Math. Proc. Camb. Phil. Soc. (1991), **110**, 327
Printed in Great Britain

Dimensions associated with recurrent self-similar sets

BY ANCA DELIU, J. S. GERONIMO, R. SHONKWILER
Georgia Institute of Technology, School of Mathematics, Atlanta, Georgia 30332, *U.S.A.*

AND D. HARDIN
Vanderbilt University, Department of Mathematics, Nashville, Tennessee 37235, *U.S.A.*

(*Received* 3 *October* 1990; *revised* 18 *December* 1990)

Abstract

The Hausdorff and box dimensions for measures associated with recurrent self-similar sets generated by similitudes is explicitly given. The box dimension of the attractor associated with a class of two-dimensional affine maps is also computed.

1. *Introduction*

In the study of fractal sets the notion of dimension plays an important role, Mandelbrot [**15**]. Along with the fractal set there may be a natural measure that determines the intensity of a colour or level of greyscale at various points on the set. To this measure one can also associate various dimensions, Tricot [**18**], Young [**17**], Mandelbrot [**15**], Bohr and Rand [**6**], Cawley and Mauldin [**9**]. Two dimensions that are in common use at this time are the Hausdorff and box dimensions. Here we consider measures and attractors associated with recurrent iterated function systems on $\mathbb{R}^n$, Barnsley, Elton, and Hardin, (BEH) [**1**], computed with affine maps w_i, $i = 1, 2, \ldots, N$. In this case

$$w_i(x) = B_i x + b_i,$$

where $B_i, i = 1, 2, \ldots, N$ are $n \times n$ matrices and $b_i \in \mathbb{R}^n$. When these maps are contracting similitudes whose law of composition is given by a probability matrix P we give an explicit formula for the Hausdorff and box dimensions of the measure μ associated with P. This generalizes the work of Geronimo and Hardin [**13**]. We also compute the box dimension in the case when the B_i are two-dimensional diagonal matrices. Similar situations have also been considered by Bedford [**6**], Bedford and Urbanski [**7**], BEH [**1**] and McMullen [**16**]. Falconer [**11**] has also considered this problem for general affine maps. Unfortunately his formula is not a closed formula and more importantly it applies only for almost every point in the parameter space containing the b_i's. In computing the box dimension one applies compositions of affine maps to a square containing the attractor and one uses the number of boxes in this covering to get estimates on the minimum number of boxes needed to cover the attractor at various levels. One of the main problems is to know where the attractor is in order to count only those boxes that are needed. A number of

https://doi.org/10.1017/S0305004100070407 Published online by Cambridge University Press

conditions have been imposed such as connectedness of the attractor, as in fractal functions, or a Darboux condition [7] in order to overcome this problem. Another method introduced by Bedford [4], which we exploit here, is to project the attractor onto a convenient axis and relate the dimension of the projected attractor to the dimension of the original set.

We begin in Section 2 by collecting the relevant facts and setting the notation. Then in Section 3 we derive the formula for various measure dimensions for a class of measures associated with similitudes satisfying an open set condition. In Section 4 using a convenient axis to project onto we give a formula for the box dimension of the attractor in terms of the box dimension of the projected attractor. Finally in Section 5 we give some examples.

2. *Preliminaries*

Let d be a metric on $\mathbb{R}^n$ and $w_i: \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, 2, \ldots, N$ be strict contractions in this metric. Let $P = (p_{ij})$ be an $N \times N$ row stochastic matrix (i.e. $p_{ij} \geqslant 0$ and $\sum_{j=1}^{N} p_{ij} = 1$). Then, following BEH [1] we call $\{P, w_i, i = 1, \ldots, N\}$ a recurrent iterated function system (R.I.F.S.). Even though some of the entries in P may be zero, we assume that P is irreducible; i.e. for any i, j there exist $i_1, i_2, \ldots, i_n$ such that $p_{ii_1} p_{i_1 i_2} \cdots p_{i_n j} > 0$. Since P is irreducible it follows (Feller [12]) that there is a unique probability vector $m = (m_i)$ such that

$$\sum_{i=1}^{N} m_i p_{ij} = m_j. \tag{1}$$

Let $i_0, i_1, \ldots$ be the Markov chain in $\{1, \ldots, N\}$ with probability transition matrix P and initial distribution m where m satisfies (1). Associated with the Markov chain $\{i_k\}$ is a random walk on $\mathbb{R}^n$ given by

$$Z_k = w_{i_k}(Z_{k-1}) = w_{i_k} \circ w_{i_{k-1}} \circ \ldots \circ w_{i_1}(Z_0),$$

where Z_0 is a given random vector in $\mathbb{R}^n$. While Z_k is not a Markov process on $\mathbb{R}^n$, $\tilde{Z}_k = (Z_k, i_k)$ is a Markov process on $\mathbb{R}^n \times \{1, \ldots, N\}$ which converges in distribution to a unique stationary distribution $\tilde{\mu}$: see BEH [1]. The distribution $\tilde{\mu}$ satisfies the stationary condition

