r, \ell_r, m_r, b_r, n_{r+1}$ have already been listed. Since $f \restriction \ell_{r+1}$ is a normal $\overline{\beta}$-regular extension of $f \restriction n_{r+1}$, it is the first element of the sequence $f \restriction q$, $q > n_{r+1}$, belonging to $\mathscr{R}_{\overline{\beta}}$. In this way we get ℓ_{r+1}. To determine m_{r+1}, we have to consider the value y of $f(n_{r+1})$:

(a) $y = \langle 0, \beta, e\rangle + 1$. Then $m_{r+1} = \mathrm{lh}(\mu^X_{\overline{\beta}}(f \restriction (\ell_{r+1}+1), \langle e, \ell_{r+1}\rangle))$.

(b) $y = \langle 1, \beta, p\rangle + 1$. Then $m_{r+1} = \mathrm{lh}(\mu^X_{\overline{\beta}}(f \restriction (\ell_{r+1}+1), p))$.

(c) Otherwise, m_{r+1} is equal to the least $q > \ell_{r+1}$ such that $f \restriction q \in \mathscr{R}_{\overline{\beta}}$.

Since $f \restriction b_{r+1}$ is a normal $\overline{\beta}$-regular extension of $f \restriction m_{r+1}$ and $f \restriction n_{r+2}$ is a normal $\overline{\beta}$-regular extension of $f \restriction b_{r+1}$, we can find b_{r+1} and n_{r+2} in the same way as above.

So $B^f_{\overline{\alpha}}$ is recursive in $f \oplus \mathscr{P}'_\beta$. Hence, $B^*_\alpha \leqslant_{\mathrm{e}} f \oplus \mathscr{P}'_\beta$ since $B^*_\alpha = f(B^f_\alpha)$.

Suppose now that $\alpha = \lim \alpha(p)$ is a limit ordinal. Clearly, for every p the set $\mathscr{P}_{\alpha(p)}$ is recursive in $\mathscr{P}_{<\alpha}$ uniformly in p. Let $\overline{\alpha}$ be the α-predecessor of $\overline{\zeta}$, and for every p let $\overline{\alpha(p)}$ be the $\alpha(p)$-predecessor of $\overline{\alpha}$. We may think of f as an infinite union of $\overline{\alpha}$-regular finite parts. So there exists an infinite sequence of natural numbers

$$n_0 < b_0 < m_0 < n_1 < b_1 < m_1 < \ldots < n_r < b_r < m_r < n_{r+1} \ldots$$

such that for every r the finite part $f \restriction n_{r+1}$ is $\overline{\alpha}$-regular of rank $r+1$ and $n_0, b_0, m_0 \ldots$, n_r, b_r, m_r are the elements of $\mathrm{dom}(f \restriction n_{r+1})$ satisfying the conditions **l_a–l_c** from the definition of the $\overline{\alpha}$-regular finite part. As in the previous case, there exists a recursive in $f \oplus \mathscr{P}_{<\alpha}$ way to list the numbers $n_0, b_0, m_0 \ldots$ in an increasing order. To show this, all we need to know is that for every p the set $\mathscr{R}_{\overline{\alpha(p)}}$ is uniformly recursive $\mathscr{P}'_{\alpha(p)}$, and hence it is uniformly recursive in $\mathscr{P}_{<\alpha}$.

Assertions (4) and (5) follow easily from (1), (2) and (3). □

Remark 5.4. An inspection of the above proof shows there exists a recursive function φ such that, for every sequence $\{B_\alpha\}_{\alpha \leqslant \zeta}$ of sets of natural numbers and every regular with respect to this sequence enumeration f, we have

$$B_\alpha = W^{f^{(\alpha)}}_{\varphi(\alpha)}.$$

Let f be a total mapping on $\mathbb{N}$. We now define for every recursive ordinal α, $e, x \in \mathbb{N}$ the relations $f \models_\alpha F_e(x)$ and $f \models \neg F_e(x)$ by means of transfinite recursion on α.

Definition 5.5.

(i) Let $\alpha = 0$. Then

$$f \models_0 F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(f((u)_0) = (u)_1)).$$

(ii) Let $\alpha = \beta + 1$. Then

$$\begin{aligned} f \models_\alpha F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)((u = \langle e_u, x_u, 0\rangle \ \&\ f \models_\beta F_{e_u}(x_u)) \\ \vee (u = \langle e_u, x_u, 1\rangle \ \&\ f \models_\beta \neg F_{e_u}(x_u)))). \end{aligned}$$

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

(iii) Let $\alpha = \lim \alpha(p)$ be a limit ordinal. Then

$$f \models_\alpha F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(u = \langle p_u, e_u, x_u\rangle \ \&\ f \models_{\alpha(p_u)} F_{e_u}(x_u))).$$

(iv) $f \models_\alpha \neg F_e(x) \iff f \not\models_\alpha F_e(x)$.

Following the definition of the enumeration jump and the above definition, we can define a recursive function h such that for every recursive ordinal α and every enumeration operator Φ_z the following equivalence is true:

$$x \in \Phi_z(f^{(\alpha)}) \iff f \models_\alpha F_{h(\alpha,z)}(x).$$

Therefore we have the following lemma.

Lemma 5.6. Let f be a total mapping on $\mathbb{N}$ and α be a recursive ordinal. Then $A \leqslant_e f^{(\alpha)}$ if and only if there exists an e such that for all x, we have $x \in A \iff f \models_\alpha F_e(x)$.

Our next goal is to prove the Truth Lemma. Note that for all $\overline{\alpha} \preceq \overline{\zeta}$ the relation $\Vdash_{\overline{\alpha}}$ is monotone, that is, if $\tau \subseteq \rho$ are $\overline{\alpha}$-regular and $\tau \Vdash_{\overline{\alpha}} (\neg)F_e(x)$, then $\rho \Vdash_{\overline{\alpha}} (\neg)F_e(x)$.

Suppose that f is a regular enumeration.

Lemma 5.7. Let $\alpha < \zeta$ and let $\overline{\alpha}$ be the α-predecessor of $\overline{\zeta}$. Assume also that

$$f \models_\alpha F_e(x) \iff (\exists \tau \subseteq f)(\tau \text{ is } \overline{\alpha}\text{-regular } \&\ \tau \Vdash_{\overline{\alpha}} F_e(x)).$$

Then

$$f \models_\alpha \neg F_e(x) \iff (\exists \tau \subseteq f)(\tau \text{ is } \overline{\alpha}\text{-regular } \&\ \tau \Vdash_{\overline{\alpha}} \neg F_e(x)).$$

Proof. Assume that $f \models_\alpha \neg F_e(x)$ and for all $\overline{\alpha}$-regular $\tau \subseteq f$, we have $\tau \nVdash_{\overline{\alpha}} \neg F_e(x)$. Then for all $\overline{\alpha}$-regular finite parts τ of f there exists an $\overline{\alpha}$-regular $\rho \supseteq \tau$ such that $\rho \Vdash_{\overline{\alpha}} F_e(x)$.

Since f is regular, there exists an $\overline{\alpha+1}$-regular finite part $\tau \subseteq f$ such that $\langle 1, \alpha, \langle e, x\rangle\rangle + 1 \in \tau(N^\tau_{\overline{\alpha+1}})$. Let $|\tau_{\overline{\alpha+1}}| = r + 1$, and let

$$0 < n_0 < \ell_0 < m_0 < b_0 < \ldots < n_r < \ell_r < m_r < b_r < n_{r+1}$$

be the natural numbers satisfying the conditions **s_a–s_d** of the definition of the $\overline{\alpha+1}$-regular finite parts. Then $N^\tau_{\overline{\alpha+1}} = \{n_0, \ldots, n_r\}$. Let $\tau(n_j) \simeq \langle 1, \alpha, \langle e, x\rangle\rangle + 1$. Hence $\tau\restriction b_j = \mu_{\overline{\alpha}}(\tau\restriction(\ell_j + 1), X^{\overline{\alpha}}_{\langle e,x\rangle})$.

