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Badly approximable vectors, C^1 curves and number fields

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Abstract. We show that the set of points on C^1 curves which are badly approximable by rationals in a number field form a winning set in the sense of Schmidt. As a consequence, we obtain a number field version of Schmidt's conjecture in Diophantine approximation.

1. *Introduction*

Recall that a real number x is *badly approximable* if there exists $c > 0$ such that

$$|qx - p| > \frac{c}{q} \tag{1}$$

for all $q \in \mathbb{N}$ and $p \in \mathbb{Z}$. It is well known that badly approximable vectors have zero Lebesgue measure and full Hausdorff dimension (Jarnik [**15**] for $n = 1$ and Schmidt [**22**, **23**] for arbitrary n). In fact, Schmidt showed that they are *winning* for a certain game, a stronger and more versatile property than having full Hausdorff dimension.

1.1. *Schmidt's game.*

In [**22**], Schmidt introduced the following game. Two players, say Player A and Player B, start with a complete metric space X, a subset $W \subseteq X$, and two parameters $0 < \alpha, \beta < 1$. Player A begins by choosing an arbitrary ball $A_0 = B(x_0, \rho)$. Player B then chooses a ball $B_0 = B(y_1, \alpha\rho)$ contained in A_0. Player A makes his next move by choosing a ball $A_1 \subset B_0$ of radius $\alpha\beta\rho$. The nth step of the game consists of first Player A choosing a ball $A_n = B(x_n, (\alpha\beta)^n\rho) \subset B_{n-1}$, then Player B following by choosing the next ball $B_n = B(y_n, \alpha(\alpha\beta)^n\rho) \subset A_n$. As the radii of the balls are shrinking to zero and X is complete, at the end of the infinite game Player A and Player B are left with a single point $\{x_\infty\} = \bigcap_n A_n$. We say that Player B has won this (α, β) game if $x_\infty \in W$. The set W is called (α, β)-winning if Player B can find a

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winning strategy, α-winning if it is (α, β)-winning for all $0 < \beta < 1$ and winning if it is α-winning for some $\alpha > 0$. Schmidt games have the following properties (cf. [**9**, **22**]).

(1) A winning subset of $\mathbb{R}^n$ is *thick*, i.e. the intersection of a winning set with every open set in $\mathbb{R}^n$ has Hausdorff dimension n.
(2) A countable intersection of α-winning sets is α-winning.
(3) Winning sets are preserved by bi-Lipschitz homeomorphisms of $\mathbb{R}^n$.
(4) The set of badly approximable vectors is (α, β)-winning whenever $2\alpha < 1 + \alpha\beta$; in particular, it is α-winning for any $0 < \alpha \leq \frac{1}{2}$.

1.2. *Diophantine approximation in number fields.* Let K be a number field of degree d with r real and s complex embeddings. Denote by S the set of Galois embeddings σ, where for the complex embeddings one chooses one of the pair σ and $\bar{\sigma}$. Let $\mathcal{O}_K$ be the ring of integers of K. Denote by $K_S := \mathbb{R}^r \times \mathbb{C}^s \cong \mathbb{R}^d$. We denote by τ the twisted diagonal embedding of K into K_S by

$$\tau(x) = (\sigma(x))_{\sigma \in S},$$

where we identify each coordinate of K_S with an element of S. The notation will be extended to vectors and matrices and τ will be omitted in the notation when it causes no confusion.

It is natural to ask if analogues of the traditional theorems in Diophantine approximation hold in the setting of number fields. More precisely, we wish to approximate elements in K_S using ratios of elements in $\mathcal{O}_K$. Analogues of Dirichlet's theorem in this setting have been established by several authors (cf. [**7**, **14**, **21**, **24**]) using appropriate adaptations of the geometry of numbers. Moreover, [**7**, **14**] also show the existence of badly approximable vectors† in this setting.

Say that a vector $\mathbf{x} = (x^\sigma)_{\sigma \in S} \in K_S$ is K-badly approximable ($\mathbf{x} \in \mathrm{Bad}(K)$) if there exists $c > 0$ such that for all $p, q \in \mathcal{O}_K$ with $q \neq 0$

$$\max_{\sigma \in S}\{|\sigma(p) + x^\sigma \sigma(q)|\} \max_{\sigma \in S}\{|\sigma(q)|\} > c. \tag{2}$$

Here and in the rest of the paper, $|\ |$ will be used to denote both real and complex absolute values depending on context.

Dani [**8**] showed that a real number is badly approximable if and only if a lattice associated with the number has a bounded trajectory in the space $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ under the action of a certain subsemigroup. A version of the Dani correspondence (§3) states that a vector $\mathbf{x}$ is K-badly approximable if and only if the trajectory

$$\left\{\mathrm{SL}_2(\mathcal{O}_K)\left(\begin{pmatrix} 1 & x^\sigma \\ & 1 \end{pmatrix}\right)_{\sigma \in S} g_t \mid t \geq 0\right\}$$

for the flow

$$g_t := \left\{\left(\begin{pmatrix} e^{-t} & \\ & e^{t} \end{pmatrix}\right)_{\sigma \in S} \mid t \geq 0\right\} \tag{3}$$

† We note, however, that our notion of badly approximable vectors differs slightly from the notion considered elsewhere as we do not square the absolute value at the complex places in our definition. The proofs of the main theorems can be adapted to this case as well.

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is bounded in the quotient space $\mathrm{SL}_2(\mathcal{O}_K)\backslash\mathrm{SL}_2(K_S)$. In conjunction with the Moore ergodicity theorem, we can conclude that K-badly approximable vectors have zero Lebesgue measure. Nevertheless, they constitute a winning set for Schmidt's game and therefore have full Hausdorff dimension.

1.3. *Main results.* We show that the set of badly approximable vectors are winning even when the game is restricted to a curve. We need slightly separate conditions on the curve in question in different cases. We recall that $r + s$ is the number of simple factors of $\mathrm{SL}_2(K_S)$.