$$\tilde{\mu}(B, i) = \sum_{j=1}^{n} p_{ji} \tilde{\mu}(w_i^{-1}(B), j) \tag{2}$$

for any Borel set B and $i = 1, \ldots, N$. Let

$$\mu(B) = \tilde{\mu}(B, \{1, \ldots, N\})$$
$$A_i = \operatorname{supp}(\tilde{\mu}(\cdot, i)), \quad i = 1, \ldots, N \tag{3}$$

and

$$A = \operatorname{supp}(\mu) = \bigcup_{i=1}^{N} A_i,$$

where $\operatorname{supp}(\nu)$ denotes the support of a measure ν. The sets $A_1, \ldots A_N$ are the unique non-empty compact sets satisfying

$$A_i = \bigcup_{\{j : p_{ji} > 0\}} w_i(A_j), \quad i = 1, \ldots N \tag{4}$$

https://doi.org/10.1017/S0305004100070407 Published online by Cambridge University Press

(see [1]). Let the *connection matrix* $C = (c_{ij})$ be defined by

$$c_{ij} = \begin{cases} 0 & p_{ji} = 0 \\ 1 & p_{ji} > 0. \end{cases} \tag{5}$$

Observe that (4) and (5) imply that the sets $A_1, \ldots, A_N$ depend only on the connection matrix C and the maps $w_1, \ldots w_N$.

Let Ω be the 'code' space of admissible codes $\underline{i} = (i_{-1}, i_{-2}, \ldots)$ such that $c_{i_{-k} i_{-k-1}} = 1$ for $k = 1, 2, \ldots$. Since each w_i is contractive, it easily follows that for each $\underline{i} \in \Omega$ there is a unique $x(\underline{i}) \in A$ such that

$$x(\underline{i}) = \lim_{q \to \infty} w_{i_{-1}} \circ w_{i_{-2}} \circ \ldots \circ w_{i_{-q}}(y) \tag{6}$$

independent of $y \in \mathbb{R}^n$. Let P be the probability measure on Ω that is defined on 'thin cylinders'

$$P(\underline{i}(q)) = P\{\underline{j} \in \Omega : j_k = i_k, k = 1, \ldots, q\} = m_{i_{-q}} p_{i_{-q} i_{-q+1}} \cdots p_{i_{-2} i_{-1}}. \tag{7}$$

From (7) and (2) it follows that

$$\mu(B) = P\{x^{-1}(B)\} = P\{\underline{i} \in \Omega : x(\underline{i}) \in B\}. \tag{8}$$

Let F be a bounded set in $\mathbb{R}^n$ and $0 \leqslant s \leqslant n$; then the *s-dimensional Hausdorff outer measure* of F is defined by

$$\mathscr{H}^s(F) = \liminf_{\delta \to 0} \left\{ \sum_{i=1}^{\infty} |U_i|^s : F \subset \bigcup_{i=1}^{\infty} U_i ; 0 < |U_k| < \delta \right\},$$

where $|\ |$ denotes the diameter of a set. The Hausdorff dimension of F is then given by

$$\dim_H(F) = \inf\{s : \mathscr{H}^s(F) < \infty\} = \sup\{s : \mathscr{H}^s(F) > 0\}.$$

If μ is a Borel probability measure on $\mathbb{R}^n$ then the Hausdorff dimension of μ is defined as

$$\dim_H(\mu) = \inf\{\dim_H(Y) : Y \subset \mathbb{R}^n, \mu(Y) = 1\}.$$

Another useful dimension is the box dimension defined in the following way. Let $n(F, \epsilon)$ denote the least number of ϵ-balls needed to cover F and define the upper and lower box dimensions of F by

$$\overline{\dim}_B(F) = \limsup_{\epsilon \to 0} \frac{\log(n(F, \epsilon))}{\log(1/\epsilon)}$$

and

$$\underline{\dim}_B(F) = \liminf_{\epsilon \to 0} \frac{\log(n(F, \epsilon))}{\log(1/\epsilon)}$$

respectively. In the event that $\overline{\dim}_B(F) = \underline{\dim}_B(F)$ we define the *box dimension of F*, $\dim_B(F)$ to be this common value. Note that the box dimension is independent of the norm chosen, therefore we will use the sup norm in which case the ϵ-balls will be squares.

3. *Measure dimensions for similitudes*

In this section we assume that the maps w_i are similitudes, i.e.

$$d(w_i(x), w_i(y)) = s_i d(x, y) \quad \text{for } i = 1, \ldots, N.$$

https://doi.org/10.1017/S0305004100070407 Published online by Cambridge University Press

We also assume that the maps satisfy the 'open set condition', that is, there exist open sets $O_1, \ldots, O_N$ such that

$$\bigcup_{\{j: p_{ji}>0\}} w_i(O_j) \subset O_i \quad \text{for } i = 1, \ldots, N$$

and $$w_i(O_j) \cap w_i(O_k) = \varnothing \quad \text{for } j \neq k \quad \text{and} \quad p_{ji}\, p_{ki} > 0.$$

With the assumptions above we now compute $\dim_H(\mu)$ and $\dim_B(\mu)$.