Clearly, there exists an $\overline{\alpha}$-regular extension τ_1 of τ such that $\tau_1 \subseteq f$. Therefore, there exists an $\overline{\alpha}$-regular extension ρ of $\tau\restriction(\ell_j + 1)$ in $X^{\overline{\alpha}}_{\langle e,x\rangle}$. Then $\tau\restriction b_j \in X^{\overline{\alpha}}_{\langle e,x\rangle}$. Clearly, $\tau\restriction b_j$ is an $\overline{\alpha}$-regular finite part of f, and hence $f \models_\alpha F_e(x)$, which gives a contradiction.

Assume now that $\tau \subseteq f$ is $\overline{\alpha}$-regular, $\tau \Vdash_{\overline{\alpha}} \neg F_e(x)$ and $f \models_\alpha F_e(x)$. Then there exists an $\overline{\alpha}$-regular $\rho \subseteq f$ such that $\rho \Vdash_{\overline{\alpha}} F_e(x)$. Using the monotonicity of $\Vdash_{\overline{\alpha}}$, we can assume that $\tau \subseteq \rho$ and again get a contradiction. □

Lemma 5.8. Let f be a regular enumeration. Then:

(1) For all $\overline{\alpha} \preceq \overline{\zeta}$, $f \models_\alpha F_e(x) \iff (\exists \tau \subseteq f)(\tau \text{ is } \overline{\alpha}\text{-regular } \&\ \tau \Vdash_{\overline{\alpha}} F_e(x))$.
(2) For all $\overline{\alpha} \prec \overline{\zeta}$, $f \models_\alpha \neg F_e(x) \iff (\exists \tau \subseteq f)(\tau \text{ is } \overline{\alpha}\text{-regular } \&\ \tau \Vdash_{\overline{\alpha}} \neg F_e(x))$.

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

Proof. We use transfinite induction on α. Condition (1) is obviously true for $\alpha = 0$, and hence according to the above lemma, (2) is also true in this case.

Let $\alpha = \beta + 1$. The truth of (1) for α follows easily from the induction hypothesis. The truth of (2) follows from the above lemma.

Suppose that $\alpha \preceq \zeta$ and $\alpha = \lim \alpha(p)$ is limit ordinal. It is sufficient to show that (1) is true for α. Assume that $f \models_\alpha F_e(x)$. Then there exists a pair $\langle v, x\rangle \in W_e$ such that if $u \in D_v$, then $u = \langle p_u, e_u, x_u\rangle$ and $f \models_{\alpha(p_u)} F_{e_u}(x_u)$. By induction, for every $u \in D_v$ there exists an $\overline{\alpha(p_u)}$-regular finite part $\tau_u \subseteq f$ such that $\tau_u \Vdash_{\overline{\alpha(p_u)}} F_{e_u}(x_u)$. Clearly, there exists an $\overline{\alpha}$-regular finite part τ of f such that for all $u \in D_v$, we have $\tau_u \subseteq \tau$ and τ is $\overline{\alpha(p_u)}$-regular. Then $\tau \Vdash_{\overline{\alpha}} F_e(x)$.

To prove (1) in the reverse direction, assume that $\tau \subseteq f$ and $\tau \Vdash_{\overline{\alpha}} F_e(x)$. Again there exists an element $\langle v, x\rangle$ of W_e such that for all $u \in D_v$, we have $u = \langle p_u, e_u, x_u\rangle$ and $\tau \Vdash_{\overline{\alpha(p_u)}} F_{e_u}(x_u)$. Without loss of generality, we may assume that τ is $\overline{\alpha(p_u)}$-regular for every $u \in D_v$. By induction, $f \models_{\alpha(p_u)} F_{e_u}(x_u)$ for all $u \in D_v$. So $f \models_\alpha F_e(x)$. □

6. Regular extensions

Given a finite mapping τ defined on $[0, q-1]$, we use $\tau * z$ to denote the extension ρ of τ defined on $[0, q]$ and such that $\rho(q) \simeq z$.

Proposition 6.1. Let $\alpha \leqslant \zeta$ and let $\overline{\alpha}$ be an approximation of α. Then:

(1) For every $\overline{\alpha}$-regular finite part τ and every $y \in \mathbb{N}$ there exists a normal $\overline{\alpha}$-regular extension ρ of τ such that $\rho(\mathrm{lh}(\tau)) \simeq y$.
(2) For every $\overline{\delta} \prec \overline{\alpha}$, every $\overline{\delta}$-regular τ of rank 1 and every $y \in \mathbb{N}$ there exists a $\overline{\delta}, \alpha$-regular extension ρ of τ of rank 1 and such that $\rho(\mathrm{lh}(\tau)) \simeq y$.

Proof. We shall prove (1) and (2) together by means of transfinite induction on α. Note that since $B^*_\alpha = \mathbb{N} \oplus B_\alpha$, we have $0 \in B^*_\alpha$.

(a) $\alpha = 0$.
In this case (2) is trivial. To prove (1) suppose that τ is 0-regular and $y \in \mathbb{N}$. Define ρ as follows:

$$\rho(x) \simeq \begin{cases} \tau(x) & \text{if } x < \mathrm{lh}(\tau) \\ y, & \text{if } x = \mathrm{lh}(\tau) \\ 0, & \text{if } x = \mathrm{lh}(\tau) + 1 \\ \text{undefined}, & \text{if } x > \mathrm{lh}(\tau) + 1. \end{cases}$$

(b) Let $\alpha = \beta + 1$ and $\overline{\beta}$ be the β-predecessor of $\overline{\alpha}$.

To prove (1), suppose that we are given an $\overline{\alpha}$-regular τ and $y \in \mathbb{N}$. Let $\mathrm{dom}(\tau) = [0, q-1]$ and $|\tau|_{\overline{\alpha}} = r + 1$. Set $n_{r+1} = q$. By induction, there exists a $\overline{\beta}$-normal extension ρ_0 of $\tau * y$. Set $\ell_{r+1} = \mathrm{lh}(\rho_0)$. Clearly, there exists a normal $\overline{\beta}$-regular extension μ of $\rho_0 * 0$, and hence for every $p \in \mathbb{N}$ the function $\mu^X_{\overline{\beta}}(\rho_0 * 0, p)$ is defined.

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

Now, let

$$\rho_1 = \begin{cases} \mu^X_{\overline{\beta}}(\rho_0 * 0, \langle e, \ell_{r+1}\rangle) & \text{if } y = \langle 0, \beta, e\rangle + 1 \\ \mu^X_{\overline{\beta}}(\rho_0 * 0, p) & \text{if } y = \langle 1, \beta, p\rangle + 1 \\ \mu & \text{otherwise.} \end{cases}$$

Set $m_{r+1} = \mathrm{lh}(\rho_1)$. Let ρ_2 be a normal $\overline{\beta}$-regular extension of ρ_1 and ρ be a normal $\overline{\beta}$-regular extension of $\rho_2 * 0$. Clearly, ρ is a normal $\overline{\alpha}$-regular extension of τ and $\rho(\mathrm{lh}(\tau)) \simeq y$.