THEOREM 1.1. *Let $\phi = (\phi_\sigma)_{\sigma\in S} : [0, 1] \to K_S$ be a continuously differentiable map. We assume that $\phi'_\sigma(x) \neq 0$ for all but finitely many $x \in [0, 1]$ and for all σ in a subset $S' \subset S$ (possibly depending on x) with*

$$|\{\sigma \in S' : \sigma \text{ is real}\}| + 2|\{\sigma \in S' : \sigma \text{ is complex}\}| > \left\lfloor \frac{d}{2} \right\rfloor.$$

Define

$$\Phi(x) := \left(\begin{pmatrix} 1 & \phi_\sigma(x) \\ & 1 \end{pmatrix}\right)_{\sigma\in S}, \tag{4}$$

and let g_t as defined in (3) act on $\mathrm{SL}_2(\mathcal{O}_K)\backslash\mathrm{SL}_2(K_S)$ by right multiplication. Let $\Lambda = \mathrm{SL}_2(\mathcal{O}_K)g \in \mathrm{SL}_2(\mathcal{O}_K)\backslash\mathrm{SL}_2(K_S)$. Then the set

$$\{x \in [0, 1] \mid \Lambda\Phi(x) \text{ has bounded trajectory under the flow } g_t\} \tag{5}$$

is winning in the sense of Schmidt, and hence has Hausdorff dimension one.

We note that the condition on the curve in the theorem above is not of a technical nature. If Φ simply parameterizes a line segment that is parallel to a coordinate axis (which are special directions as they correspond to the simple factors of the ambient Lie group), then there may not be any points with bounded trajectory on the line segment if e.g. $d = r = 3$, see §4.2.

Theorem 1.1 coupled with Dani's correspondence gives us the following.

COROLLARY 1.2. *Let $\phi : [0, 1] \to K_S$ be as in Theorem 1.1. Then the set*

$$\{x \in [0, 1] \mid \phi(x) \in \mathrm{Bad}(K)\}$$

is winning in the sense of Schmidt, and hence has Hausdorff dimension one.

Now we let K be a real quadratic extension of $\mathbb{Q}$ where $d = r = 2$. In this case, we can choose different directions in the two-dimensional Cartan subgroup.

THEOREM 1.3. *Let K be a real quadratic field. Let $\phi = (\phi_\sigma)_{\sigma\in S} : [0, 1] \to K_S$ be a continuously differentiable map such that $\phi'_\sigma(x) \neq 0$ for all but finitely many $x \in [0, 1]$ and every $\sigma \in S$. Let $\mathbf{r} \in \mathbb{R}^2$ be a real vector with $r_\sigma \geq 0$ for $\sigma \in S$ and $\sum_\sigma r_\sigma = 1$. For $t \geq 0$, let*

$$g(\mathbf{r})_t := \left(\begin{pmatrix} e^{-r_\sigma t} & \\ & e^{r_\sigma t} \end{pmatrix}\right)_{\sigma\in S} \tag{6}$$

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act on $\mathrm{SL}_2(\mathcal{O}_K)\backslash \mathrm{SL}_2(K_S)$ *by right multiplication. Let* $\Lambda = \mathrm{SL}_2(\mathcal{O}_K)g \in \mathrm{SL}_2(\mathcal{O}_K)\backslash \mathrm{SL}_2(K_S)$. *Then the set*

$$\{x \in [0,1] \mid \Lambda\Phi(x) \text{ has bounded trajectory under the flow } g(\mathbf{r})_t\} \tag{7}$$

is winning in the sense of Schmidt, and hence has Hausdorff dimension one.

The α in the winning statement above does not depend on $\mathbf{r}$. Taking intersections over rational vectors $\mathbf{r}$ and using the fact that countable intersections of α-winning sets are α-winning therefore shows us the following.

COROLLARY 1.4. *With notation as in Theorem 1.3,*

$$\bigcap_{\mathbf{r}\in\mathbb{Q}^2_+} \{x \in [0,1] \mid \Lambda\Phi(x) \text{ has bounded trajectory under the flow } g(\mathbf{r})_t\} \tag{8}$$

is winning.

Remarks.

(1) Corollary 1.4 provides an analogue of Schmidt's conjecture for real quadratic fields, a theorem of Badziahin *et al* [**4**] in the real case. See also the works [**1**, **2**] of An for stronger results in this vein.
(2) In contrast to Corollary 1.4, we note that it follows from results in [**10**] (see also [**11**]) that bounded orbits for the full Cartan subgroup have *zero* Hausdorff dimension.
(3) Using Theorem 1.1 and the Marstrand slicing theorem (see [**13**, Theorem 5.8]), we see that $\mathrm{Bad}(K)$ has full Hausdorff dimension.
(4) As far as we are aware, the only other result regarding abundance of badly approximable vectors in the context of number fields is for certain quadratic extensions whose rings of integers have unique factorization [**12**].
(5) Theorems 1.1, 1.3 and Corollary 1.2 are among the very few existing results which show that badly approximable points on curves are winning. In a recent work, Beresnevich [**5**] shows that badly approximable vectors on 'non-degenerate' manifolds have full Hausdorff dimension. See also the works [**18–20**] for results regarding badly approximable vectors on certain classes of fractals.
(6) We refer to the r_i which appear in $g(\mathbf{r})_t$ as weights. Thus Theorem 1.1 deals with equal weights and Theorem 1.3 with unequal weights. The equal weights version of our results is closely related to a result from [**6**] (see Proposition 4.9). We note, however, that in the context of this paper certain special directions (e.g. line segments parallel to a coordinate axis corresponding to one of three real factors) may fail to have any badly approximable points on them while line segments in general directions are covered by Theorem 1.1.
(7) In a related earlier result [**3**], it is shown that points on C^1 curves in rank one locally symmetric spaces which have bounded orbits under the geodesic flow are winning. The result of this paper may be viewed as a generalization of this result to certain quotients of higher rank groups with $\mathbb{Q}$-rank one.