THEOREM 1. *Given the above R.I.F.S.* $(P, w_i, i = 1, \ldots, N)$ *with invariant measure* μ, *suppose that*

(1) w_i *is a similitude* $i = 1, 2, \ldots, N$ *with* $0 < s_1 \leqslant s_2 \leqslant \ldots \leqslant s_N < 1$

(2) *the open set condition is satisfied and*

(3) *the row stochastic matrix* P *is irreducible.*

Then for μ-a.e. $x \in \mathbb{R}^n$,

$$\lim_{\rho \to 0} \frac{\log(\mu(B_\rho(x))}{\log(\rho)} = \alpha,$$

where

$$\alpha = \frac{\Sigma_i \Sigma_j\, m_i\, p_{ij} \log(p_{ij})}{\Sigma_i\, m_i \log(s_i)}. \tag{9}$$

Here m_i *is the stationary distribution associated with* P (*see previous section*).

Proof. Let $A_1, \ldots, A_N$ denote the components of the attractor A (see (3) and (4)). Let $\underline{i} \in \Omega$ (the space of admissible codes) and consider the ball $B_\rho(x(\underline{i}))$. Let l be the least integer such that

$$w_{i_{-1}} \circ w_{i_{-2}} \circ \ldots \circ w_{i_{-l+1}}(A_{i_{-l}}) \subset B_\rho(x(\underline{i})).$$

To obtain a lower bound on $\mu(B_\rho(x(\underline{i})))$ we find from the stationarity condition (2), that

$$\mu(B_\rho(x(\underline{i}))) \geqslant \mu(w_{i_{-1}} \circ \ldots \circ w_{i_{-l+1}}(A_{i_{-l}})) \geqslant m_{i_{-l}}\, p_{i_{-l} i_{-l+1}} \cdots p_{i_{-2} i_{-1}}.$$

The definition of l and the ordering of the s_i's imply

$$\left(\prod_{j=1}^{l} s_{i_{-j}}\right) \operatorname{diam}(A_{i_{-l}}) > \rho s_1.$$

Therefore

$$\mu(B_\rho(x(\underline{i}))) \geqslant \frac{(\prod_{j=1}^{l-1} p_{i_{-j-1} i_{-j}})\, m_{i_{-l}}}{\prod_{j=1}^{l} (s_{i_{-j}})^\alpha} \frac{\rho^\alpha s_1^\alpha}{(\operatorname{diam} A_{i_{-l}})^\alpha}$$

$$\geqslant c_1 \rho^\alpha \prod_{j=1}^{l-1} \frac{p_{i_{-j-1} i_{-j}}}{s_{i_{-j}}^\alpha}, \tag{10}$$

where $c_1 = (s_1/(s_N \operatorname{diam} A))^\alpha > 0$.

We next obtain an upper bound for $\mu(B_\rho(x(\underline{i})))$. For $\rho > 0$ and $\underline{j} \in \Omega$ let q be the least integer such that

$$\prod_{m=-1}^{-q} s_{j_m} < \rho. \tag{11}$$

Let $S(\rho)$ be the set of such finite codes $(j_{-1}, \ldots, j_{-q})$. By making the identification of $(j_{-1}, \ldots, j_{-q})$ with the corresponding cylinder set in Ω then $S(\rho)$ generates a partition

https://doi.org/10.1017/S0305004100070407 Published online by Cambridge University Press

of Ω. For each $(j_{-1},\dots,j_{-q})\in S(\rho)$, set $O_{j_{-1},\dots,j_{-q}} = w_{j_{-1}}\circ\dots\circ w_{j_{-q+1}}(O_{j_{-q}})$. For fixed $j_{-1}\in\{1,\dots,N\}$ the sets $\mathscr{C}_{j_{-1}}(\rho) = \{O_{j_{-1}\dots j_{-q}}: (j_{-1},\dots,j_{-q})\in S(\rho)\}$ are open disjoint sets. From (11) and the fact that the w_i's are similitudes there are constants $c_1 > 0$ and $c_2 > 0$ independent of ρ and j_{-1} such that each $O_{j_{-1}\dots j_{-q}}$ contains a ball of radius $c_1\rho$ and is contained in a ball of radius $c_2\rho$. Consequently lemma 5·3·1 of Hutchinson [**14**] implies that there are at most $((1+2c_2)/c_1)^n$ elements of $\mathscr{C}_{j_{-1}}(\rho)$ whose closure meets $B_\rho(x(\underline{i}))$. Let $A_{j_{-1},\dots,j_{-q}} = w_{j_{-1}}\circ\dots\circ w_{j_{-q+1}}(A_{j_{-q}})$ and let $I_{\underline{i}}(\rho)$ denote the set of $(j_{-1},\dots,j_{-q})\in S(\rho)$ such that $A_{j_{-1}\dots j_{-q}}$ meets $B_\rho(\underline{i})$. The open set condition implies that $A_{j_{-1}\dots j_{-q}}\subset \bar{O}_{j_{-1}\dots j_{-q}}$ and thus $I_{\underline{i}}(\rho)$ contains at most $N((1+2c_2)/c_1)^n$ elements. Using (8) we compute