To prove (2), let $\overline{\delta} \prec \overline{\alpha}$ and τ be a $\overline{\delta}$-regular finite part of rank 1 and $y \in \mathbb{N}$.
Suppose that $\delta = \beta$. Then $\overline{\beta} = \overline{\delta}$. Note that the β-predecessor of $\overline{\delta}, \alpha$ is $\overline{\beta}$.
Let $n_0 = \mathrm{lh}(\tau)$. Clearly, we can find a normal $\overline{\beta}$-regular extension ρ_0 of $\tau * y$. We then obtain the $\overline{\beta}$-regular extensions ρ_1, ρ_2 and ρ as above. Then ρ is a $\overline{\beta}, \alpha$-regular extension of τ of rank 1.
Suppose that $\delta < \beta$. Then the β-predecessor of $\overline{\delta}, \alpha$ is $\overline{\delta}, \beta$ and $\overline{\delta} \prec \overline{\beta}$. Using the induction hypothesis, we extend $\tau * y$ to a $\overline{\delta}, \beta$-regular finite part ρ_1 of rank 1. We then extend ρ_1 to a $\overline{\delta}, \alpha$-regular finite part ρ of rank 1 in the same way as in the previous case.

(c) Let $\alpha = \lim \alpha(p)$ be a limit ordinal.
Let $\overline{\alpha} = \alpha_1, \ldots, \alpha_n, \alpha$ and $p_0 = \mu p[\alpha_n < \alpha(p)]$. For every p we use $\overline{\alpha(p)}$ to denote the $\alpha(p)$-predecessor of $\overline{\alpha}$.

To prove (1), suppose that τ is an α-regular finite part of rank $r+1$ and $y \in \mathbb{N}$. Then τ is an $\overline{\alpha(p_0 + 3r + 2)}$-regular finite part of rank 1. By induction, there exists an $\overline{\alpha(p_0 + 3r + 2)}, \alpha(p_0 + 3r + 3)$-regular extension ρ_0 of τ of rank 1 and such that $\rho_0(\mathrm{lh}(\tau)) \simeq y$. Set $b_{r+1} = \mathrm{lh}(\rho_0)$. Applying the induction hypothesis again, we obtain an $\overline{\alpha(p_0 + 3r + 2)}, \alpha(p_0 + 3r + 3), \alpha(p_0 + 3r + 4)$-regular extension ρ_1 of $\rho_0 * 0$ of rank 1. Set $m_{r+1} = \mathrm{lh}(\rho_1)$ and then extend ρ_1 to an $\overline{\alpha(p_0 + 3r + 2)}, \alpha(p_0 + 3r + 3), \alpha(p_0 + 3r + 4), \alpha(p_0 + 3r + 5)$-regular finite part of rank 1. Clearly, ρ is an $\overline{\alpha}$-regular finite part of rank $r + 2$.

To prove (2), let $\overline{\delta} \prec \overline{\alpha}$ and τ be a $\overline{\delta}$-regular finite part of rank 1 and $y \in \mathbb{N}$. Let $p_\delta = \mu p[\delta < \alpha(p)]$. By induction, there exists a $\overline{\delta}, \alpha(p_\delta)$-regular extension ρ_1 of τ of rank 1 and such that $\rho_1(\mathrm{lh}(\tau)) \simeq y$. Set $b_0 = \mathrm{lh}(\rho_1)$. Applying the induction hypothesis twice, we get a $\overline{\delta}, \alpha(p_\delta), \alpha(p_\delta + 1), \alpha(p_\delta + 2)$-regular extension ρ of $\rho_1 * 0$ of rank 1. Clearly, ρ is a $\overline{\delta}, \alpha$-regular extension of τ of rank 1. □

We fix a pair of total functions σ and ν on $\mathbb{N}$ such that for every $\alpha \leqslant \zeta$, $\sigma(\alpha) \in B_\alpha$.

Definition 6.2. Let $\alpha \leqslant \zeta$ and $\overline{\alpha}$ be an ordinal approximation of α. A finite part τ is $\overline{\alpha}$ *complete* (with respect to σ, ν) if

$$\overline{\beta} \in \mathrm{Reg}(\tau, \overline{\alpha}) \Rightarrow \sigma(\beta) \in \tau(B^\tau_{\overline{\beta}})$$

and

$$\overline{\beta} \in \mathrm{Reg}(\tau, \overline{\alpha}) \Rightarrow \nu(\beta) \in \tau(N^\tau_{\overline{\beta}}).$$

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

Proposition 6.3. Let $\alpha \leqslant \zeta$ and $\overline{\alpha}$ be an ordinal approximation of α.

(1) For every $\overline{\alpha}$-regular finite part τ there exist a normal $\overline{\alpha}$-regular extension ρ of τ which is $\overline{\alpha}$ complete.
(2) For every $\overline{\delta} \prec \overline{\alpha}$ and every $\overline{\delta}$-regular τ of rank 1 there exists a $\overline{\delta}, \alpha$-regular extension ρ of τ which is of rank 1 and $\overline{\delta}, \alpha$ complete.

Proof. We use transfinite induction on α.

(a) $\alpha = 0$.
Given a 0-regular τ, define ρ as follows:

$$\rho(x) \simeq \begin{cases} \tau(x), & \text{if } x < \mathrm{lh}(\tau) \\ \nu(0), & \text{if } x = \mathrm{lh}(\tau) \\ \sigma(0), & \text{if } x = \mathrm{lh}(\tau) + 1 \\ \text{undefined}, & \text{if } x > \mathrm{lh}(\tau) + 1. \end{cases}$$

(b) $\alpha = \beta + 1$.
To prove (1), suppose we are given an $\overline{\alpha}$-regular τ. Let $\mathrm{dom}(\tau) = [0, q-1]$ and $|\tau|_{\overline{\alpha}} = r + 1$. Set $n_{r+1} = q$. By the previous proposition, there exists a $\overline{\beta}$-normal extension ρ_0 of $\tau * \nu(\alpha)$. Set $\ell_{r+1} = \mathrm{lh}(\rho_0)$. Let μ be a normal $\overline{\beta}$-regular extension of ρ_0. Now, let

$$\rho_1 = \begin{cases} \mu^X_{\overline{\beta}}(\rho_0 * 0, \langle e, \ell_{r+1}\rangle) & \text{if } \nu(\alpha) = \langle 0, \beta, e\rangle + 1 \\ \mu^X_{\overline{\beta}}(\rho_0 * 0, p) & \text{if } \nu(\alpha) = \langle 1, \beta, p\rangle + 1 \\ \mu & \text{otherwise.} \end{cases}$$

Set $m_{r+1} = \mathrm{lh}(\rho_1)$. By induction, there exists a normal $\overline{\beta}$-regular extension ρ_2 of ρ_1 that is $\overline{\beta}$ complete. Set $b_{r+1} = \mathrm{lh}(\rho_2)$ and let ρ be a normal $\overline{\beta}$-regular extension of $\rho_2 * \sigma(\alpha)$. Clearly, ρ is a normal $\overline{\alpha}$-regular extension of τ and ρ is $\overline{\alpha}$ complete.

To prove (2), let $\overline{\delta} \prec \overline{\alpha}$ and τ be a $\overline{\delta}$-regular finite part of rank 1.
Suppose that $\delta = \beta$. Then $\overline{\beta} = \overline{\delta}$ and $\overline{\beta}$ is the β-predecessor of $\overline{\delta}, \alpha$.
Let $n_0 = \mathrm{lh}(\tau)$. Clearly, we can find a normal $\overline{\beta}$-regular extension ρ_0 of $\tau * \nu(\alpha)$. We then obtain consecutively the $\overline{\beta}$-regular extensions ρ_1, ρ_2 and ρ as above. So ρ is a $\overline{\beta}, \alpha$-regular extension of τ of rank 1 that is $\overline{\alpha}$ complete.
Suppose that $\delta < \beta$. Then the β-predecessor of $\overline{\delta}, \alpha$ is $\overline{\delta}, \beta$ and $\overline{\delta} \prec \overline{\beta}$. Using the induction hypothesis, we extend τ to a $\overline{\delta}, \beta$-regular finite part τ_1 of rank 1. We then extend τ_1 to a $\overline{\delta}, \alpha$-regular finite part ρ that is of rank 1 and $\overline{\delta}, \alpha$ complete in the same way as in the previous case.