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2. *Mahler's compactness criterion*

In this section, we state and prove Mahler's compactness criterion for S-adic homogeneous spaces. Theorem 2.2, which is the main result in this section, is almost certainly well known to experts; see, for instance, [**16**, **17**]. We provide a proof for completeness. The original statement of Mahler's compactness criterion is as follows.

THEOREM 2.1. *A subset $A \subset \mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R})$ is relatively compact if and only if there exists $\varepsilon > 0$ such that for all $\mathrm{SL}_n(\mathbb{Z})g \in A$ and for all non-zero $v \in \mathbb{Z}^n g$, $\|v\| > \varepsilon$.*

Here we take $\|v\|$ to be the maximum norm. We wish to rephrase this theorem for the space $\mathrm{SL}_2(\mathcal{O}_K)\backslash \mathrm{SL}_2(K_S)$. To do this, we will use restriction of scalars to map $\mathrm{SL}_2(\mathcal{O}_K)\backslash \mathrm{SL}_2(K_S)$ to $\mathrm{SL}_{2d}(\mathbb{Z})\backslash \mathrm{SL}_{2d}(\mathbb{R})$. More concretely, we define the embedding as follows.

Since the degree of K over $\mathbb{Q}$ is d, we may view K as a d-dimensional vector space over $\mathbb{Q}$. We choose a basis $\{a_1, \ldots, a_d\}$ of K over $\mathbb{Q}$ so that the $\mathbb{Z}$-span of these elements is $\mathcal{O}_K$. Left multiplication by an element of K is a $\mathbb{Q}$-linear transformation on K. We have an algebraic embedding $\iota : K \to \mathrm{Mat}_{d,d}(\mathbb{Q})$ with respect to this chosen basis. We then have an induced map

$$\begin{aligned} \iota : \mathrm{SL}_2(K) &\to \mathrm{SL}_{2d}(\mathbb{R}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \begin{pmatrix} \iota(a) & \iota(b) \\ \iota(c) & \iota(d) \end{pmatrix}. \end{aligned}$$

Thus, we have defined the algebraic subgroup

$$\mathrm{Res}_{K/\mathbb{Q}}\,\mathrm{SL}_2 = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;\middle|\; A, B, C, D \in \mathrm{Im}(\iota),\ AD - BC = \mathrm{Id} \right\},$$

such that $\iota(\mathrm{SL}_2(K)) = (\mathrm{Res}_{K/\mathbb{Q}}\,\mathrm{SL}_2)(\mathbb{Q}) \subseteq \mathrm{SL}_{2d}(\mathbb{Q})$.

To give another idea of the structure of this space, consider the basis of K given by $\{1, \xi, \xi^2, \ldots, \xi^{d-1}\}$, where ξ is a primitive element of K, that is, $K = \mathbb{Q}(\xi)$. (Note that the transformation from this basis to that previously mentioned is rational.) Then as a $\mathbb{Q}$-linear transformation of K, left multiplication by ξ is represented by the companion matrix

$$T_\xi = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -c_0 & -c_1 & -c_2 & \cdots & -c_{d-1} \end{pmatrix},$$

where the $p_{\min}(x) = x^d + c_{d-1}x^{d-1} + \cdots + c_1 x + c_0$ is the minimal polynomial of ξ over $\mathbb{Q}$. It is well known that this is conjugated by the Vandermonde matrix V_ξ (associated with the various Galois embeddings of ξ) to the diagonal form

$$V_\xi^{-1} T_\xi V_\xi = \mathrm{diag}(\sigma(\xi))_{\sigma \in S}.$$

Since the elements of $\mathrm{Im}(\iota)$ commute, they are simultaneously diagonalizable. Moreover, for any $\zeta \in K$ there exists $p_\zeta(x) \in \mathbb{Q}[x]$ of degree less than $d-1$ with $\zeta = p_\zeta(\xi)$, and so the transformation of left multiplication by ζ is given by $T_\zeta = p_\zeta(T_\xi)$ and is conjugate to

$$\mathrm{diag}(p_\zeta(\sigma(\xi)))_{\sigma \in S} = \mathrm{diag}(\sigma(\zeta))_{\sigma \in S}$$

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(since the maps σ are ring homomorphisms). After a simple change of bases, we have an embedding $\mathrm{SL}_2(K) \to \mathrm{SL}_{2d}(\mathbb{R})$ given by

$$\begin{pmatrix} \zeta_1 & \zeta_2 \\ \zeta_3 & \zeta_4 \end{pmatrix} \mapsto \mathrm{diag}\left(\begin{pmatrix} \sigma(\zeta_1) & \sigma(\zeta_2) \\ \sigma(\zeta_3) & \sigma(\zeta_4) \end{pmatrix}\right)_{\sigma \in S}.$$

This is precisely $\iota(\mathrm{SL}_2(K)) \subset \mathrm{SL}_2(K_S)$, where the latter is sitting in $\mathrm{SL}_{2d}(\mathbb{R})$ in block diagonal form and $\iota(\mathrm{SL}_2(K))$ forms the $\mathbb{Q}$-points of the variety defined above after conjugation with the appropriate change of basis matrix.

As with the identification of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ with the space of covolume one lattices in $\mathbb{R}^2$, we have an identification of $\mathrm{SL}_2(\mathcal{O}_K)\backslash\mathrm{SL}_2(K_S)$ with the set, denoted X, of discrete (as subsets of K_S^2) rank two $\mathcal{O}_K$-modules with the property that for each $\Lambda \in X$ there exists a basis $\{v, w\}$ of Λ so that, for each σ, $\sigma(v)$ and $\sigma(w)$ form the sides of a parallelepiped of area one (in $\mathbb{R}^2$ or $\mathbb{C}^2$, as appropriate). Now that we have a proper embedding $\mathrm{SL}_2(\mathcal{O}_K)\backslash\mathrm{SL}_2(K_S) \to \mathrm{SL}_{2d}(\mathbb{Z})\backslash\mathrm{SL}_{2d}(\mathbb{R})$, we use Mahler's compactness criterion on the second space to derive the following statement.

