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A generalized inverse for arbitrary operators between Hilbert spaces†

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Abstract. A function analytic approach to the generalized inversion problem, for arbitrary operators between Hilbert spaces, is investigated in the present paper.

The generalized inverse is defined as the inverse of the largest invertible restriction of the given operator and it is extended by zero to the orthogonal complement of the range of the invertible restriction. This domain-dense operator satisfies, on some restricted manifolds, the defining properties of the generalized inverse in some special cases, and it provides the least extremal solution of possibly inconsistent linear equations.

If one applies consecutively twice the rule of the generalized inversion to the given T, one obtains an operator T^{++} with a closed kernel and which agrees with the original T on a dense decomposable domain. Some further properties of the generalized inverse, its adjoint, and the generalized inverse of the adjoint of a preclosed operator are developed to the extent of unbounded and not necessarily closed operators.

Notations

$\mathfrak{H}$, $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}'$, $\mathfrak{H}''$ represent Hilbert spaces over a given scalar field.

For vectors x, y in a Hilbert space: (x, y) denotes the inner product; $\langle x, y\rangle$ indicates an ordered pair; and $\|x\|$ stands for the vector norm.

$\mathfrak{H}_1 \times \mathfrak{H}_2$ designates the Cartesian product of $\mathfrak{H}_1$ and $\mathfrak{H}_2$, i.e.

$$\mathfrak{H}_1 \times \mathfrak{H}_2 = \{\langle x_1, x_2\rangle : x_1 \in \mathfrak{H}_1, x_2 \in \mathfrak{H}_2\},$$

with the inner product defined by $(\langle x_1, x_2\rangle, \langle y_1, y_2\rangle) = (x_1, y_1) + (y_2, y_2)$.

For linear manifolds $\mathfrak{M}, \mathfrak{N} \subset \mathfrak{H}$, we write: $\overline{\mathfrak{M}}$ for the closure of $\mathfrak{M}$; $\mathfrak{M}^{\perp}$ for the orthogonal complement of $\mathfrak{M}$ in $\mathfrak{H}$; $\mathscr{P}_{\overline{\mathfrak{M}}}$ for the orthogonal projector on $\overline{\mathfrak{M}}$;

$$\mathfrak{M} + \mathfrak{N} = \{x + y : x \in \mathfrak{M} \quad \text{and} \quad y \in \mathfrak{N}\}; \quad \mathfrak{M} \oplus \mathfrak{N} = \mathfrak{M} + \mathfrak{N} \quad \text{with} \quad \mathfrak{M} \cap \mathfrak{N} = \{0\};$$

$$\mathfrak{M} \setminus \mathfrak{N} = \{x : x \in \mathfrak{M} \quad \text{and} \quad x \notin \mathfrak{N}\} \quad \text{with} \quad \mathfrak{M} \supset \mathfrak{N}.$$

For an operator T: $\mathfrak{H}_1 \to \mathfrak{H}_2$, by which we mean any linear mapping from $\mathfrak{H}_1$ into $\mathfrak{H}_2$ with domain in $\mathfrak{H}_1$ and range in $\mathfrak{H}_2$, we use the following notations: $\mathscr{D}(T)$ for the domain;

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$\mathscr{R}(T)$ for the range; $\operatorname{Ker} T$ for the kernel (null manifold); $\mathscr{C}(T)$ for the carrier (5), i.e. $\mathscr{C}(T) = \mathscr{D}(T) \cap (\operatorname{Ker} T)^{\perp}$; T^* for the adjoint; T^+ for the generalized inverse; $T|\mathfrak{M}$ for the restriction of T to $\mathfrak{M} \subset \mathscr{D}(T)$; $T_0 = T|\mathscr{D}_0$, where $\mathscr{D}_0 = \operatorname{Ker} T \oplus \mathscr{C}(T)$; $T[\mathfrak{M}]$ for the image of $\mathfrak{M}$ under T, $\mathfrak{M} \subset \mathscr{D}(T)$; $\Gamma(T)$ for the graph; and $\Gamma^{-1}(T)$ for the inverse graph.