$$\begin{aligned}\mu(B_\rho(x(\underline{i}))) &= P\{x^{-1}(B_\rho(x(\underline{i})))\}\\ &\leqslant P\{I_{\underline{i}}(\rho)\}\\ &= \sum_{I_{\underline{i}}(\rho)} m_{j_{-q}} p_{j_{-q}j_{-q+1}}\cdots p_{j_{-2}j_{-1}}\\ &= \sum_{I_{\underline{i}}(\rho)} \frac{p_{j_{-q}j_{-q+1}}\cdots p_{j_{-2}j_{-1}}}{s^\alpha_{j_{-q}}\cdots s^\alpha_{j_{-1}}} m_{j_{-q}} s^\alpha_{j_{-q}}\cdots s^\alpha_{j_{-1}}\\ &\leqslant c\rho^\alpha \sum_{I_{\underline{i}}(\rho)}\left(\prod_{i=1}^{q-1}\frac{p_{j_{-i-1}j_{-i}}}{s^\alpha_{j_{-i}}}\right), \qquad (12)\end{aligned}$$

where the last inequality follows from (11) where $c = \max(m_{-q}/s_{-q})$. It follows from the Ergodic Theorem that for P-a.e. $\underline{j}\in\Omega$

$$\begin{aligned}\lim_{q\to\infty}\frac{1}{q}\log\sum_{i=-1}^{-q+1} p_{j_{i-1}j_i} &= \lim_{q\to\infty}\frac{1}{q}\sum_{i=-1}^{-q+1}\log p_{j_{i-1}j_i}\\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} m_i p_{ij}\log(p_{ij}) \qquad (13)\end{aligned}$$

and similarly that

$$\lim_{q\to\infty}\frac{1}{q}\log\prod_{i=-1}^{-q+1} s^\alpha_{j_i} = \alpha\sum_i m_i\log s_i.$$

Since

$$s_1\rho \leqslant s_{i_{-1}}\dots s_{i_{-q}} < \rho \qquad (14)$$

we see that

$$\lim_{\rho\to 0}\frac{\log\rho}{q} = \sum_{i=1}^{N} m_i\log s_i \qquad (15)$$

for P-a.e. $\underline{i}\in\Omega$.

From (13), (14), and (15) it follows that

$$\lim_{\rho\to 0}\log\left(\sum_{I_{\underline{i}}(\rho)}\prod_{i=-1}^{-q+1}\frac{p_{j_{i-1}j_i}}{s^\alpha_{j_i}}\right)\Big/\log\rho = 0. \qquad (16)$$

From (16), (10) and the Ergodic Theorem we have

$$\lim_{\rho\to 0}\frac{(\mu(B_\rho(x(\underline{i}))))}{\log\rho} = \alpha$$

for P-a.e. $\underline{i}\in\Omega$ and therefore from the relationship (8) between μ and P the result follows μ-a.e. ∎

https://doi.org/10.1017/S0305004100070407 Published online by Cambridge University Press

The following corollary now follows from Young [16].

COROLLARY 2. *Given the hypothesis of Theorem* 1 *the box dimension and Hausdorff dimension of* μ *equals* α.

Since $A = \operatorname{supp}(\mu)$ we have $\dim_H(\mu) \leqslant \dim_H(A)$. We will show that there is actually some choice of (p_{ij}) such that $\dim_H(\mu) = \dim_H(A)$. Under the hypotheses of Theorem 1 the Hausdorff dimension and the box dimension of A are the same and may be calculated in the following way (Bedford [5], Falconer [10], BEH [1]). Let $S(d) = \operatorname{diag}(s_1^d, \ldots, s_N^d)$. Note that $S(d)\,C$ is an $N \times N$ irreducible non-negative matrix. The Perron–Frobenius Theorem (Walters [18], p. 16) implies

(*a*) the spectral radius $\lambda(d)$ of $S(d)C$ is an eigenvalue of $S(d)C$ with strictly positive left and right eigenvectors, and

(*b*) $\lambda(d)$ is a strictly decreasing continuous function of d.

Since C is irreducible consisting of 1's and 0's we have $\lambda(0) \geqslant 1$. Furthermore, $\lim_{d \to \infty} \lambda(d) = 0$ and so, by (*b*), there is a unique value d^* such that $\lambda(d^*) = 1$. Then $\dim_B(A) = \dim_H(A) = d^*$.

We next construct a row stochastic matrix P with connection matrix C such that $\alpha = d^*$ (here α is given by (9)). Let $\underline{v}$ be a strictly positive left eigenvector of $S(d^*)\,C$ with eigenvalue 1 and let

$$p_{ij} = \frac{v_j}{v_i} s_j^{d^*} c_{ji}. \tag{17}$$

Substituting (17) into (9), using the fact that $p_{ij} \log c_{ji} = 0$ for all i, j, and using

$$\sum_{ij} m_i p_{ij} \log v_j = \sum_k m_k \log v_k = \sum_{ij} m_i p_{ij} \log v_i$$

we find $\alpha = d^*$.