THEOREM 2.2. *A subset $A \subset X$ is relatively compact if and only if there exists $\varepsilon > 0$ such that, for all $\Lambda \in A$ and for all non-zero vectors $v \in \Lambda = \tau(\mathcal{O}_K^2)g$, $\|v\| > \varepsilon$.*

For a vector v in K_S^2, denote by v^σ the projection of v onto the factor associated with the embedding σ. We define a height function $\mathrm{H} : K_S^2 \to \mathbb{R}$ by

$$\mathrm{H}(v) := \prod_\sigma \|v^\sigma\|_\sigma = \prod_{\sigma \text{ real}} \|v^\sigma\| \prod_{\sigma \text{ complex}} \|v^\sigma\|^2,$$

where we write $\|\cdot\|_\sigma$ for the norm when the place σ is real and the square of the norm when σ is complex. It will be useful to think of this height function as a measure of depth into the cusp. We wish to say that a set A is relatively compact if and only if the height function is uniformly bounded below by a positive constant over all $\mathcal{O}_K$-modules in A. To prove this statement, we first need some properties of the function H.

LEMMA 2.3. *Let $\Lambda = \tau(\mathcal{O}_K)g \in X$ and $v \in \Lambda\backslash\{0\}$. Then:*

(1) $\mathrm{H}(v) \neq 0$*; and*

(2) *for $\xi \in \mathcal{O}_K^\times$, $\mathrm{H}(\xi v) = \mathrm{H}(v)$.*

Proof. For the first property, suppose $\mathrm{H}(v) = 0$. Then $\|v^\sigma\| = \max\{|v_1^\sigma|, |v_2^\sigma|\} = 0$ for some σ. Since $v = \tau(a, b)g$ for $a, b \in \mathcal{O}_K$, and since g is invertible, we have $(\sigma(a), \sigma(b)) = (0, 0)$. Thus, $(a, b) = (0, 0)$ and hence $v = 0$. The other property follows from the product formula $\prod_{\sigma \text{ real}} |\sigma(\xi)| \prod_{\sigma \text{ complex}} |\sigma(\xi)|^2 = 1$ for units $\xi \in \mathcal{O}^*$. □

The following lemma is essentially taken from the preprint [**16**] of Kleinbock and Tomanov.

LEMMA 2.4. *There exists a constant C such that if $v \in K_S^2$ with $\mathrm{H}(v) \neq 0$ then there exists a unit ξ with*

$$C^{-1}\,\mathrm{H}(v)^{1/d} \leq \|\xi v\| \leq C\,\mathrm{H}(v)^{1/d}.$$

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Proof. Let

$$Z = \left\{ \mathbf{x} = (x^\sigma)_{\sigma \in S} \in \mathbb{R}_{>0}^{r+s} \;\middle|\; \prod_{\sigma \text{ real}} x^\sigma \prod_{\sigma \text{ complex}} (x^\sigma)^2 = 1 \right\}.$$

The morphism $\xi \mapsto (|\sigma(\xi)|)_{\sigma \in S}$ sends $\mathcal{O}_K^*$ to a subgroup of the multiplicative group Z. By the proof of the Dirichlet Unit Theorem, this is a cocompact lattice. Thus, there exists a constant C so that for any $(x^\sigma) \in Z$ there exists $\xi \in \mathcal{O}_K^*$ with

$$C^{-1} \leq |\sigma(\xi)| x^\sigma \leq C.$$

Let $v \in K_S^2$ with $\mathrm{H}(v) \neq 0$. Then the vector $(\|v^\sigma\| / \mathrm{H}(v)^{1\backslash d})$ is in Z. Applying the previous lemma, we have the claim. □

PROPOSITION 2.5. *A subset $A \subset X$ is relatively compact if and only if there exists $\delta > 0$ such that for all $\Lambda \in A$ and for all non-zero vectors $v \in \Lambda = \tau(\mathcal{O}_K^2)g$, $\mathrm{H}(v) > \delta$.*

Proof. The first implication follows from the previous lemma along with Theorem 2.2. The reverse implication is immediate from continuity of H. □

3. *Dani's correspondence*

We prove a version of Dani's correspondence for number fields. As in the introduction, we will consider here the notion of badly approximable vectors with equal weights, which, as we will now show, corresponds to the dynamics of the flow g_t.

PROPOSITION 3.1. *A vector $\mathbf{x} \in K_S$ is K-badly approximable, that is, there exists $c > 0$ such that for all $p, q \in \mathcal{O}_K$ with $q \neq 0$*

$$\max_{\sigma \in S} \{|\sigma(q)x^\sigma + \sigma(p)|\} \max_{\sigma \in S} \{|\sigma(q)|\} > c, \tag{9}$$

if and only if the trajectory

$$\left\{ \mathrm{SL}_2(\mathcal{O}_K) \left(\begin{pmatrix} 1 & x^\sigma \\ & 1 \end{pmatrix} \right)_{\sigma \in S} g_t \mid t \geq 0 \right\} \tag{10}$$

in the quotient space $\mathrm{SL}_2(\mathcal{O}_K)\backslash \mathrm{SL}_2(K_S)$ is bounded.

Proof. By Mahler's compactness criterion, we know that boundedness of the trajectory is equivalent to the existence of positive c' such that

$$\max_{\sigma \in S} \left\| \tau(q, p) \left(\begin{pmatrix} 1 & x^\sigma \\ & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & \\ & e^t \end{pmatrix} \right)_{\sigma \in S} \right\| > c'$$

or, equivalently,

$$\max_{\sigma \in S} \|(e^{-t}\sigma(q), e^t(\sigma(q)x^\sigma + \sigma(p)))\| > c'$$

for all $t \geq 0$ and all non-zero pairs $(q, p) \in \mathcal{O}_K^2$.