(c) Let $\alpha = \lim \alpha(p)$ be a limit ordinal.
Let $\overline{\alpha} = \alpha_1, \ldots, \alpha_n, \alpha$ and $p_0 = \mu p[\alpha_n < \alpha(p)]$. For every p we use $\overline{\alpha(p)}$ to denote the $\alpha(p)$-predecessor of $\overline{\alpha}$.

To prove (1), suppose that τ is an α-regular finite part of rank $r+1$. Then τ is an $\overline{\alpha(p_0 + 3r + 2)}$-regular finite part of rank 1. Set $n_{r+1} = \mathrm{lh}(\tau)$. By the previous proposition, there exists an $\overline{\alpha(p_0 + 3r + 2)}, \alpha(p_0 + 3r + 3)$-regular extension ρ_0 of $\tau * \nu(\alpha)$ of rank 1. Set $b_{r+1} = \mathrm{lh}(\rho_0)$. There exists an $\overline{\alpha(p_0 + 3r + 2)}, \alpha(p_0 + 3r + 3), \alpha(p_0 + 3r + 4)$-regular extension ρ_1 of $\rho_0 * \sigma(\alpha)$ of rank 1. Let $m_{r+1} = \mathrm{lh}(\rho_1)$. Finally, by induction,

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

there exists an $\overline{\alpha(p_0+3r+2)}, \alpha(p_0+3r+3), \alpha(p_0+3r+4), \alpha(p_0+3r+5)$-regular extension ρ of ρ_1 that is of rank 1 and $\overline{\alpha(p_0+3r+2)}, \alpha(p_0+3r+3), \alpha(p_0+3r+4), \alpha(p_0+3r+5)$ complete. So we have constructed an $\overline{\alpha(p_0+3r+5)}$ extension ρ of τ of rank 1 that is $\overline{\alpha(p_0+3r+5)}$ complete and such that $\rho(n_{r+1}) \simeq \nu(\alpha)$ and $\rho(b_{r+1}) \simeq \sigma(\alpha)$. Clearly, ρ is $\overline{\alpha}$-regular of rank $r+2$. To see that ρ is $\overline{\alpha}$ complete, consider an element $\overline{\beta}$ of $\mathrm{Reg}(\rho, \alpha)$. Then, by Lemma 4.7,

$$\overline{\beta} = \overline{\alpha} \vee \overline{\beta} \in \mathrm{Reg}(\rho, \overline{\alpha(p_0+3r+5)}).$$

In both cases it follows from the construction of ρ that it satisfies the completeness conditions with respect to $\overline{\beta}$.

To prove (2), let $\overline{\delta} \prec \overline{\alpha}$ and τ be a $\overline{\delta}$-regular finite part of rank 1. Let $p_\delta = \mu p[\delta < \alpha(p)]$. There exists a $\overline{\delta}, \alpha(p_\delta)$-regular extension ρ_1 of τ that is of rank 1 and such that $\rho_1(\mathrm{lh}(\tau)) \simeq \nu(\alpha)$. Set $b_0 = \mathrm{lh}(\rho_1)$. We then get an $\overline{\delta}, \alpha(p_\delta), \alpha(p_\delta+1)$-regular extension ρ_2 of $\rho_1 * \sigma(\alpha)$ that is of rank 1. Finally, we extend ρ_2 to a $\overline{\delta}, \alpha(p_\delta), \alpha(p_\delta+1), \alpha(p_\delta+2)$-regular finite part of rank 1 that is complete. Clearly, ρ is a $\overline{\delta}, \alpha$-regular extension of τ of rank 1 that is $\overline{\alpha}$ complete. □

Proposition 6.4. Let $\alpha \leqslant \zeta$ and let $\overline{\alpha}$ be an ordinal approximation of α.

(1) Suppose that τ is an $\overline{\alpha}$-regular finite part, $\overline{\gamma} \in \mathrm{Reg}(\tau, \overline{\alpha})$ and μ is a normal $\overline{\gamma}$-regular extension of τ. Then there exists a normal $\overline{\alpha}$-regular extension ρ of τ such that $\mu \subseteq \rho$.
(2) Let $\overline{\delta} \prec \overline{\alpha}$ and τ be $\overline{\delta}$-regular of rank 1. Let $\overline{\gamma} \in \mathrm{Reg}(\tau, \overline{\delta})$. Then every normal $\overline{\gamma}$-regular extension μ of τ can be extended to a $\overline{\delta}, \alpha$-regular extension ρ of τ such that $|\rho|_{\overline{\delta},\alpha} = 1$.

Proof. We use transfinite induction on α. (1) and (2) are trivial for $\alpha = 0$.

Suppose that $\alpha = \beta + 1$. Let $\overline{\alpha}$ be an approximation of α and $\overline{\beta}$ be the β-predecessor of $\overline{\alpha}$.

We start with the proof of (1). Consider an $\overline{\alpha}$-regular finite part τ, let $\overline{\gamma} \in \mathrm{Reg}(\tau, \overline{\alpha})$ and μ be a normal $\overline{\gamma}$-regular extension of τ. If $\gamma = \alpha$, then μ is an $\overline{\alpha}$-regular extension of τ. Set $\rho = \mu$. Otherwise, $\overline{\gamma} \in \mathrm{Reg}(\tau, \overline{\beta})$. By induction, μ can be extended to a normal $\overline{\beta}$-regular extension ρ_1 of τ that can be extended to a normal $\overline{\alpha}$-regular extension ρ of τ by means of Proposition 6.1.

The proof of (2) is similar. Let $\overline{\delta} \prec \overline{\alpha}$ and τ be a $\overline{\delta}$-regular finite part of rank 1. Let $\overline{\gamma} \in \mathrm{Reg}(\tau, \overline{\delta})$ and μ be a normal $\overline{\gamma}$-regular extension of τ. Now, we have to consider two cases.

(a) $\delta = \beta$.
In this case $\overline{\delta} = \overline{\beta}$ and $\overline{\beta}$ is the β predecessor of $\overline{\delta}, \alpha$. By induction, we can extend μ to a normal $\overline{\beta}$-regular extension ρ_1 of τ, and by Proposition 6.1, we can extend ρ_1 to a $\overline{\delta}, \alpha$-regular finite part of rank 1.
(b) $\delta < \beta$.
Then $\overline{\delta} \prec \overline{\beta}$ and the β predecessor of $\overline{\delta}, \alpha$ is $\overline{\delta}, \beta$. By induction, there exists a $\overline{\delta}, \beta$-regular extension ρ_1 of τ such that $\mu \subseteq \rho_1$. Then we can extend ρ_1 to a $\overline{\delta}, \alpha$-regular finite part of rank 1 by Proposition 6.1.

Suppose now that $\alpha = \lim \alpha(p)$ is a limit ordinal. Let $\overline{\alpha} = \alpha_1, \ldots, \alpha_n, \alpha$ and $p_0 = \mu p[\alpha(p) > \alpha_n]$.

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

To prove (1), let τ be an $\overline{\alpha}$-regular finite part of rank $r+1$, let $\overline{\gamma} \in \mathrm{Reg}(\tau, \overline{\alpha})$ and let μ be a normal $\overline{\gamma}$-regular extension of τ.

Clearly, τ is an $\overline{\alpha(p_0 + 3r + 2)}$-regular finite part of rank 1. By Lemma 4.7,

$$\overline{\gamma} = \overline{\alpha} \vee \overline{\gamma} \in \mathrm{Reg}(\tau, \overline{\alpha(p_0 + 3r + 2)}).$$

The case $\overline{\gamma} = \overline{\alpha}$ is trivial. Let $\overline{\gamma} \in \mathrm{Reg}(\tau, \overline{\alpha(p_0 + 3r + 2)})$. By induction, we can extend μ to an $\overline{\alpha(p_0 + 3r + 2)}, \alpha(p_0 + 3r + 3)$-regular extension ρ_1 of τ that is of rank 1. We can then extend ρ_1 to a normal $\overline{\alpha}$-regular extension ρ of τ using Proposition 6.1.