1. *Preliminaries.* Let $T\colon \mathfrak{H}_1 \to \mathfrak{H}_2$ be a densely defined operator,† and let

$$\mathfrak{H} = \mathfrak{H}_1 \times \mathfrak{H}_2.$$

Then $\mathfrak{H}_1$ and $\mathfrak{H}_2$ may be identified, under isometric isomorphisms J_1 and J_2, with

$$\mathfrak{H}' = \mathfrak{H}_1 \times \{0\} = \{\langle x, 0\rangle : x \in \mathfrak{H}_1\}, \quad \text{and} \quad \mathfrak{H}'' = \{0\} \times \mathfrak{H}_2 = \{\langle 0, y\rangle : y \in \mathfrak{H}_2\},$$

respectively.

The graph $\Gamma(T)$ of T is the linear manifold of $\mathfrak{H}$ defined by

$$\Gamma(T) = \{\langle x, Tx\rangle : x \in \mathscr{D}(T)\}. \tag{1·1}$$

Conversely, a linear manifold $\mathfrak{T}$ of the product space $\mathfrak{H}$ is the graph of an operator $T\colon \mathfrak{H}_1 \to \mathfrak{H}_2$ iff $\mathfrak{T} \cap \mathfrak{H}'' = \langle 0, 0\rangle$.

For a graph (1·1) the domain, the range, and the kernel of T are given by:

$$\mathscr{D}(T) = J_1^{-1} \mathscr{P}_{\mathfrak{H}'} \Gamma(T), \tag{1·2}$$

$$\mathscr{R}(T) = J_2^{-1} \mathscr{P}_{\mathfrak{H}''} \Gamma(T), \tag{1·3}$$

$$\operatorname{Ker} T = J_1^{-1} \Gamma(T) \cap \mathfrak{H}', \tag{1·4}$$

respectively (8).

The inverse graph of an operator $R\colon \mathfrak{H}_2 \to \mathfrak{H}_1$ is obtained by the homeomorphic mapping: $\langle y, x\rangle \to \langle x, y\rangle$ of $\mathfrak{H}_2 \times \mathfrak{H}_1$ onto $\mathfrak{H}_1 \times \mathfrak{H}_2$,

$$\Gamma^{-1}(R) = \{\langle Ry, y\rangle \in \mathfrak{H} : y \in \mathscr{D}(R)\}.$$

For an invertible operator T, $\Gamma^{-1}(T^{-1}) = \Gamma(T)$.

The annihilator of a linear manifold

$$\mathfrak{T} = \{\langle x, y\rangle\}$$

is defined by the orthocomplementation of $\mathfrak{T}$ in $\mathfrak{H}$.

$$\mathfrak{T}^{\perp} = \{\langle u, v\rangle \in \mathfrak{H} : (u, x) + (v, y) = 0, \langle x, y\rangle \in \mathfrak{T}\}.$$

The adjoint T^*, defined by $\quad \Gamma^{-1}(-T^*) = \Gamma(T)^{\perp}$,
is a closed operator satisfying

$$(y, Tx) = (T^*y, x), \quad \text{for all} \quad x \in \mathscr{D}(T), y \in \mathscr{D}(T^*),$$

with

$$\mathscr{D}(T^*) = \operatorname{Ker} T^* \oplus \mathscr{C}(T^*), \tag{1·5}$$

and

$$\operatorname{Ker} T^* = \mathscr{R}(T)^{\perp}.$$

† There is not much loss of generality in assuming that T is densely defined on its original space $\mathfrak{H}_1$. In fact, if T is not domain-dense, it could be regarded as an operator from

$$\mathscr{K}_1 = \overline{\mathscr{D}(T)} \subset \mathfrak{H}_1$$

into $\mathfrak{H}_2$. We shall henceforth consider T with dense domain in $\mathfrak{H}_1$.

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A vector $x_0 \in \mathscr{D}(T)$ is said to be the least extremal (or best approximate) solution of equation

$$Tx = y, \quad y \in \mathfrak{H}_2 \tag{1·6}$$

if for all $x \in \mathscr{D}(T)$, either

(i) $\|Tx_0 - y\| < \|Tx - y\|$, or

(ii) $\|Tx_0 - y\| = \|Tx - y\|$ and $\|x_0\| \leqslant \|x\|$.

If T is not invertible, it is always possible to define a generalized inverse operator such that some of its relevant properties be closely akin to those of T^{-1}. This concept, under the name of 'general reciprocal', arose in a work by Moore (6).