Assume first that the orbit is unbounded. Then there exists for every $c' > 0$ some $t \geq 0$ and some non-zero vector $(q, p) \in \mathcal{O}_K^2$ with

$$\max_{\sigma \in S} \|(e^{-t}\sigma(q), e^t(\sigma(q)x^\sigma + \sigma(p)))\| < c'.$$

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Since $t \geq 0$, note that $q = 0$ would contradict this inequality (at least for small enough c', since $p \in \mathcal{O}_K$ cannot be small at all places $\sigma \in S$). Hence $q \neq 0$. By splitting the above inequality into two inequalities for the first and second coordinates of the vectors involved and taking the product, we obtain

$$\max_{\sigma \in S}\{|\sigma(q)x^\sigma + \sigma(p)|\} \max_{\sigma \in S}\{|\sigma(q)|\} < (c')^2.$$

As c' was arbitrary, we see that the vector $(x^\sigma)_{\sigma \in S}$ is not badly approximable.

Assume now that $\mathbf{x} = (x^\sigma)_{\sigma \in S}$ is not badly approximable. Then we have by definition that for every $c > 0$ there exists $p, q \in \mathcal{O}_K$ with $q \neq 0$ such that

$$\max_{\sigma \in S}\{|\sigma(p) + x^\sigma \sigma(q)|\} \max_{\sigma \in S}\{|\sigma(q)|\} < c.$$

We choose $t = -\frac{1}{2}\log c + \log \max_{\sigma \in S}\{|\sigma(q)|\}$, which will be positive only if c is sufficiently small (as the second summand is bounded from below for $q \in \mathcal{O}_K \setminus \{0\}$). Note that this gives $\max_{\sigma \in S} e^{-t}|\sigma(q)| = c^{1/2}$. Dividing our assumed inequality by the latter equality we also get $\max_{\sigma \in S} e^t |\sigma(p) + x^\sigma \sigma(q)| < c^{1/2}$. As c was arbitrary, this shows that the orbit is not bounded. □

Notice that it is sufficient that the trajectory is bounded for a discrete sequence of times t_n where the consecutive differences are uniformly bounded.

4. *A special case*

In this section, we give proofs of Theorems 1.1 and 1.3 in a simplified linear case. In the next section, we show that a modification of this simplified argument suffices for the general C^1 case as well.

4.1. *A special case of Theorem 1.1.* We fix notation as in Theorem 1.1. Fix $0 < \alpha < \frac{1}{2}$ and let $0 < \beta < 1$. Notice that we choose α to be independent of β as required by the definition of winning. Fix the base point $\mathrm{SL}_2(\mathcal{O}_K)g \in \mathrm{SL}_2(\mathcal{O}_K)\backslash \mathrm{SL}_2(K_S)$ and denote by $\Lambda = \tau(\mathcal{O}_K^2)g$ the associated discrete $\mathcal{O}_K$-module viewed as a subset of $\mathbb{R}^{2d}$. In the simplified case, we suppose that the function Φ is such that $\phi_\sigma(x) = a_\sigma x$ with $a_\sigma \in \mathbb{R}$ and $a_\sigma \neq 0$ for sufficiently many $\sigma \in S$ as required in the theorem; to be precise, at least one half of the factors should satisfy $a_\sigma \neq 0$ with the complex places counting double. In this setting we will describe the strategy of Player B and prove that it is indeed winning.

The ultimate goal of Player B is to have the point x_∞ which remains at the end of the game satisfy that the set $\{v \in \Lambda\Phi(x_\infty)g_t \setminus \{0\} \mid t \geq 0\}$ is bounded away from zero. As remarked before, it is sufficient to have that $\{v \in \Lambda\Phi(x_\infty)g_{t_n} \setminus \{0\} \mid n \in \mathbb{N}\}$ is bounded away from zero where t_n is a positive sequence tending to infinity with bounded gaps. During each round of the game, Player B will want to monitor the short vectors that will appear in the module $\Lambda\Phi(x_n)g_{t_n}$ and play in such a way that these vectors are expanding under g_t after a finite amount of time (which needs to be independent of the short vector and the length of time the game has already been played). The initial step of the game taken by Player A may force a vector to be short for a long time, and the initial step of Player B will only make sure that the perturbed vector grows at some point in the future. However, for

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the later steps of the game it is important that we give a uniform lower bound on how short the perturbed vectors can become.

After the initial steps of the game, the nth round of the game plays out as follows: Player A has chosen a subinterval $A_n = B(x_n, \rho(\alpha\beta)^n) \subseteq B_{n-1}$. This corresponds to the collection of $\mathcal{O}_K$-modules

$$\{\Lambda\Phi(x_n + x)g_{t_n}) \mid x \in B(0, \rho(\alpha\beta)^n)\}$$

where $t_n = \frac{1}{2}\log(1/\rho(\alpha\beta)^n)$. Player B focuses on the modules associated to the midpoint of A_n, namely, $\Lambda\Phi(x_n)g_{t_n}$. If this module contains no short vectors, i.e. no non-zero $v \in \Lambda\Phi(x_n)g_{t_n}$ with $\mathrm{H}(v) < 1$, then Player B chooses the new ball B_n heedlessly, as allowed by the rules of the game. Suppose there does exist a non-zero $v \in \Lambda\Phi(x_n)g_{t_n}$ with $\mathrm{H}(v) < 1$. The strategy Player B employs in choosing the ball B makes use of the following phenomenon†.

LEMMA 4.1. *Let $v = (v_1, v_2)$ and $w = (w_1, w_2)$ be two non-zero vectors of an $\mathcal{O}_K$-module $\Delta = \tau(\mathcal{O}_K^2)h \in X$. Suppose $\mathrm{H}(v)\,\mathrm{H}(w) < 1$. Then $Kv = Kw$.*

Proof. We may write $v = (a_1, a_2)h$ and $w = (b_1, b_2)h$ with $a_1, a_2, b_1, b_2 \in \mathcal{O}_K$. Recall that Δ admits a basis as an $\mathcal{O}_K$-module whose projection to each factor corresponding to some σ determines a parallelogram with area one. Consider, then, the parallelograms formed by the projections of v and w to each factor. On the one hand, the product of their areas (respectively the areas squared for the complex places) is given by

$$\begin{aligned}\prod_\sigma \left|\det\begin{pmatrix} v^\sigma \\ w^\sigma \end{pmatrix}\right|_\sigma &= \prod_\sigma \left|\det\begin{pmatrix} v_1^\sigma & v_2^\sigma \\ w_1^\sigma & w_2^\sigma \end{pmatrix}\right|_\sigma \\ &\le \prod_\sigma \max\{|v_1^\sigma|_\sigma, |v_2^\sigma|_\sigma\}\max\{|w_1^\sigma|_\sigma, |w_2^\sigma|_\sigma\} \\ &= \mathrm{H}(v)\,\mathrm{H}(w) < 1.\end{aligned}$$