To prove (2) in this case, let $\overline{\delta} \prec \overline{\alpha}$ and τ be $\overline{\delta}$-regular finite part of rank 1. Let $\overline{\gamma} \in \mathrm{Reg}(\tau, \overline{\delta})$ and μ be a normal $\overline{\gamma}$-regular extension of τ. Set $p_\delta = \mu p[\alpha(p) > \delta]$.

By induction, we can extend μ to a $\overline{\delta}, \alpha(p_\delta)$-regular ρ_1 of rank 1, and then, using Proposition 6.1, we can extend ρ_1 to a $\overline{\delta}, \alpha$-regular ρ of rank 1. □

Definition 6.5. Let τ be an $\overline{\alpha}$-regular finite part. An extension ρ of τ is *canonical* if $(\forall x \in \mathrm{dom}(\rho))(x > \mathrm{lh}(\tau) \Rightarrow \rho(x) \simeq 0)$.

Definition 6.6. Given an $\overline{\alpha}$-regular finite part, define the $\overline{\alpha}$-characteristic $D^\tau_{\overline{\alpha}}$ to be the set

$$\{\langle \overline{\beta}, |\tau|_{\overline{\beta}} \rangle : \overline{\beta} \in \mathrm{Reg}(\tau, \overline{\alpha})\}.$$

Note that by Lemma 4.7 the set $\mathrm{Reg}(\tau, \overline{\alpha})$ is finite, so the set $D^\tau_{\overline{\alpha}}$ is also finite.

Definition 6.7. Let $\alpha \leqslant \zeta$. A natural number y is α-*nice* if $y = 0$ or $y - 1$ is not of the form $\langle 0, \beta, e\rangle$ or $\langle 1, \beta, p\rangle$ for any $\beta < \alpha$.

Clearly, if y is α-nice, it is β-nice for every $\beta < \alpha$.

Proposition 6.8. There exist recursive functions ext, ext_ℓ, chs and chs_ℓ such that for every approximation $\overline{\alpha}$ of an ordinal $\alpha \leqslant \zeta$ the following assertions hold:

(1) For every $\overline{\alpha}$-regular finite part τ and every α-nice y, we have $\mathrm{ext}(\overline{\alpha}, \tau, D^\tau_{\overline{\alpha}}, y)$ is a canonical $\overline{\alpha}$-regular extension of $\tau * y$ with $\overline{\alpha}$-characteristic $\mathrm{chs}(\overline{\alpha}, \tau, D^\tau_{\overline{\alpha}}, y)$ and $\overline{\alpha}$-rank equal to $|\tau_\alpha| + 1$.

(2) For every $\overline{\delta} \prec \overline{\alpha}$, every $\overline{\delta}$-regular finite part τ of rank 1 and every α-nice y,

$$\mathrm{ext}_\ell((\overline{\delta}, \alpha), \tau, D^\tau_{\overline{\delta}}, y)$$

is a $\overline{\delta}, \alpha$-regular and canonical extension of $\tau * y$ of rank 1 with characteristic

$$\mathrm{chs}_\ell((\overline{\delta}, \alpha), \tau, D^\tau_{\overline{\delta}}, y).$$

Proof. We shall construct the functions ext, chs, ext_ℓ and chs_ℓ simultaneously by means of effective transfinite recursion on α.

Let $\alpha = 0$. Then $\overline{\alpha} = 0$. Let τ be a 0-regular finite part. Using D^τ_0, we can determine the 0-rank $r + 1$ of τ. Let

$$\rho(x) \simeq \begin{cases} \tau(x), & \text{if } x < \mathrm{lh}(\tau) \\ y, & \text{if } x = \mathrm{lh}(\tau) \\ 0, & \text{if } x = \mathrm{lh}(\tau) + 1 \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

Set $\mathrm{ext}(\overline{\alpha},\tau,D^{\tau}_{\overline{\alpha}},y)=\rho$ and $\mathrm{chs}(\overline{\alpha},\tau,D^{\tau}_{\overline{\alpha}},y)=\{\langle 0,r+2\rangle\}$. Let $\alpha=\beta+1$. Given an approximation $\overline{\alpha}$ of α, first find the β predecessor $\overline{\beta}$ of $\overline{\alpha}$. Let τ be an $\overline{\alpha}$-regular finite part and y be α-nice. Using the $\overline{\alpha}$-characteristic $D^{\tau}_{\overline{\alpha}}$ of τ, we can find the $\overline{\beta}$-characteristic $D^{\tau}_{\overline{\beta}}$ of τ and the $\overline{\alpha}$-rank $r+1$ of τ. Now let

$$\rho_0=\mathrm{ext}(\overline{\beta},\tau,D^{\tau}_{\overline{\beta}},y),\rho_1=\mathrm{ext}(\overline{\beta},\rho,D^{\rho_0}_{\overline{\beta}},0),\ldots,\rho_3=\mathrm{ext}(\overline{\beta},\rho_2,D^{\rho_2}_{\overline{\beta}},0).$$

Set $\mathrm{ext}(\overline{\alpha},\tau,D^{\tau}_{\overline{\alpha}},y)=\rho_3$ and $\mathrm{chs}(\overline{\alpha},\tau,D^{\tau}_{\overline{\alpha}},y)=D^{\rho_3}_{\overline{\beta}}\cup\{\langle\overline{\alpha},r+2\rangle\}$.

Suppose now that $\overline{\delta}\prec\overline{\alpha}$ and let τ be a $\overline{\delta}$-regular finite part of rank 1 and let y be α-nice.

(a) $\overline{\delta}=\overline{\beta}$.

Set $\rho_0=\mathrm{ext}(\overline{\beta},\tau,D^{\tau}_{\overline{\beta}},y),\ldots,\rho_3=\mathrm{ext}(\overline{\beta},\rho_2,D^{\rho_2}_{\overline{\beta}},0)$.

Let $\mathrm{ext}_{\ell}((\overline{\delta},\alpha),\tau,D^{\tau}_{\overline{\delta}},y)=\rho_3$ and $\mathrm{chs}_{\ell}((\overline{\delta},\alpha),\tau,D^{\tau}_{\overline{\delta}},y)=D^{\rho_3}_{\overline{\beta}}\cup\{\langle(\overline{\delta},\alpha),1\rangle\}$.

(b) $\overline{\delta}\prec\overline{\beta}$.

Set $\tau'=\mathrm{ext}_{\ell}((\overline{\delta},\beta),\tau,D^{\tau}_{\overline{\delta}},y)$,

$$\rho_0=\mathrm{ext}((\overline{\delta},\beta),\tau',D^{\tau'}_{\overline{\delta},\beta},0),\ldots,\rho_3=\mathrm{ext}((\overline{\delta},\beta),\rho_2,D^{\rho_2}_{\overline{\delta},\beta},0).$$

Let $\mathrm{ext}_{\ell}((\overline{\delta},\alpha),\tau,D^{\tau}_{\overline{\delta}})=\rho_3$ and $\mathrm{chs}_{\ell}((\overline{\delta},\alpha),\tau,D^{\tau}_{\overline{\delta}})=D^{\rho_3}_{\overline{\delta},\beta}\cup\{\langle(\overline{\delta},\alpha),1\rangle\}$.

Let $\alpha=\lim\alpha(p)$ be a limit ordinal. Consider an approximation $\overline{\alpha}=\alpha_1,\ldots,\alpha_n,\alpha$ of α. Let $p_0=\mu p[\alpha(p)>\alpha_n]$. For every $p\geqslant p_0$ let $\overline{\alpha(p)}$ be the $\alpha(p)$-predecessor of $\overline{\alpha}$.