4. *Box dimension associated with 2-dimensional affine maps*

The result of this section relates the box dimension of certain self-affine sets in the plane to the box dimension of their orthogonal projection onto an axis. For $i = 1, \ldots, N$ let $w_i : \mathbb{R}^2 \to \mathbb{R}^2$ be of the form

$$w_i(x, y) = (a_i x + c_i, b_i y + d_i), \tag{18}$$

where $0 < |a_i| < 1$ and $0 < |b_i| < 1$. By rescaling and translating x and y, we may assume that each w_i maps $S = [0, 1] \times [0, 1]$ into itself. Given an $N \times N$ irreducible connection matrix C (i.e. $c_{ij} = 0$ or 1), let A denote the attractor of the R.I.F.S. $(C, w_i, i = 1, \ldots, N)$ with components $A_1, \ldots, A_n$ (see Section 2).

For $B \subset \mathbb{R}^2$ let πB denote the orthogonal projection of B onto the y-axis:

$$\pi B = \{y \mid (x, y) \in B \text{ for some } x\}.$$

Note that πA is the attractor for the one-dimensional R.I.F.S. $(C, g_i, i = 1, \ldots, N)$ where $g_i : \mathbb{R} \to \mathbb{R}$ is given by $g_i(y) = b_i y + d_i$. By example 3 of Falconer [10] it follows that $\dim_B(\pi A)$ exists (and, in fact, equals $\dim_H(\pi A)$).

THEOREM 3. *Let* $(C, w_i, i = 1, 2, \ldots, N)$ *be a R.I.F.S. with attractor* A *such that* $w_i : S \to S$ *is of the form* (18). *We assume the following open set condition*:

$$w_i(S^0) \cap w_j(S^0) = \varnothing \quad \text{for all } i \neq j,$$

https://doi.org/10.1017/S0305004100070407 Published online by Cambridge University Press

where S^0 is the interior of S. Let d^* be the unique number such that $\mathrm{sp}(\mathrm{diag}\,(|a_i|^{d^*})\,C) = 1$ (*where* sp *denotes spectral radius*) *and D be determined by the formula*

$$\mathrm{sp}(\mathrm{diag}\,(|b_i|^{D_\pi}\cdot|a_i|^{D-D_\pi})\,C) = 1,$$

where $D_\pi = \dim_B(\pi A)$. *If*

$$\mathrm{sp}(\mathrm{diag}\,(b_i^{D_\pi}|a_i|^{d^*-D_\pi})\,C) > 1 \tag{19}$$

then $\dim_B(A) = D$.

Remark. Condition (19) insures that almost all of the boxes are located in long columns.

Proof. Let $A_1, \ldots, A_N$ denote the components of A as in (4). Since $A = \bigcup_{i=1}^N A_i$, we have $\dim_B(A) = \max_i(\dim_B(A_i)) = \dim_B(A_j)$. The irreducibility of C implies that each A_i contains a non-singular affine image of A_j and thus $\dim_B(A_i) = \dim_B(A_j) = \dim_B(A)$ for $i = 1, \ldots, N$.

For $\epsilon > 0$ let Ω_ϵ be the set of finite codes $\underline{i} = i_{-1} \ldots i_{-q}$ such that

$$|a_{i_{-1}} \ldots a_{i_{-q}}| \leqslant \epsilon \tag{20}$$

and

$$|a_{i_{-1}} \ldots a_{i_{-q+1}}| > \epsilon \tag{21}$$

($\underline{i} = i_1 \in \Omega_\epsilon$ if $|a_{i_1}| \leqslant \epsilon$). Note that Ω_ϵ generates a partition of Ω into cylinder sets $\underline{i}$. For $\underline{i} \in \Omega_\epsilon$ we define the following notation:

$$a_{\underline{i}} = |a_{i_{-1}} \ldots a_{i_{-q}}|, \quad b_{\underline{i}} = |b_{i_{-1}} \ldots b_{i_{-q}}|$$

and

$$w_{\underline{i}} = w_{i_{-1}} \circ \ldots \circ w_{i_{-q}}, \quad A_{\underline{i}} = w_{i_{-1}} \circ \ldots \circ w_{i_{-q+1}}(A_{i_{-q}}).$$

For $i = 1, \ldots, N$ let $F_i = \bigcup_{c_{ij}>0} A_j$ and note that $A_{\underline{i}} = w_{\underline{i}}(F_{i_{-q}})$. In the following we consider coverings of closed $\epsilon \times \epsilon$ squares with sides parallel to the coordinate axes. If B is a bounded set in $\mathbb{R}^2$ let $n_2(B, \epsilon)$ denote the least number of such squares needed to cover B. If B is a bounded subset of $\mathbb{R}$ then let $n_1(B, \epsilon)$ denote the least number of closed intervals of length ϵ needed to cover B. The germ of the proof is contained in the following geometrical observations (compare with Bedford's thesis [**4**]).

Let $\epsilon > 0$ and $\underline{i} \in \Omega_\epsilon$.

(1) $A_{\underline{i}} \subset w_{\underline{i}}(S)$ and $w_{\underline{i}}(S)$ is a rectangle $a_{\underline{i}}$ wide and $b_{\underline{i}}$ high.