On the other hand, using the fact that $h \in \mathrm{SL}_2(K_S)$ and so $\det(h^\sigma) = 1$ for all σ, we have

$$\prod_\sigma \left|\det\begin{pmatrix} v^\sigma \\ w^\sigma \end{pmatrix}\right|_\sigma = \prod_\sigma \left|\det\begin{pmatrix} a_1^\sigma & a_2^\sigma \\ b_1^\sigma & b_2^\sigma \end{pmatrix}\right|_\sigma = \prod_\sigma |\sigma(a_1b_2 - b_1a_2)|_\sigma = |N_{K|\mathbb{Q}}(a_1b_2 - b_1a_2)|,$$

which is an integer as $a_1b_2 - b_1a_2 \in \mathcal{O}_K$. Therefore,

$$\prod_\sigma \det\begin{pmatrix} v^\sigma \\ w^\sigma \end{pmatrix} = 0,$$

implying that a and b are K-multiples of each other. As multiplication by elements of K on K_S^2 commutes with h, we see also that v and w are K-linearly dependent. □

Thus, in any round of the game Player B need only worry about a single K-span of short vectors. Define $\Phi_n(x) = g_{t_n}^{-1}\Phi(x)g_{t_n}$ so that $\Lambda\Phi(x_n + x)g_{t_n} = \Lambda\Phi(x_n)g_{t_n}\Phi_n(x)$ and the vectors in $\Lambda\Phi(x_n + x)g_{t_n}$ corresponding to the short vector $v \in \Lambda\Phi(x_n)g_{t_n}$ are $v\Phi_n(x)$. Player B wishes to choose B_n so that, for $x \in B_n$, the neighbors $v\Phi_n(x)$ are all eventually

† This is where we use that our quotient has $\mathbb{Q}$-rank one.

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expanding (or at least not further contracting) under g_{t_n}, where the notion of 'eventually' depends only on α, β, and Φ, and not on v, n, or the game play up to this point. If the height of $v\Phi_n(x)$ is growing (in a uniform way for all $x \in B_n$) under g_{t_m} for $m \geq n$, then the same holds for all other vectors in $K v\Phi_n(x)$. Moreover, by the lemma we know that for the period of time that the height of our vector is < 1 there cannot be any other vector outside the K-span with height < 1. Also note that $\Phi_n(x)$ is uniformly bounded for $|x| < \rho(\alpha\beta)^n$, which implies that all vectors $w \in \Lambda\Phi(x_n)g_{t_n}\Phi_n(x)$ of sufficiently small norm are, by the lemma, in $K v\Phi_n(x)$ and so controlled by the move of Player B.

Recall that under an application of g_t, v_1^σ is contracted by e^{-t} and v_2^σ is expanded by e^t. Consider the ratio between the expanding direction in the factor corresponding to some $\sigma \in S$ of $v\Phi_n(x)$ with $a_\sigma \neq 0$ and the norm of v^σ:

$$\frac{\left|v_1^\sigma a_\sigma e^{2t_n}x + v_2^\sigma\right|}{\|v^\sigma\|} = \left|a_\sigma(\rho(\alpha\beta)^n)^{-1}x\frac{v_1^\sigma}{\|v^\sigma\|} + \frac{v_2^\sigma}{\|v^\sigma\|}\right|,$$

where a_σ denotes the slope of Φ at σ. Also recall that x is restricted, at this stage in the game, to $x \in (-\rho(\alpha\beta)^n, \rho(\alpha\beta)^n)$. This shows that it is possible to choose x such that this ratio is bounded below by a constant $\varepsilon_\sigma > 0$ (depending on a_σ and α only). In fact, depending on the signs of v_1^σ and v_2^σ, we can choose a subinterval $B \subset (-\rho(\alpha\beta)^n, \rho(\alpha\beta)^n)$ of radius α such that the ratio is uniformly bounded from below by α. We may choose $B = (\rho(\alpha\beta)^n(1-2\alpha), \rho(\alpha\beta)^n)$ if the real part of v_2^σ/v_1^σ is non-negative, or choose $B = (-\rho(\alpha\beta)^n, -\rho(\alpha\beta)^n(1-2\alpha))$ if the real part of v_2^σ/v_1^σ is negative.

If we make this choice, then it follows that the expanding component of $(v\Phi_n(x))^\sigma$ is of norm at least $\varepsilon_\sigma\|v^\sigma\|$ for all x in the new ball, and this remains true in the future even if the vector is initially contracted. Moreover, if we know that the coordinate in the expanding direction is of norm at least $\varepsilon_\sigma\|v^\sigma\|$, then after time $t = |\log \varepsilon_\sigma|$ the expanding direction of the factor $(v\Phi_n(x)g_t)^\sigma$ will be at least as big as $\|v^\sigma\|$ for all $x \in B$.

Take $\varepsilon = \min_\sigma \varepsilon_\sigma$. If half of the factors with non-zero a_σ agree in the choice of B, then B is chosen to be the rightmost (respectively, leftmost) subinterval of $(-\rho(\alpha\beta)^n, \rho(\alpha\beta)^n)$. After this, Player A makes his move. Now we again look at all components and repeat the vote among those factors that voted differently the first time. After a uniformly bounded number of repetitions of this voting procedure, say m repetitions (during which the game moves on), we have ensured that the norm of the expanding component v_1^σ is at least $\varepsilon e^{-2t_m}\|v^\sigma\|$ for all places σ with $a_\sigma \neq 0$. Let $k = n + m$ and let B_k be the ball that is chosen by Player B at the last step; then it follows that the height of $v\Phi_k(x)g_t$ is uniformly bounded away from zero for all $x \in B_k$ and all $t \geq t_k$. Indeed, for all places $\sigma \in S$ with $a_\sigma \neq 0$ we have ensured that the expanding direction is significant in size, and even if all the remaining directions are contracted by g_t the height will be bounded away from zero (depending on ε and $\mathrm{H}(v)$).