For a densely defined operator $T: \mathfrak{H}_1 \to \mathfrak{H}_2$, Tseng (7) defined the (maximal) generalized inverse $T^+: \mathfrak{H}_2 \to \mathfrak{H}_1$ by the following conditions:

$$\overline{\mathscr{D}(T^+)} = \mathfrak{H}_2;$$

$$\mathscr{R}(T) \subset \mathscr{D}(T^+), \quad \mathscr{R}(T^+) \subset \mathscr{D}(T);$$

$$TT^+ = \mathscr{P}_{\overline{\mathscr{R}(T)}} \text{ restricted to } \mathscr{D}(T^+);$$

$$T^+T = \mathscr{P}_{\overline{\mathscr{R}(T^+)}} \text{ restricted to } \mathscr{D}(T);$$

$$\operatorname{Ker} T^+ = \mathscr{R}(T)^\perp.$$

This generalized inverse is defined on

$$\mathscr{D}(T^+) = \mathscr{R}(T) \oplus \mathscr{R}(T)^\perp,$$

and it exists iff $\mathscr{D}(T)$ is 'decomposable (with respect to the kernel)' (see (7), (1) and (2)), i.e.

$$\mathscr{D}(T) = \operatorname{Ker} T \oplus \mathscr{C}(T). \tag{1·7}$$

Implicit in a work by M. R. Hestenes (5) is a definition of the generalized inverse suitable for arbitrary operators. Given T, Hestenes defined the operator $\tilde{T}$ by

$$\mathscr{D}(\tilde{T}) = \mathscr{D}(T) + \overline{\operatorname{Ker} T},$$

$$\operatorname{Ker} \tilde{T} = \overline{\operatorname{Ker} T}.$$

Since $\tilde{T}$ possesses a closed kernel its domain is decomposable

$$\mathscr{D}(\tilde{T}) = \operatorname{Ker} \tilde{T} \oplus \mathscr{C}(\tilde{T}) = \overline{\operatorname{Ker} T} \oplus \mathscr{C}(T)$$

and hence Tseng's condition (1·7) does hold for $\tilde{T}$. Hestenes then identified the generalized inverse of T with that of $\tilde{T}$.

A generalized inverse for arbitrary operators between Hilbert spaces based on the least extremal solution property of linear equations (1·6) was deduced in (4).

2. *The definition of the generalized inverse.* Given $T: \mathfrak{H}_1 \to \mathfrak{H}_2$. The graph of its invertible restriction $T|\mathscr{C}(T)$ is

$$\mathfrak{T} = \{\langle x, Tx\rangle : x \in \mathscr{C}(T)\}.$$

The annihilator $\mathfrak{T}^\perp$ of $\mathfrak{T}$,

$$\mathfrak{T}^\perp = \{\langle y, z\rangle : (y, x) + (z, Tx) = 0, x \in \mathscr{C}(T)\}$$

confined to $(T[\mathscr{C}(T)])^\perp$, determines the manifold

$$\mathfrak{T}' = \mathfrak{T}^\perp \cap \mathfrak{H}'' = \{\langle 0, z\rangle : z \in (T[\mathscr{C}(T)])^\perp\}.$$

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The graph

$$\Gamma^{-1}(T^+) = \mathfrak{T} \oplus \mathfrak{T}' = \{\langle x, Tx+z\rangle : x \in \mathscr{C}(T), \quad z \in (T[\mathscr{C}(T)])^\perp\}, \tag{2·1}$$

by property

$$\mathfrak{T}^+ \cap \mathfrak{H}' = \langle 0, 0\rangle,$$

defines an operator T^+: $\mathfrak{H}_2 \to \mathfrak{H}_1$ that we shall call the generalized inverse of T.