(2) Since $a_{\underline{i}} \leqslant \epsilon$, any ϵ-cover of $A_{\underline{i}}$ may be arranged so that each ϵ-square meets both vertical sides of $w_{\underline{i}}(S)$ as shown in Figure 1 since each ϵ-square is wider than $w_{\underline{i}}(S) \supset A_{\underline{i}}$. If B is such an ϵ-square, then $w_{\underline{i}}^{-1}(B)$ is a rectangle of height $\epsilon/b_{\underline{i}}$ which meets both vertical sides of S, i.e. the lines $x = 0$ and $x = 1$. Hence B meets $A_{\underline{i}}$ if and only if $w_{\underline{i}}^{-1}(B)$ meets $\pi F_{i_{-q}}$. Thus there is a one-to-one correspondence between ϵ-covers of $A_{\underline{i}}$ and $\epsilon/b_{\underline{i}}$-covers of $\pi F_{i_{-q}}$. Consequently

$$n_2(A_{\underline{i}}, \epsilon) = n_1(\pi F_{i_{-q}}, \epsilon/b_{\underline{i}}). \tag{22}$$

In order to estimate the cardinality of Ω_ϵ we introduce the following probability measure. Let $\underline{v}$ be a positive right eigenvector of $\mathrm{diag}\,|a_i|^{d^*} C$ and define the row-stochastic irreducible matrix $Q = (q_{ij})$ by

$$q_{ij} = |a_i|^{d^*} c_{ij} \frac{v_j}{v_i}.$$

Let $\underline{m} = (m_i)$ be the unique stationary distribution associated with Q, i.e. $\underline{m}Q = \underline{m}$

https://doi.org/10.1017/S0305004100070407 Published online by Cambridge University Press

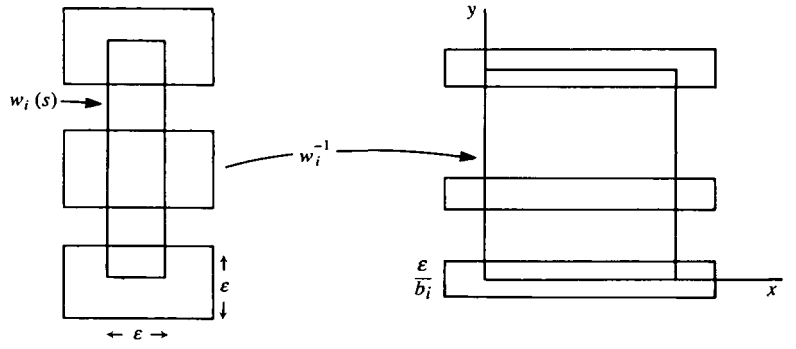


Fig. 1

and $\sum_{i=1}^{N} m_i = 1$ and let P_Q be the probability measure on Ω generated by Q with initial distribution $\underline{m}$ given above. For $\underline{i} \in \Omega_\epsilon$ we have $k_1 |a_{\underline{i}}|^{d^*} \leqslant P_Q(\underline{i}) \leqslant k_2 |a_{\underline{i}}|^{d^*}$ where $k_1 = \dfrac{\min v_i}{\max v_i}$ and $k_2 = \dfrac{\max v_i}{\min v_i}$. Then (20) and (21) imply

$$c_1 \epsilon^{d^*} \leqslant P_Q(\underline{i}) \leqslant c_2 \epsilon^{d^*}$$

for some positive constants c_1 and c_2. Consequently

$$P_Q(\Omega_\epsilon) = 1 \geqslant c_1 \epsilon^{d^*} |\Omega_\epsilon|. \tag{23}$$

We now construct a row stochastic matrix which will be useful in estimating certain sums. For $0 \leqslant \beta \leqslant 1$ let α be the unique number such that

$$\mathrm{sp}[\mathrm{diag}\,(b_i^\beta a_i^{\alpha-\beta})\, C] = 1 \tag{24}$$

and let $\underline{v}$ be a positive left eigenvector of $\mathrm{diag}\,(b_i^\beta a_i^{\alpha-\beta})\, C$ with eigenvalue 1. Let $P = (p_{ij})$ where p_{ij} is given by

$$p_{ij} = \frac{v_j}{v_i} b_i^\beta a_i^{\alpha-\beta} c_{ji}.$$

Given an initial distribution $p_i = 1/N$ the induced probability measure R on code space satisfies

$$R(\underline{i}) = \frac{1}{N} p_{i_{-1} i_{-2}} \cdots p_{i_{-q+1} i_{-q}} = \frac{1}{N} \frac{v_{i_{-q}}}{v_{i_{-1}}} b_{\underline{i}}^\beta a_{\underline{i}}^{\alpha-\beta}.$$

Thus there exist constants

$$k_1 = \frac{1}{N} \frac{\min v_i}{\max v_i} > 0 \quad \text{and} \quad k_2 = \frac{1}{N} \frac{\max v_i}{\min v_i} > 0$$

such that

$$k_1 R(\underline{i}) \leqslant b_{\underline{i}}^\beta a_{\underline{i}}^{\alpha-\beta} \leqslant k_2 R(\underline{i}). \tag{25}$$