If, during the above procedure (or later in the game), a new vector becomes of height less than one, its height is bounded away from zero by a constant depending on m. We then repeat the procedure with the new vector.

Notice that the strategy for Player B has not depended on the short vector v but on the direction of v. Moreover, this is indeed a winning strategy for Player B. The resulting point $x_\infty \in \cap A_n$ satisfies that for all $v \in \Lambda\Phi(x_\infty)$, either $\mathrm{H}(vg_{t_n}) \geq 1$ for all n, or in the first

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round n with $\mathrm{H}(vg_{t_n}) < 1$, we have that $\mathrm{H}(vg_{t_{n+m}})$ is increasing for positive n, and hence $\mathrm{H}(vg_t)$ is bounded away from zero for all $t \geq t_n$. Thus, no vector in $\{v \in \Lambda\Phi(x_\infty)g_{t_n} \backslash \{0\} \mid n \in \mathbb{N}\}$ is ever shorter than a constant depending on α, β, ε and m, except for perhaps the short vectors which appear in $\Lambda\Phi(x_\infty)$ initially.

4.2. *A counterexample.* Note that we are crucially using the fact that $a_\sigma \neq 0$ for at least half of the places σ (counting complex places double). Indeed, suppose $d = r = 3$ and, in two of the factors, we have $a_{\sigma_1} = a_{\sigma_2} = 0$. In this case, it may happen that the vector v considered above satisfies that v^{σ_1} and v^{σ_2} are contracted eigenvectors. However, in that case Player B will always lose—regardless of the choices in the game, the height of the vector corresponding to v will go to zero. As we will see below, this behavior becomes even more significant in the unequal weights case.

4.3. *Proof of Theorem 1.3 in a special case.*

Proof. The proof of Theorem 1.3, i.e. the weighted result for quadratic extensions in the linear case, follows along the same general lines as above. We explain the strategy at the nth stage following the notation in Theorem 1.3. In particular, we are now acting by $g(\mathbf{r})_t$. As before, Player A has chosen a subinterval $A_n = B(x_n, \rho(\alpha\beta)^n) \subseteq B_{n-1}$. This corresponds to the collection of $\mathcal{O}_K$-modules

$$\{\Lambda\Phi(x_n + x)g(\mathbf{r})_{t_n}) \mid x \in B(0, \rho(\alpha\beta)^n)\}$$

where $t_n = (1/2r)\log(1/\rho(\alpha\beta)^n)$ and $r = \max_{\sigma\in S} r_\sigma$.

We assume that there exists non-zero $v \in \Lambda\Phi(x_n)g(\mathbf{r})_{t_n}$ with $\mathrm{H}(v) < 1$. Let $\sigma_0 \in S$ be a place with $r_{\sigma_0} = r$. By Lemma 4.1, we again have to worry about at most one direction. Define $\Phi_n(x) = g(\mathbf{r})_{t_n}^{-1}\Phi(x)g(\mathbf{r})_{t_n}$ so that $\Lambda\Phi(x_n + x)g(\mathbf{r})_{t_n} = \Lambda\Phi(x_n)g(\mathbf{r})_{t_n}\Phi_n(x)$, denote by v the short vector, and consider $v\Phi_n(x)$. Under an application of $g(\mathbf{r})_t$, $v_1^{\sigma_0}$ is contracted by e^{-rt} and $v_2^{\sigma_0}$ is expanded by e^{rt}. In the component corresponding to σ_0 consider the ratio between the expanding coordinate $v\Phi_n(x)$ and the norm of the vector:

$$\left|\frac{v_1^{\sigma_0}a_{\sigma_0}e^{2t_n r}x + v_2^{\sigma_0}}{\|v^{\sigma_0}\|}\right| = \left|a_{\sigma_0}(\rho(\alpha\beta)^n)^{-1}x\frac{v_1^{\sigma_0}}{\|v^{\sigma_0}\|} + \frac{v_2^{\sigma_0}}{\|v^{\sigma_0}\|}\right|.$$

This ratio can be guaranteed to be larger than $|a_{\sigma_0}|(1-2\alpha)$ for all x in a subinterval $B \subset (-\rho(\alpha\beta)^n, \rho(\alpha\beta)^n)$ of radius α, choosing $B = (\rho(\alpha\beta)^n(1-2\alpha), \rho(\alpha\beta)^n)$ if $(v_2^{\sigma_0}/v_1^{\sigma_0})$ is non-negative, or choosing $B = (-\rho(\alpha\beta)^n, -\rho(\alpha\beta)^n(1-2\alpha))$ if $(v_2^{\sigma_0}/v_1^{\sigma_0})$ is negative. (If $v_1^{\sigma_0} = 0$ then the vector v^{σ_0} is an expanding eigenvector anyway and the ratio is one.)

We now argue as before. As the modified vector $v\Phi_n(x)$ for x in the new subinterval B has a significant expanding component in the place σ_0, and since this place is the one with the faster dynamics, it follows that the height of $v\Phi_n(x)g(\mathbf{r})_t$ will be ≥ 1 for $x \in B$ and for $t \geq t_0 = t_0(a_{\sigma_0}, \alpha, \mathbf{r})$. As before, since $\Phi_n(x)$ is bounded, no vector in $\Lambda\Phi(x_n + x)g(\mathbf{r})_{t_n} = \Lambda\Phi(x_n)g(\mathbf{r})_{t_n}\Phi_n(x)$ will have height much smaller than v. Once more, we obtain a winning strategy for Player B. □

In this case, once again the lower bound on derivatives which amounts to $a_\sigma \neq 0$ for *both* σ is crucial. If not, we may consider a similar example to the one in §4.2, i.e. suppose

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$a_{\sigma'} = 0$ for one factor. If now $\sigma' = \sigma_0$ for some weight $\mathbf{r}$, then the above strategy fails. Moreover, in that case it may happen that our lattice contains a vector $v = (v^{\sigma_0}, v^{\sigma_1})$ with v^{σ_0} being contracted by $e^{r_{\sigma_0} t}$ under the dynamics of $g(\mathbf{r})_t$. Even if v^{σ_1} is now expanded by $e^{r_{\sigma_1} t}$, then the height of the vector $vg(\mathbf{r})_t$ will still go to zero and this remains true for every possible outcome of the game. Hence in this case there cannot exist a winning strategy.