With the aid of (1·2), (1·3), and (1·4) from (2·1) we deduce respectively:

$$\mathscr{D}(T^+) = T[\mathscr{C}(T)] \oplus (T[\mathscr{C}(T)])^\perp, \tag{2·2}$$

$$\mathscr{R}(T^+) = \mathscr{C}(T), \quad \overline{\mathscr{R}(T^+)} = (\operatorname{Ker} T)^\perp, \tag{2·3}$$

$$\operatorname{Ker} T^+ = (T[\mathscr{C}(T)])^\perp. \tag{2·4}$$

Thus T^+ is densely defined in $\mathfrak{H}_2$ and possesses a closed kernel. Furthermore,

$$\mathscr{R}(T^+) \subset \mathscr{D}(T)$$

but, in general

$$R(T) \not\subset \mathscr{D}(T^+).$$

3. *Properties of the generalized inverse.*

THEOREM 3·1. $TT^+ = \mathscr{P}_{\overline{\mathscr{R}(T)}}$, *restricted to* $\mathscr{D}(T^+)$. (3·1)

Proof. Let $y \in \mathscr{D}(T^+)$. In view of (2·2) and (2·4),

$$y = y_1 + y_2, \quad \text{where} \quad y_1 \in T[\mathscr{C}(T)], \quad y_2 \in \operatorname{Ker} T^+.$$

Since $T|\mathscr{C}(T)$ is invertible, there exists a unique $x_1 \in \mathscr{C}(T)$ such that $y_1 = Tx_1$. According to the graph (2·1), $T^+y_1 = x_1$. Then

$$TT^+y = TT^+y_1 = Tx_1 = y_1 = \mathscr{P}_{\overline{\mathscr{R}(T)}}y.$$

COROLLARY 3·1. $T^+TT^+ = T^+$, *restricted to* $D(T^+)$.

THEOREM 3·2. $T^+T = \mathscr{P}_{\overline{\mathscr{R}(T^+)}}$, *restricted to* $\mathscr{D}_0 = \operatorname{Ker} T \oplus \mathscr{C}(T)$. (3·2)

Proof. Let $x \in D_0$. Then $x = x_1 + x_2$, with $x_1 \in \mathscr{C}(T)$, $x_2 \in \operatorname{Ker} T$. In view of (2·1) and (2·3), we obtain successively:

$$T^+Tx = T^+Tx_1 = x_1 = \mathscr{P}_{\overline{\mathscr{C}(T)}}x = \mathscr{P}_{\overline{\mathscr{R}(T^+)}}x.$$

COROLLARY 3·2. $TT^+T = T$, *restricted to* $\mathscr{D}_0$.

THEOREM 3·3. $x_0 = T^+y$ (3·3)

is the least extremal solution of equation

$$Tx = y, \text{ restricted to } y \in T[\mathscr{C}(T)] \oplus \mathscr{R}(T)^\perp \subset \mathscr{D}(T^+).$$

Proof. Following F. J. Beutler in (3), we write

$$Tx - y = (Tx_0 - y) + (Tx - Tx_0),$$

and we express the term $Tx_0 - y$, with the help of (3·3) and (3·1), as follows:

$$Tx_0 - y = TT^+y - y = \mathscr{P}_{\overline{\mathscr{R}(T)}}y - y = -\mathscr{P}_{\mathscr{R}(T)^\perp}y.$$

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Then $$Tx-y = -\mathscr{P}_{\mathscr{R}(T)^\perp} y + T(x-x_0),$$

and hence $$\|Tx-y\|^2 = \|\mathscr{P}_{\mathscr{R}(T)^\perp} y\|^2 + \|T(x-x_0)\|^2.$$

Thus $$\|Tx-y\| \geqslant \|Tx_0-y\|$$

with equality iff $$x = x_0+z, \quad z \in \operatorname{Ker} T.$$

Since by (3·3) and (2·3), $$x_0 = T^+ y \in \mathscr{R}(T^+) \subset (\operatorname{Ker} T)^\perp,$$

we have $$\|x\|^2 = \|x_0\|^2 + \|z\|^2,$$

and hence $$\|x_0\| < \|x\| \quad \text{if} \quad x_0 \neq x.$$

4. *The operator* T^{++}. $\mathscr{D}(T^+)$ being dense in $\mathfrak{H}_2$, the generalized inversion rule expressed by the graph (2·1) is applicable to T^+. We construct the graph of $(T^+)^+ = T^{++}$ by direct substitution and graph inversion:

$$\Gamma(T^{++}) = \{\langle T^+y+z, y\rangle : y \in \mathscr{C}(T^+), z \in (T^+[\mathscr{C}(T)^+])^\perp\}.$$