Let τ be an $\overline{\alpha}$-regular finite part with characteristic $D^{\tau}_{\overline{\alpha}}$ and y be α-nice. Let $|\tau|_{\overline{\alpha}}=r+1$. Then τ is an $\overline{\alpha(p_0+3r+2)}$-regular finite part of rank 1. By Lemma 4.7,

$$D=D^{\tau}_{\overline{\alpha(p_0+3r+2)}}=\{\langle\overline{\beta},k\rangle:\overline{\beta}\preceq\overline{\alpha(p_0+3r+2)}\ \&\ \langle\overline{\beta},k\rangle\in D^{\tau}_{\overline{\alpha}}\}.$$

Set

$$\begin{aligned}\rho_0&=\mathrm{ext}_{\ell}((\overline{\alpha(p_0+3r+2)},\alpha(p_0+3r+3)),\tau,D,y),\ldots\\ \rho_2&=\mathrm{ext}_{\ell}((\overline{\alpha(p_0+3r+4),0},\alpha(p_0+3r+5)),\rho_1,D^{\rho_1}_{\overline{\alpha(p_0+3r+4)}},0).\end{aligned}$$

Let $\mathrm{ext}(\overline{\alpha},\tau,D^{\tau}_{\overline{\alpha}},y)=\rho_2$ and $\mathrm{chs}(\overline{\alpha},\tau,D^{\tau}_{\overline{\alpha}},y)=D^{\rho_2}_{\overline{\alpha(p_0+3r+5)}}\cup\{\langle\overline{\alpha},r+2\rangle\}$.

Now suppose that $\overline{\delta}\prec\overline{\alpha}$ and let τ be a $\overline{\delta}$-regular finite part of rank 1. Set $l=\mu p[\alpha(p)>\delta]$. Let

$$\begin{aligned}\rho_0&=\mathrm{ext}_{\ell}((\overline{\delta},\alpha(l)),\tau,D^{\tau}_{\overline{\delta}},y)\\ \rho_1&=\mathrm{ext}_{\ell}((\overline{\delta},\alpha(l),\alpha(l+1)),\rho_0,D^{\rho_0}_{\overline{\delta},\alpha(l)},0)\\ \rho_2&=\mathrm{ext}_{\ell}((\overline{\delta},\alpha(l),\alpha(l+1),\alpha(l+2)),\rho_1,D^{\rho_1}_{\overline{\delta},\alpha(l),\alpha(l+1)}0).\end{aligned}$$

Let $\mathrm{ext}_{\ell}((\overline{\delta},\alpha),\tau,D^{\tau}_{\overline{\delta}},y)=\rho_2$ and

$$\mathrm{chs}_{\ell}((\overline{\delta},\alpha),\tau,D^{\tau}_{\overline{\delta}},y)=D^{\rho_2}_{\overline{\delta},\alpha(l),\alpha(l+1),\alpha(l+2)}\cup\{\langle(\overline{\delta},\alpha),1\rangle\}.$$ □

Corollary 6.9. There exists an effective way to construct for every ordinal approximation $\overline{\alpha}$ and every $r\geqslant 0$ an $\overline{\alpha}$-regular finite part $\tau_{\overline{\alpha}}$ of rank $r+1$ and such that $(\forall x\in\mathrm{dom}(\tau_{\overline{\alpha}}))(\tau_{\overline{\alpha}}(x)\simeq 0)$.

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

7. The proofs

In this section we present the proofs of Theorems 2.9 and 2.14.

Proof of Theorem 2.9. Let two sequences $\mathscr{A} = \{A_\alpha\}_{\alpha\leqslant\zeta}$ and $\mathscr{B} = \{B_\alpha\}_{\alpha\leqslant\zeta}$ of sets of natural numbers be given. Suppose that $\mathscr{A} \not\leqslant_{\mathrm{u}} \mathscr{B}$, that is, there is no recursive function g such that

$$(\forall\alpha)(A_\alpha = \Phi_{g(\alpha)}(\mathscr{P}_\alpha(\mathscr{B}))).$$

To prove the theorem it is sufficient to construct a total set X such that:

(1) for some recursive h, $(\forall\alpha\leqslant\zeta)(B_\alpha = \Phi_{h(\alpha)}(X^{(\alpha)})$; and

(2) there is no recursive function g such that $(\forall\alpha)(A_\alpha = \Phi_{g(\alpha)}(X^{(\alpha)}))$.

If for some $\alpha\leqslant\zeta$ we have $A_\alpha \not\leqslant_{\mathrm{e}} \mathscr{P}_\alpha(\mathscr{B})$, we can get the desired X by means of Theorem 2.5.

Assume that $(\forall\alpha)(A_\alpha \leqslant_{\mathrm{e}} \mathscr{P}_\alpha(\mathscr{B}))$.

We shall construct X as a graph of a regular enumeration f. In this way, by Theorem 5.3, we will automatically ensure that X satisfies (1).

Let $\overline{\zeta}$ be an ordinal approximation of ζ. For every $\alpha\leqslant\zeta$, set $B^*_\alpha = \mathbb{N}\oplus B_\alpha$ and let $\mathscr{P}_\alpha$ denote the set $\mathscr{P}_\alpha(\mathscr{B})$. Fix two total functions $\sigma(\alpha,s)$ and $\nu(\alpha,s)$ such that:

(i) $(\forall\alpha\leqslant\zeta)(\lambda s.\sigma(\alpha,s)$ enumerates the set $B^*_\alpha)$.

(ii) $(\forall\alpha\leqslant\zeta)(\lambda s.\nu(\alpha,s)$ enumerates the set $\mathbb{N})$.

For every $s\in\mathbb{N}$, set $\sigma_s = \lambda\alpha.\sigma(\alpha,(s)_0)$ and $\nu_s = \lambda\alpha.\nu(\alpha,(s)_0)$.

The construction of f will be carried out in steps. At each step s we shall construct a $\overline{\zeta}$-regular finite part τ_s so that $\tau_s\subseteq\tau_{s+1}$ and $|\tau_s|_{\overline{\zeta}} < |\tau_{s+1}|_{\overline{\zeta}}$. We shall ensure that for every s the finite part τ_{s+1} is complete with respect to σ_s,τ_s. We shall then define $f = \bigcup\tau_s$. The f obtained in this way is a regular enumeration. Indeed, since f contains finite parts of arbitrary large rank, for every $\rho\subseteq f$ there exists a $\overline{\zeta}$-regular finite part τ_s of f such that $\rho\subseteq\tau_s$.

Let $\alpha\leqslant\zeta$ and $z\in B^*_\alpha$. Let s be so large that $\overline{\alpha}\in \mathrm{Reg}(\tau_{s+1},\overline{\zeta},)$ and such that $\sigma(\alpha,(s)_0)\simeq z$. Therefore, $\sigma_s(\alpha)\simeq z$, so, since τ_{s+1} is complete with respect to σ_s,ν_s, we have $z\in\tau_{s+1}(B^{\tau_{s+1}}_{\overline{\alpha}})$.

Let $p\in\mathbb{N}$. Consider an $s_1 > s$ such that $\nu(\alpha,(s_1)_0)\simeq p$. So $\nu_{s_1}(\alpha)\simeq p$, and hence, by the completeness of τ_{s_1+1} with respect to σ_{s_1},ν_{s_1}, we have $p\in\tau_{s_1+1}(N^{\tau_{s_1+1}}_{\overline{\alpha}})$.

Fix an enumeration $g_0,g_1,\ldots,g_s,\ldots$ of all total recursive functions.

Let τ_0 be a $\overline{\zeta}$-regular finite part of rank 1. Suppose that τ_s is defined and let $D = D^{\tau_s}_\zeta$ be the characteristic of τ_s.