We next show that D is an upper bound for $\dim_B(A)$. Since Ω_ϵ generates a partition of Ω we have $A = \bigcup_{\underline{i} \in \Omega_\epsilon} A_{\underline{i}}$. Thus

$$n_2(A, \epsilon) \leqslant \sum_{\underline{i} \in \Omega_\epsilon} n_2(A_{\underline{i}}, \epsilon) = \sum_{\underline{i} \in \Omega_\epsilon} n_1\left(\pi F_{i_{-q}}, \frac{\epsilon}{b_{\underline{i}}}\right),$$

https://doi.org/10.1017/S0305004100070407 Published online by Cambridge University Press

where the last equality follows from (22). Let $\beta > D_\pi$ and α be as in (24); then $\lim_{\beta \to D_\pi} \alpha = D$ from the Perron–Frobenius theory. Since $\beta > D_\pi$, there is some constant $c > 0$ such that

$$n_1(\pi F_j, \epsilon) \leqslant c\epsilon^{-\beta}$$

for all $j = 1, \ldots, N$ and $0 < \epsilon \leqslant 1$. Let $B_\epsilon = \{i_\epsilon \in \Omega_\epsilon : \epsilon/b_1 > 1\}$. Then $n_1(\pi F_j, \epsilon/b_{\underline{i}}) = 1$ for $j = 1, \ldots, N$ while if $\underline{i} \in G_\epsilon = \Omega_\epsilon \backslash B_\epsilon$ then $\epsilon/b_{\underline{i}} \leqslant 1$ and we have

$$n_1(\pi F_j, \epsilon/b_{\underline{i}}) \leqslant c\epsilon^{-\beta} b_{\underline{i}}^{\beta} \quad \text{for } j = 1, \ldots, N.$$

Hence

$$n_2(A, \epsilon) \leqslant \sum_{\underline{i} \in \Omega_\epsilon} n_2(A_{\underline{i}}, \epsilon) = \sum_{\underline{i} \in \Omega_\epsilon} n_1(\pi F_{i_{-q}}, \epsilon/b_{\underline{i}}) \leqslant c \sum_{\underline{i} \in G_\epsilon} \epsilon^{-\beta} b_{\underline{i}}^{\beta} + |B_\epsilon|$$

$$\leqslant c \sum_{\underline{i} \in \Omega_\epsilon} \epsilon^{-\beta} b_{\underline{i}}^{\beta} + |B_\epsilon| \leqslant c\Big(\sum_{\underline{i} \in \Omega_\epsilon} a_{\underline{i}}^{\alpha - \beta} b_{\underline{i}}^{\beta} \Big) \epsilon^{-\alpha} + |B_\epsilon|,$$

where we used $a_{\underline{i}} \leqslant \epsilon$ for $i \in \Omega_\epsilon$ in the last inequality. Since

$$|B_\epsilon| \leqslant |\Omega_\epsilon| \quad \text{and} \quad k_2 = k_2 R(\Omega_\epsilon) > \sum_{\underline{i} \in \Omega_\epsilon} \alpha_{\underline{i}}^{\alpha - \beta} b_{\underline{i}}^{\beta}$$

by (25), the above inequality becomes

$$n_2(A, \epsilon) \leqslant \epsilon^{-\alpha} \left(ck_2 + \frac{\epsilon^{\alpha - d^*}}{c_1} \right).$$

Letting $\beta \to D_\pi$ we find $\dim_B A \leqslant D$.

Finally we show $D \leqslant \dim_B(A)$. The open set condition implies $w_{\underline{i}}(S)^0 \cap w_j(S)^0 = \varnothing$ for any $\underline{i}, \underline{j} \in \Omega_\epsilon$ with $\underline{i} \neq \underline{j}$. Thus any $\epsilon \times \epsilon$ square meets at most $l = 2a^{-1} + 4$ of the rectangles $\{w_{\underline{i}}(S) : \underline{i} \in G_\epsilon\}$. Hence we have

$$n_2(A, \epsilon) \geqslant \frac{1}{l} \sum_{\underline{i} \in G_\epsilon} n_2(A_{\underline{i}}, \epsilon) = \frac{1}{l} \Big(\sum_{\underline{i} \in \Omega_\epsilon} n_2(A_{\underline{i}}, \epsilon) - |B_\epsilon| \Big)$$

since $n_2(A_{\underline{i}}, \epsilon) = 1$ for $\underline{i} \in B_\epsilon$. Choosing $\beta < D_\pi$ the lower bound follows in the same way as the upper bound. ∎

5. *Examples*

(a) *Iterated function systems*

Iterated function systems are special cases of recurrent iterated function systems when the connection matrix C has only 1's for entries. In this case $\operatorname{diag}(\lambda_i) C$ with $\lambda_i > 0$ has maximal eigenvalue $\operatorname{Tr} \operatorname{diag} \lambda_i$ with eigenvector $\bar{v} = (\lambda_i)$. Thus (19) becomes

$$\sum_{i=1}^{N} |a_i|^{d^* - D_\pi} |b_i|^{D_\pi} > 1, \tag{26}$$

where $\sum_{i=1}^{N} |a_i|^{d^*} = 1$ and Theorem 2 becomes the following corollary.