With the help of (2·4), (2·2) and (3·2) we obtain successively:

$$(\operatorname{Ker} T^+)^\perp = \overline{T[\mathscr{C}(T)]},$$

$$\mathscr{C}(T^+) = \mathscr{D}(T^+) \cap \overline{T[\mathscr{C}(T)]} = T[\mathscr{C}(T)],$$

$$(T^+[\mathscr{C}(T^+)])^\perp = (T^+T[\mathscr{C}(T)])^\perp = \mathscr{C}(T)^\perp = \overline{\operatorname{Ker} T},$$

$$\Gamma(T^{++}) = \{\langle T^+y+z, y\rangle : y \in T[\mathscr{C}(T)], z \in \overline{\operatorname{Ker} T}\}. \tag{4·1}$$

The application of the generalized inversion (2·1) to the term

$$\{\langle T^+y, y\rangle : y \in T[\mathscr{C}(T)]\}$$

gives $$\{\langle x, Tx\rangle : x \in \mathscr{C}(T)\}$$

and then (4·1) becomes

$$\Gamma(T^{++}) = \{\langle x+z, Tx\rangle : x \in \mathscr{C}(T), z \in \overline{\operatorname{Ker} T}\}. \tag{4·2}$$

A quick check. $$\Gamma(T^{++}) \cap \mathfrak{H}'' = \langle 0, 0\rangle$$

shows that $\Gamma(T^{++})$ defines T^{++}: $\mathfrak{H}_1 \to \mathfrak{H}_2$ on

$$\mathscr{D}(T^{++}) = \overline{\operatorname{Ker} T} \oplus \mathscr{C}(T)$$

and with $$R(T^{++}) = T[\mathscr{C}(T)],$$

$$\operatorname{Ker} T^{++} = \overline{\operatorname{Ker} T}.$$

THEOREM 4·1. $$T^{++} \supset T_0;$$

$$T^+ = T_0^+ \tag{4·3}$$

where $$T_0 = T|\mathscr{D}_0. \tag{4·4}$$

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Proof. We have

$$\begin{aligned} \operatorname{Ker} T_0 &= \operatorname{Ker} T; \\ \mathscr{C}(T_0) &= \mathscr{C}(T); \\ \mathscr{R}(T_0) &= T[\mathscr{C}(T)] \\ \mathscr{D}(T_0) &= \mathscr{D}_0 \subset \mathscr{D}(T^{++}). \end{aligned} \tag{4·5}$$

Furthermore,

$$\Gamma(T_0) = \{\langle x+z, Tx\rangle : x \in \mathscr{C}(T), \quad z \in \operatorname{Ker} T\}, \tag{4·6}$$

$$\Gamma(T_0^+) = \{\langle x, Tx+z\rangle : x \in \mathscr{C}(T), \quad z \in \mathscr{R}(T_0)^\perp\}. \tag{4·7}$$

Now compare the graphs (4·2) with (4·6) and (2·1) with (4·7), then property (4·5) completes the proof.

COROLLARY 4·1. *$T^{++} \supset T$ if $\mathscr{D}(T)$ is decomposable,*
$T^{++} = T$ iff $\operatorname{Ker} T$ is closed.

5. *The adjoint of the generalized inverse.* Since $\mathscr{D}(T)^+$ is dense in $\mathfrak{H}_2$, the operator $(T^+)^* = T^{+*}$ is defined by the annihilator of $\Gamma^{-1}(T^+)$, (2·1):

$$\Gamma^{-1}(T^+)^\perp = \{\langle u, v\rangle : (u,x) + (v, Tx+z) = 0, \quad x \in \mathscr{C}(T), \quad z \in (T[\mathscr{C}(T)])^\perp\}.$$

Here x and z are independent:

$$\left.\begin{aligned} (v,z) = 0 &\Rightarrow v \in \overline{T[\mathscr{C}(T)]}, \\ (u,x) + (v,Tx) = 0, \quad x \in \mathscr{C}(T), &\quad v \in \overline{T[\mathscr{C}(T)]}. \end{aligned}\right\} \tag{5·1}$$