Fix an $\overline{\alpha}\prec\overline{\zeta}$ and let $\overline{\alpha+1}$ be the $\alpha+1$-predecessor of $\overline{\zeta}$. Construct an $\overline{\alpha}$-regular extension δ_α of τ_s as follows. First let $k = k(\overline{\zeta},\overline{\alpha+1})$ be the number such that every $\overline{\zeta}$-regular finite part of rank greater than k is also $\overline{\alpha+1}$-regular. Using the functions ext and chs from Proposition 6.8, find a canonical $\overline{\zeta}$-regular extension δ_0 of τ_s that is of $\overline{\zeta}$-rank equal to $\max(|\tau_s|_\zeta,k)$. Clearly, δ_0 is $\overline{\alpha+1}$-regular, so it is also $\overline{\alpha}$-regular. Let $g_s(\alpha) = e_\alpha$. Using Proposition 6.8 again, we define recursively a canonical $\overline{\alpha}$-regular extension δ_1 of $\delta_0 * (\langle 0,\alpha,e_\alpha\rangle + 1)$ such that $|\delta_1|_{\overline{\alpha}} = |\delta_0|_{\overline{\alpha}} + 1$. Set $\delta_\alpha = \delta_1$. Let $\ell_\alpha = \mathrm{lh}(\delta_\alpha)$ and set

$$C_\alpha = \{x : (\exists\rho\supseteq\delta_\alpha)(\rho \text{ is } \overline{\alpha}\text{-regular, } \rho(\ell_\alpha)\simeq x \text{ and } \rho\Vdash_{\overline{\alpha}} F_{e_\alpha}(\ell_\alpha))\}.$$

Set $C_\zeta = \varnothing$.

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

From the construction of δ_α and Proposition 4.11, it follows that there exist a recursive function h such that $(\forall\alpha \leqslant \zeta)(C_\alpha = \Phi_{h(\alpha)}(\mathscr{P}_\alpha))$. Hence there exists an α such that $A_\alpha \neq C_\alpha$. Actually, there exist infinitely many such α, otherwise, since $(\forall\alpha \leqslant \zeta)(A_\alpha \leqslant_e \mathscr{P}_\alpha)$, we could redefine the function h and get $\mathscr{A} \leqslant_u \mathscr{B}$. So we may assume that there exists an $\alpha < \zeta$ such that $A_\alpha \neq C_\alpha$. Let α_s be the least $\alpha \leqslant \zeta$ such that $A_\alpha \neq C_\alpha$.

Now let x_s be the least natural number such that

$$(x_s \in A_{\alpha_s} \ \&\ x_s \notin C_{\alpha_s}) \vee (x_s \notin A_{\alpha_s} \ \&\ x_s \in C_{\alpha_s}).$$

Set $\delta_2 = \mu_{\overline{\alpha_s}}(\delta_{\alpha_s} * x_s, X^{\overline{\alpha_s}}_{\langle e_{\alpha_s}, \ell_{\alpha_s}\rangle})$. Then let δ_3 be a normal $\overline{\alpha_s}$-regular extension of δ_2, and δ_4 be a normal $\overline{\alpha_s}$-regular extension of $\delta_3 * 0$. Clearly, δ_4 is a normal $\overline{\alpha_s + 1}$-regular extension of δ_0. By Proposition 6.4, there exists a normal $\overline{\zeta}$-regular extension τ of δ_0 such that $\delta_4 \subseteq \tau$. Let τ_{s+1} be a normal $\overline{\zeta}$-regular extension of τ that is complete with respect to σ_s, ν_s.

To show a contradiction, assume that there exists a recursive function g such that $(\forall\alpha \leqslant \zeta)(A_\alpha = \Phi_{g(\alpha)}(f^{(\alpha)}))$. Then there exists a recursive function h such that $(\forall\alpha \leqslant \zeta)(f^{-1}(A_\alpha) = \Phi_{h(\alpha)}(f^{(\alpha)}))$, so there exists a recursive function g_s such that

$$(\forall\alpha \leqslant \zeta)(f^{-1}(A_\alpha) = \{n : f \models_\alpha F_{g_s(\alpha)}(n)\}).$$

Consider the step s of the construction. Clearly, $f(\ell_{\alpha_s}) \simeq x_s$ and $g_s(\alpha_s) = e_{\alpha_s}$. Hence

$$x_s \in A_{\alpha_s} \iff f \models F_{e_{\alpha_s}}(\ell_{\alpha_s}).$$

Now, assume that $x_s \in A_{\alpha_s}$. Then $f \models F_{e_{\alpha_s}}(\ell_{\alpha_s})$. Clearly, $\delta_{\alpha_s} \subseteq f$. Hence, by Lemma 5.8, there exists an $\overline{\alpha_s}$-regular $\rho \supseteq \delta_{\alpha_s}$ such that

$$\rho(\ell_{\alpha_s}) \simeq x_s \ \&\ \rho \Vdash_{\overline{\alpha_s}} F_{e_{\alpha_s}}(\ell_{\alpha_s}).$$

Therefore $x_s \in C_{\alpha_s}$, which gives a contradiction.

So, $x_s \notin A_{\alpha_s}$. Thus $x_s \in C_{\alpha_s}$, and hence there exists an $\overline{\alpha}$-regular extension ρ of δ_{α_s} such that $\rho(\ell_{\alpha_s}) \simeq x_s$ and $\rho \Vdash_{\overline{\alpha_s}} F_{e_{\alpha_s}}(\ell_{\alpha_s})$. From here, by the construction, it follows that $\delta_2 \Vdash_{\overline{\alpha_s}} F_{e_{\alpha_s}}(\ell_{\alpha_s})$, and hence $f \models F_{e_{\alpha_s}}(\ell_{\alpha_s})$, which gives a contradiction.

So we have proved that there is no recursive function g such that $(\forall\alpha \leqslant \zeta)(A_\alpha = \Phi_{g(\alpha)}(f^{(\alpha)}))$, which concludes the proof Theorem 2.9. □

Proof of Theorem 2.14. Let ζ be a recursive ordinal and $\Gamma : \mathscr{S}_\zeta \to \mathscr{S}_\zeta$ be a uniform operator. Suppose that g is a function on the natural numbers such that given a recursive function φ with index a, $g(a)$ is an index of a recursive function ψ such that for all $X \subseteq \mathbb{N}$,

$$\Gamma(\{W^{X^{(\alpha)}}_{\varphi(\alpha)}\}_{\alpha\leqslant\zeta}) = \{W^{X^{(\alpha)}}_{\psi(\alpha)}\}_{\alpha\leqslant\zeta}.$$

Given $\mathscr{B} \in \mathscr{S}_\zeta$ and $\alpha \leqslant \zeta$, we shall use $\Gamma(\mathscr{B})_\alpha$ to denote the α-th member of the sequence $\Gamma(\mathscr{B})$.

Our goal is to show that there exists a recursive function h such that for every $\mathscr{B} \in \mathscr{S}_\zeta$ and every $\alpha \leqslant \zeta$,

$$\Gamma(\mathscr{B})_\alpha = \Phi_{h(\alpha)}(\mathscr{P}_\alpha(\mathscr{B})).$$

Fix an ordinal approximation $\overline{\zeta + 1}$ of $\zeta + 1$. For every $\alpha \leqslant \zeta$ let $\overline{\alpha}$ be the α-predecessor of $\overline{\zeta + 1}$.

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

From now on, given a sequence $\mathscr{B} \in \mathscr{S}_\zeta$, we shall call f a *regular enumeration with respect to* $\mathscr{B}$ if it is regular with respect to $\overline{\zeta+1}$ and the sequence $\mathscr{B}$ extended to a sequence of length $\zeta+1$ by setting $B_{\zeta+1} = \varnothing$.