COROLLARY 4. *Let $w_i : i = 1, 2, \ldots, N$ be an I.F.S. with attractor A such that $w_i : S \to S$ is of the form* (18). *Suppose the open set condition*

$$w_i(S^0) \cap w_j(S^0) = \varnothing \quad \textit{for all } i \neq j$$

https://doi.org/10.1017/S0305004100070407 Published online by Cambridge University Press

and (26) *holds where* $D_\pi = \dim_B(\pi A)$. *Let* D *be determined by the formula*

$$\sum_{i=1}^{N} |b_i|^{D_\pi} |a_i|^{D-D_\pi} = 1. \tag{27}$$

Then $\dim_B(A) = D$.

(*b*) *Fractal functions*

Let $0 = x_0 < x_1 < \ldots < x_N = 1$, $y_i \in \mathbb{R}$, $i = 0, \ldots, N$ with $y_0 = 0$ and $y_N = 1$. Choose $w_i : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix},$$

where $a_i = x_i - x_{i-1}$ and $b_i = y_i - y_{i-1}$ for $i = 1, \ldots, N$. Then the attractor A of the I.F.S. $(w_i, i = 1, \ldots, N)$ is the graph of a continuous 'fractal interpolation function' f^* (see Barnsley and Harrington [3]) such that $f^*(x_i) = y_i$. (These are a special case of the more general fractal interpolation functions considered in [3].) In this case $D_\pi = 1$ since f^* is continuous and $d^* = 1$ since $\sum_{i=1}^{N} a_i = 1$. Hence (19) becomes $\sum_{i=1}^{N} |b_i| > 1$ in which case (27) becomes $\sum_{i=1}^{N} |a_i|^{D-1} |b_i| = 1$, which recovers the result of theorem 4 of Barnsley, Elton, Hardin, Massopust [2] for this special case.

REFERENCES

[1] M. F. BARNSLEY, J. ELTON and D. HARDIN. Recurrent iterated function systems. *Constr. Approx.* **5** (1989), 3–31.

[2] M. F. BARNSLEY, J. H. ELTON, D. P. HARDIN and P. R. MASSOPUST. Hidden-variable fractal interpolation functions. *SIAM J. Math. Anal.* **20** (1989), 1218–1242.

[3] M. F. BARNSLEY and A. N. HARRINGTON. The calculus of fractal interpolating functions. *J. Approx. Theory* **57** (1989), 14–34.

[4] T. BEDFORD. Crinkly curves, Markov partitions, and dimension. Ph. Thesis, Warwick University (1984).

[5] T. BEDFORD. Dimension and dynamics for fractal recurrent sets. *J. London Math. Soc.* **33** (1986), 89–100.

[6] T. BEDFORD. Hausdorff dimension and box dimension in self-similar sets. In *Proc. Conf. Topology and Measure V* (*Binz, GDR*, 1987), Wissenschaftliche Beiträge der Ernst-Moritz-Arndt Universität, Griefswald (1988).

[7] T. BEDFORD and M. URBANSKI. *The Box and Hausdorff Dimensions of Self-affine Sets.* Delft University report 88-80 (1988).

[8] T. BOHR and D. RAND. The entropy function for characteristic exponents. *Physica* **25D** (1987), 387–393.

[9] R. CAWLEY and R. D. MAULDIN. Multifractal decompositions of Moran fractals. Preprint.

[10] K. J. FALCONER. Dimensions and measures of quasi-self-similar sets. *Proc. Amer. Math. Soc.* **106** (1989), 543–554.

[11] K. J. FALCONER. The Hausdorff dimension of self-affine fractals. *Math. Proc. Cambridge Philos. Soc.* **103** (1988), 339–350.

[12] W. FELLER. *An Introduction to Probability Theory and Its Applications* (John Wiley and Sons, 1950).

[13] J. S. GERONIMO and D. P. HARDIN. An exact formula for the measure dimensions associated with a class of piecewise linear maps. *Constr. Approx.* **5** (1989), 89–98.

[14] J. HUTCHINSON. Fractals and self-similarity. *Indiana Univ. Math. J.* **30** (1981), 713–747.

[15] B. MANDELBROT. *The Fractal Geometry of Nature* (W. H. Freeman and Co., 1982).

[16] C. MCMULLEN. The Hausdorff dimension of general Sierpinski carpets. *Nagoya Math. J.* **96** (1984), 1–9.

[17] L.-S. YOUNG. Dimension, entropy and Lyapunov exponents. *Ergodic Theory Dynamical Systems* **2** (1982), 109–129.

[18] C. TRICOT. Two definitions of fractional dimension. *Math. Proc. Cambridge Philos. Soc.* **91** (1982), 57–74.

[19] P. WALTERS. *An Introduction to Ergodic Theory* (Springer-Verlag, 1982).

https://doi.org/10.1017/S0305004100070407 Published online by Cambridge University Press