5. *Proof in the general case*

We now discuss the proof of Theorem 1.1 in the general case of ϕ in C^1. Recall that we have assumed that we have a subset $S' \subset S$ with $|\{\sigma \in S' \mid \sigma \text{ is real}\}| + 2|\{\sigma \in S' \mid \sigma \text{ is complex}\}| > \lfloor d/2 \rfloor$ and, for each $\sigma \in S'$, only finitely many points x have $\phi'_\sigma(x) = 0$. Here we use linear approximations to ϕ, and the strategy explained above for linear functions. At the beginning of the game, Player B acts by moving the playing field away from all points x with $\phi'_\sigma(x) = 0$ for any $\sigma \in S'$. Since this is assumed to be a finite set, this takes only finitely many rounds of the game. Thus, the game arrives at round N in the situation that for all $\sigma \in S'$, $\phi'_\sigma(x) \neq 0$ for all $x \in \overline{B_N}$. Since ϕ'_σ is uniformly continuous on $\overline{B_N}$, for each $\sigma \in S'$ there exists $m_\sigma, M_\sigma \in \mathbb{R}$ such that $0 \notin [m_\sigma, M_\sigma]$, while for all $x \in B_N$, $\phi'_\sigma(x) \in [m_\sigma, M_\sigma]$. Player B will use these bounds to produce the piecewise linear approximation to ϕ.

Suppose that in round $n > N$ there is a non-zero vector $v \in \Lambda\Phi(x_n)$ with $\mathrm{H}(vg_{t_n}) < 1$. (Again we choose $t_n = \frac{1}{2}\log \rho(\alpha\beta)^{-n}$.) Define

$$\hat{\Phi}_n(x) = g_{t_n}^{-1}\left(\begin{pmatrix} 1 & \phi_\sigma(x_n + x) - \phi_\sigma(x_n) \\ 0 & 1 \end{pmatrix}\right)_\sigma g_{t_n}.$$

Then, as before, Player B would like to choose B_n so that the 'angle' between $v\hat{\Phi}_n(x)$ and the contracting direction is significant. Thus for $v\hat{\Phi}_n(x) \in \Lambda\Phi(x_n + x)g_{t_n}$ and $\sigma \in S'$, Player B considers the ratio

$$\begin{aligned} &\left|\frac{v_1^\sigma(\phi_\sigma(x_n + x) - \phi_\sigma(x_n))e^{2t_n} + v_2^\sigma}{\|v^\sigma\|}\right| \\ &\quad = \left|(\phi_\sigma(x_n + x) - \phi_\sigma(x_n))(\rho(\alpha\beta)^n)^{-1}\frac{v_1^\sigma}{\|v^\sigma\|} + \frac{v_2^\sigma}{\|v^\sigma\|}\right|, \end{aligned} \tag{11}$$

and wishes to bound this quantity from below uniformly over x in the yet-to-be-determined B_n. Since ϕ is monotone, let us first suppose that ϕ is increasing, that is, we assume $0 < m_\sigma \leq M_\sigma$. Then on $[0, \rho(\alpha\beta)^n)$ we have that $\phi_\sigma(x_n + x) - \phi_\sigma(x_n) \geq m_\sigma x$ and on $(-\rho(\alpha\beta)^n, 0)$, $\phi_\sigma(x_n + x) - \phi_\sigma(x_n) \geq M_\sigma x$. Using these linear approximations of ϕ, Player B uses the same strategy as before. If the real part of v_2^σ / v_1^σ is non-negative, then for x in $((1 - 2\alpha)\rho(\alpha\beta)^n, \rho(\alpha\beta)^n)$,

$$\left|(\phi_\sigma(x_n + x) - \phi_\sigma(x_n))(\rho(\alpha\beta)^n)^{-1}\frac{v_1^\sigma}{\|v^\sigma\|} + \frac{v_2^\sigma}{\|v^\sigma\|}\right| \geq m_\sigma(1 - 2\alpha).$$

Similarly, if the real part of v_2^σ / v_1^σ is negative, then for x in $(-\rho(\alpha\beta)^n, -(1 - 2\alpha)\rho(\alpha\beta)^n)$,

$$\left|(\phi_\sigma(x_n + x) - \phi_\sigma(x_n))(\rho(\alpha\beta)^n)^{-1}\frac{v_1^\sigma}{\|v^\sigma\|} + \frac{v_2^\sigma}{\|v^\sigma\|}\right| \geq M_\sigma(1 - 2\alpha).$$

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Player B uses a similar analysis in the case that ϕ_σ is decreasing. So, for fixed $\sigma \in S'$, Player B can choose B_n so that, for all $x \in B_n$, the ratio between the coordinates in the factor corresponding to σ of any neighbor $v\hat{\Phi}_n(x)$ is greater than $\inf_{y\in B_N} |\phi'_\sigma(y)|(1-2\alpha)$.

As before, Player B should execute this strategy over several, say m, rounds of the game to ensure that $\mathrm{H}(v\hat{\Phi}_n(x)g_t)$ is increasing for all $t > t_m + |\log \varepsilon|$, where $\varepsilon = \min_{\sigma\in S'} \inf_{y\in\overline{B_N}} |\phi'_\sigma(y)|(1-2\alpha)$. The same reasoning as in the special case shows that this strategy is winning. The same strategy works for the more general case of Theorem 1.3.

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