As remarked after the proof of Proposition 5.3, there exists a recursive function φ such that for every sequence $\mathscr{B} = \{B_\alpha\}_{\alpha \leqslant \zeta}$ and every regular with respect to $\mathscr{B}$ enumeration f, we have

$$(\forall \alpha \leqslant \zeta)(B_\alpha = W^{f^{(\alpha)}}_{\varphi(\alpha)}).$$

Let a be an index of φ and $g(a)$ be the index of the recursive function ψ. Then for every sequence $\mathscr{B} \in \mathscr{S}_\zeta$, every $\alpha \leqslant \zeta$ and every regular with respect to $\mathscr{B}$ enumeration f, we have

$$\Gamma(\mathscr{B})_\alpha = W^{f^{(\alpha)}}_{\psi(\alpha)}.$$

Then there exists a recursive function χ such that for every regular sequence $\mathscr{B}$ and every regular with respect to $\mathscr{B}$ enumeration f, we have

$$(\forall \alpha \leqslant \zeta)(f^{-1}(\Gamma(\mathscr{B})_\alpha) = \{n : f \models_\alpha F_{\chi(\alpha)}(n)\}).$$

Given an $\overline{\alpha} \preceq \overline{\zeta}$, define the $\overline{\alpha}$-regular finite part δ_α as follows. Set $k = k(\overline{\zeta+1}, \overline{\alpha+1})$. Let δ_0 be a $\overline{\zeta+1}$-regular finite part of rank k such that $(\forall x \in \mathrm{dom}(\delta_0))(\delta_0(x) \simeq 0)$. Clearly, δ_0 is $\overline{\alpha+1}$-regular and hence it is also $\overline{\alpha}$-regular. Let $\chi(\alpha) = e_\alpha$. Note that $\langle 0, \alpha, e_\alpha\rangle + 1$ is α-nice. Let δ_1 be a canonical $\overline{\alpha}$-regular extension of $\delta_0 * (\langle 0, \alpha, e_\alpha\rangle + 1)$ of $\overline{\alpha}$-rank equal to $|\delta_0|_\alpha + 1$. Set $\delta_\alpha = \delta_1$. By Proposition 6.8, there exists a recursive function yielding for every $\alpha \leqslant \zeta$ (the code of) δ_α.

Let $\ell_\alpha = \mathrm{lh}(\delta_\alpha)$. We shall show that for every sequence $\mathscr{B} \in \mathscr{S}_\zeta$,

$$\Gamma(\mathscr{B})_\alpha = \{x : (\exists \rho \supseteq \delta_\alpha)(\rho \text{ is } \overline{\alpha}\text{-regular with respect to } \mathscr{B}, \ \rho(\ell_\alpha) \simeq x \text{ and } \rho \Vdash_{\overline{\alpha}} F_{e_\alpha}(\ell_\alpha))\}. \tag{3}$$

Indeed, fix a sequence $\mathscr{B} \in \mathscr{S}_\zeta$ and let $\alpha \leqslant \zeta$. We use C_α to denote the set

$$\{x : (\exists \rho \supseteq \delta_\alpha)(\rho \text{ is } \overline{\alpha}\text{-regular with respect to } \mathscr{B}, \ \rho(\ell_\alpha) \simeq x \text{ and } \rho \Vdash_{\overline{\alpha}} F_{e_\alpha}(\ell_\alpha))\}.$$

Consider a $x \in \Gamma(\mathscr{B})_\alpha$. Clearly, there exists an $\overline{\alpha+1}$-regular extension δ of $\delta_\alpha * x$ that can be extended to a $\overline{\zeta+1}$-regular finite part τ by Proposition 6.4. We can then extend τ to a regular with respect to $\mathscr{B}$ enumeration f as in the previous proof. Since $f(\ell_\alpha) \simeq x$, we have $f \models_\alpha F_{\chi(\alpha)}(\ell_\alpha)$, that is, $f \models_\alpha F_{e_\alpha}(\ell_\alpha)$. By Lemma 5.8, there exists an $\overline{\alpha}$-regular $\rho \subseteq f$ such that $\rho \Vdash_{\overline{\alpha}} F_{e_\alpha}(\ell_\alpha)$. We may assume that $\delta_\alpha \subseteq \rho$ and $\rho(\ell_\alpha) \simeq x$. Thus $x \in C_\alpha$.

Assume that $x \in C_\alpha$. Let $\delta_2 = \mu_{\overline{\alpha}}(\delta_\alpha * x, X^{\overline{\alpha}}_{\langle e_\alpha, \ell_\alpha\rangle})$. Clearly, $\delta_2 \Vdash_{\overline{\alpha}} F_{e_\alpha}(\ell_\alpha)$. Let δ_3 be a normal $\overline{\alpha}$-regular extension of δ_2, and δ_4 be a normal $\overline{\alpha}$-regular extension of $\delta_2 * 0$. Clearly, δ_4 is an $\overline{\alpha+1}$-regular extension of δ_0. Then there exists a $\overline{\zeta+1}$-regular extension τ of δ_0 such that $\delta_4 \subseteq \tau$. Finally, let f be a regular with respect to $\mathscr{B}$ enumeration such that $\tau \subseteq f$. By Lemma 5.8, $f \models_\alpha F_{e_\alpha}(\ell_\alpha)$, and hence $x = f(\ell_\alpha) \in \Gamma(B)_\alpha$.

From (1), by Proposition 4.11, it follows that there exists a recursive function h such that for all $\mathscr{B} \in \mathscr{S}_\zeta$,

$$\Gamma(\mathscr{B})_\alpha = \Phi_{h(\alpha)}(P_\alpha(\mathscr{B})).$$

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press

It is just a routine computation to show the uniformity of every operator $\Gamma : \mathscr{S}_\zeta \to \mathscr{S}_\zeta$ for which there exists a recursive function h such that for every sequence $\mathscr{B} \in \mathscr{S}_\zeta$, $\Gamma(\mathscr{B}) = \{\Phi_{h(\alpha)}(\mathscr{P}_\alpha(\mathscr{B}))\}_{\alpha \leqslant \zeta}$. □

Acknowledgments

We would like to thank the anonymous referee for suggestions that improved the exposition of the paper.

References

Ash, C. J. (1992) Generalizations of enumeration reducibility using recursive infinitary propositional sentences. *Ann. Pure Appl. Logic* **58** 173–184.

Case, J. (1974) Maximal arithmetical reducibilities. *Z. Math. Logik Grundlag. Math.* **20** 261–270.

Cooper, S. B. (1984) Partial degrees and the density problem. Part 2: The enumeration degrees of the Σ_2 sets are dense. *J. Symbolic Logic* **49** 503–513.

Cooper, S. B. (1990) Enumeration reducibilty, nondeterministic computations and relative computability of partial functions. In: Sacks, G. E., Ambos-Spies, K. and Muler, G. (eds.) Recursion theory week, Oberwolfach 1989. *Springer-Verlag Lecture Notes in Mathematics* **1432** 57–110.

McEvoy, K. (1985) Jumps of quasi-minimal enumeration degrees. *J. Symbolic Logic* **50** 839–848.

Rogers, H. Jr. (1967) *Theory of recursive functions and effective computability*, McGraw-Hill Book Company.

Sacks, G. E. (1990) *Higher recursion theory*, Springer-Verlag.

Selman, A. L. (1971) Arithmetical reducibilities (I). *Z. Math. Logik Grundlag. Math.* **17** 335–350.

Soskov, I. N. (2005) Uniform operators. In: Cooper, S. B., Loewe, B. and Torenvliet, L. (eds.) New computational paradigms *Springer-Verlag Lecture Notes in Computer Science* **3526** 441–450.

Soskov, I. N. and Baleva, V. (2002) Regular enumerations. *J. Symbolic Logic* **67** 1323–1343.

https://doi.org/10.1017/S0960129506005639 Published online by Cambridge University Press