From (5·1) we find successively:

$$(u,x) + (T^*v, x) = 0, \quad x \in \mathscr{C}(T), \quad v \in \mathscr{D}(T^*) \cap \overline{T[\mathscr{C}(T)]} \subset \mathscr{C}(T^*);$$

$$(u + T^*v, x) = 0, \quad x \in \mathscr{C}(T), \quad v \in \mathscr{D}(T^*) \cap \overline{T[\mathscr{C}(T)]};$$

$$u + T^*v \in \mathscr{C}(T)^\perp = \overline{\operatorname{Ker} T}.$$

Consequently u has the following representation:

$$u = -T^*v + w, \quad v \in \mathscr{D}(T^*) \cap \overline{T[\mathscr{C}(T)]}, \quad w \in \overline{\operatorname{Ker} T}$$

and hence the graph of T^{+*} takes the shape

$$\Gamma(-T^{+*}) = \{\langle -T^*v + w, v\rangle : v \in \mathscr{D}(T^*) \cap \overline{T[\mathscr{C}(T)]}, \quad w \in \overline{\operatorname{Ker} T}\}. \tag{5·2}$$

From (5·2) we deduce

$$\mathscr{D}(T^{+*}) = T^*[\mathscr{D}(T^*) \cap \overline{T[\mathscr{C}(T)]}] \oplus \overline{\operatorname{Ker} T}, \tag{5·3}$$

$$\mathscr{R}(T^{+*}) = \mathscr{D}(T^*) \cap \overline{T[\mathscr{C}(T)]}, \tag{5·4}$$

$$\operatorname{Ker} T^{+*} = \overline{\operatorname{Ker} T}. \tag{5·5}$$

In terms of the restriction T_0 (4·4), with the help of (4·3) and (4·5), we find successively:

$$T^{+*} = T_0^{+*},$$

$$\overline{T[\mathscr{C}(T)]} = \overline{\mathscr{R}(T_0)} = (\operatorname{Ker} T_0^*)^\perp,$$

$$\mathscr{D}(T_0^*) \cap \overline{T[\mathscr{C}(T)]} = \mathscr{D}(T_0^*) \cap (\operatorname{Ker} T_0^*)^\perp = \mathscr{C}(T_0^*), \tag{5·6}$$

$$T_0^*[\mathscr{D}(T_0^*) \cap \overline{T[\mathscr{C}(T)]}] = T_0^*[\mathscr{C}(T_0^*)] = \mathscr{R}(T_0^*). \tag{5·7}$$

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The graph (5·2) and the expressions (5·3), (5·4) through (5·6) and (5·7) become

$$\Gamma^{-1}(-T^{+*}) = \{\langle -T_0^* v + w, v\rangle : v \in \mathscr{C}(T_0^*), \quad w \in \overline{\operatorname{Ker} T}\}, \tag{5·8}$$

$$\mathscr{D}(T^{+*}) = \mathscr{R}(T_0^*) \oplus \overline{\operatorname{Ker} T}, \tag{5·9}$$

$$\mathscr{R}(T^{+*}) = \mathscr{C}(T_0^*), \quad \overline{\mathscr{R}(T^{+*})} = \overline{\mathscr{R}(T_0)}.$$

Expression (5·9) reveals that $\mathscr{D}(T^{+*})$ is dense in $\mathfrak{H}_1$.

Let T be a preclosed (closable) operator. Since $\mathscr{D}(T)$ is assumed to be dense in $\mathfrak{H}_1$, T^* is a densely defined closed operator.

The graph of the generalized inversion (2·1) applied to $-T^*$ and followed by graph inversion defines the operator $(-T^*)^+ = -T^{*+}$:

$$\Gamma(-T^{*+}) = \{\langle -T^* y + z, y\rangle : y \in \mathscr{C}(T^*), z \in \overline{\operatorname{Ker} T}\}. \tag{5·10}$$

THEOREM 5·1. *Let T be preclosed. Then*

(i) $T^{+*} \subset T^{*+}$; *and*

(ii) $T^{+*} = T^{*+}$ *if* $\mathscr{D}(T)$ *is decomposable.* (5·11)

Proof. Note that

$$\mathscr{C}(T^*) = \mathscr{D}(T^*) \cap \overline{\mathscr{R}(T)} \supset \mathscr{D}(T^*) \cap T[\mathscr{C}(T)]$$

and compare (5·2) with (5·10). If $\mathscr{D}(T)$ is decomposable then $T_0 = T$ and hence the graphs (5·8) and (5·10) coincide.

Thus for a preclosed operator T with decomposable domain we have:

$$\left.\begin{aligned} \mathscr{D}(T^{+*}) = \mathscr{D}(T^{*+}) &= \mathscr{R}(T^*) \oplus \overline{\operatorname{Ker} T}, \\ \mathscr{R}(T^{+*}) = \mathscr{R}(T^{*+}) &= \mathscr{C}(T^*), \end{aligned}\right\} \tag{5·12}$$

$$\overline{\mathscr{R}(T^{+*})} = \overline{\mathscr{R}(T^{*+})} = \overline{\mathscr{R}(T)}, \tag{5·13}$$

$$\operatorname{Ker} T^{+*} = \operatorname{Ker} T^{*+} = \overline{\operatorname{Ker} T}. \tag{5·14}$$

Moreover, from (5·12) with the aid of (5·14) we deduce

$$\mathscr{R}(T^*) = \mathscr{D}(T^{+*}) \cap (\operatorname{Ker} T^{+*})^{\perp} = \mathscr{C}(T^{+*}),$$

and since $\mathscr{D}(T^{+*})$ is dense,

$$\overline{\mathscr{R}(T^*)} = (\operatorname{Ker} T^{+*})^{\perp} = \overline{\mathscr{R}(T^+)}. \tag{5·15}$$

THEOREM 5·2. *For a preclosed operator T with decomposable domain*:

(i) $(TT^+)^* = TT^+$ *restricted to* $\operatorname{Ker} T^* \oplus \mathscr{C}(T^*) \cap \mathscr{R}(T)$;

(ii) $(T^+T)^* = T^+T$ *restricted to* $\operatorname{Ker} T \oplus \mathscr{C}(T) \cap \mathscr{R}(T^*)$.

Proof. (i) Property (3·2) applied to T^*, with the aid of (5·11) and (5·13) gives

$$T^{+*}T^* y = T^{*+}T^* y = \mathscr{P}_{\overline{\mathscr{R}(T^{*+})}} y = \mathscr{P}_{\overline{\mathscr{R}(T)}} y, \quad \text{for} \quad y \in \mathscr{D}(T^*).$$

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This, together with property (3·1), leads one to

$$T^{+*}T^{*}y = TT^{+}y, \quad \text{for} \quad y \in \mathscr{D}(T^{+}) \cap \mathscr{D}(T^{*}).$$

Next, property $$(TT^{+})^{*} \supset T^{+*}T^{*},$$

in view of (1·5), (2·2), (4·3), and (4·5), establishes (i).

(ii): Through similar algebra, this time applying (3·1) to T^{*}, and using (5·15), (3·2), (5·12) and property $$(T^{+}T)^{*} \supset T^{*}T^{+*},$$

we find (ii).

REFERENCES

(1) ARGHIRIADE, E. Sur l'inverse généralisée d'un opérateur linéaire dans les espaces de Hilbert. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. ser.* VIII, Vol. **45** (1968), 471–477.

(2) BEN-ISRAEL, A. and CHARNES, A. Contributions to the theory of generalized inverses. *J. Soc. Indust. Appl. Math.* **11** (1963), 667–699.

(3) BEUTLER, F. J. The operator theory of the pseudo-inverse. II Unbounded operators with arbitrary range. *J. Math. Anal. Appl.* **10** (1965), 471–493.

(4) ERDELYI, I. and BEN-ISRAEL, A. Least external solutions of linear equations and generalized inversion between Hilbert spaces. *J. Math. Anal. Appl.* (In Press).

(5) HESTENES, M. R. Relative self-adjoint operators in Hilbert space. *Pacific J. Math.* **11**, (1961), 1315–1357.

(6) MOORE, E. H. *General Analysis I.* Mem. Amer. Phil. Soc. I., 1935.

(7) TSENG, Y. Y. Generalized inverses of unbounded operators between two unitary spaces (in Russian). *Dokl. Akad. Nauk SSSR (N.S.)* **67** (1949), 431–434 (reviewed in *Math. Rev.* **11** (1950), 115).

(8) NEUMANN, J. VON, *Functional Operators*, Vol. II: *The Geometry of Orthogonal Spaces.* Annals of Mathematics Studies No. 22, Princeton University Press (Princeton, N Jersey, 1950).

https://doi.org/10.1017/S0305004100050246 Published online by Cambridge University Press