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A note on normal operators

BY S. J. BERNAU† AND F. SMITHIES
Churchill College, Cambridge and St John's College, Cambridge

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1. We recall that a bounded linear operator T in a Hilbert space or finite-dimensional unitary space $\mathfrak{H}$ is said to be *normal* if T commutes with its adjoint operator T^*, i.e. $TT^* = T^*T$. Most of the proofs given in the literature for the spectral theorem for normal operators, even in the finite-dimensional case, appeal to the corresponding results for Hermitian or unitary operators.

The present note begins with a direct proof of the spectral theorems for the finite-dimensional case. We then prove the relation

$$\|T\| = \sup\{|(Tx, x)| : \|x\| = 1\}; \tag{1}$$

Halmos remarked long ago ((1), p. 111) that a direct proof of (1) for normal operators would make possible a much simpler proof of the spectral theorem. We confine ourselves here to showing how (1) can be applied to obtain the spectral theorem for compact normal operators.

2. We begin with the spectral theorem for a normal operator T in an n-dimensional unitary space E. The following lemmas are well known.

LEMMA 1. $\|Tx\| = \|T^*x\|$ $(x \in E)$.
For $\|T^*x\|^2 = (T^*x, T^*x) = (TT^*x, x) = (T^*Tx, x) = \|Tx\|^2$.

LEMMA 2. *If* $Tx = \lambda x$, *then* $T^*x = \bar{\lambda}x$.
For $T - \lambda I$ is normal with T, and $(T-\lambda I)^* = T^* - \bar{\lambda}I$. Hence

$$\|(T^* - \bar{\lambda}I)x\| = \|(T-\lambda I)x\| = 0.$$

THEOREM 1. *There is an orthonormal base* $(x_1, x_2, \ldots, x_n)$ *of* E *consisting entirely of eigenvectors of* T.

We proceed by induction over $n = \dim E$. The result clearly holds when $n = 1$; we suppose that it holds when $\dim E = n-1$, and deduce it for the case $\dim E = n$.

Let λ_1 be an eigenvalue of T, and x_1 a corresponding eigenvector with $\|x_1\| = 1$; we have $Tx_1 = \lambda_1 x_1$. Let $M = \{x: (x, x_1) = 0\}$ be the orthogonal complement of the subspace spanned by x_1. If $x \in M$, then

$$(Tx, x_1) = (x, T^*x_1) = (x, \bar{\lambda}x_1) = \lambda(x, x_1) = 0,$$

i.e. $Tx \in M$; also $\quad (T^*x, x_1) = (x, Tx_1) = (x, \lambda x_1) = \bar{\lambda}(x, x_1) = 0,$

i.e. $T^*x \in M$; thus M is invariant under both T and T^*. It follows that the restriction of T to M is a normal operator in M; since $\dim M = n-1$, we can apply the inductive

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hypothesis, obtaining an orthonormal base $(x_2, \ldots, x_n)$ of M consisting of eigenvectors of T. Clearly $(x_1, x_2, \ldots, x_n)$ is an orthonormal base of E, and the result is proved.

3. We now come to the proof of the relation (1). We suppose that $\mathfrak{H}$ is a complex Hilbert space, and that T is a bounded linear operator in $\mathfrak{H}$.

LEMMA 3. *Let* $\rho(T) = \sup\{|(Tx, x)| : \|x\| = 1\}$; *then*

$$\|Tx\|^2 + |(T^2x, x)| \leqslant 2\rho(T)\,\|Tx\|\,.\,\|x\| \quad (x \in \mathfrak{H}). \tag{2}$$

Let λ and θ be real numbers, $\lambda \neq 0$. We then have the identity

$$\|Tx\|^2 + e^{2i\theta}(T^2x, x) = \tfrac{1}{2}(\lambda e^{2i\theta} T^2x + \lambda^{-1} e^{i\theta} Tx, \lambda e^{i\theta} Tx + \lambda^{-1}x) - \tfrac{1}{2}(\lambda e^{2i\theta} T^2x - \lambda^{-1} e^{i\theta} Tx, \lambda e^{i\theta} Tx - \lambda^{-1}x);$$

since $|(Ty, y)| \leqslant \rho(T)\,\|y\|^2$ for all y, it follows that

$$\begin{aligned} |\|Tx\|^2 + e^{2i\theta}(T^2x, x)| &\leqslant \tfrac{1}{2}\rho(T)(\|\lambda e^{i\theta} Tx + \lambda^{-1}x\|^2 + \|\lambda e^{i\theta} Tx - \lambda^{-1}x\|^2) \\ &= \rho(T)(\lambda^2 \|Tx\|^2 + \lambda^{-2}\|x\|^2). \end{aligned}$$

If $Tx \neq 0$, we choose θ so that $e^{2i\theta}(T^2x, x) = |(T^2x, x)|$, and λ so that $\lambda^2\|Tx\| = \|x\|$; the inequality then becomes

$$\|Tx\|^2 + |(T^2x, x)| \leqslant 2\rho(T)\,\|Tx\|\,.\,\|x\|,$$

the required result. Clearly this holds also when $Tx = 0$.

LEMMA 4. *We have*

$$(a) \quad \|T\| \leqslant 2\rho(T),$$
$$(b) \quad \rho(T^2) \leqslant [\rho(T)]^2.$$

From (2) we have at once $\|Tx\|^2 \leqslant 2\rho(T)\,\|Tx\|\,.\,\|x\|$, whence $\|Tx\| \leqslant 2\rho(T)\,\|x\|$ for all x, $\|T\| \leqslant 2\rho(T)$, which is (a). To obtain (b), we take $\|x\| = 1$ in (2), and rearrange to get

$$[\|Tx\| - \rho(T)]^2 + |(T^2x, x)| \leqslant [\rho(T)]^2,$$

whence

$$|(T^2x, x)| \leqslant [\rho(T)]^2 \quad (\|x\| = 1),$$

and therefore $\rho(T^2) \leqslant [\rho(T)]^2$.

THEOREM 2. *Let* T *be a bounded normal operator in* $\mathfrak{H}$; *then* $\|T\| = \rho(T)$.

Since $|(Tx, x)| \leqslant \|T\|\,.\,\|x\|^2$, we have $\rho(T) \leqslant \|T\|$ in any case. If T is normal, we have $\|T^2\| = \|T\|^2$; for, by Lemma 1,

$$\|T^2x\| = \|T^*Tx\|,$$

whence $\|T^2\| = \|T^*T\| = \|T\|^2$ ((2), p. 250). Hence, by induction, if $p = 2^k$, we have $\|T^p\| = \|T\|^p$; it also follows from Lemma 4(b) that $\rho(T^p) \leqslant [\rho(T)]^p$. Consequently, by Lemma 4(a),

$$\|T\| = \|T^p\|^{1/p} \leqslant [2\rho(T^p)]^{1/p} \leqslant 2^{1/p}\rho(T);$$

letting $k \to \infty$, we obtain $\|T\| \leqslant \rho(T)$, which gives the result.

4. To prove the spectral theorem for compact normal operators, the first essential step, as in the Hermitian case, is to establish the existence of eigenvalues.

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THEOREM 3. *Let T be a non-zero compact normal operator; then T has at least one non-zero eigenvalue.*

By Theorem 2, there is a sequence (x_n) in $\mathfrak{H}$ such that $\|x_n\| = 1$ for all n and

$$\lim_{n\to\infty} |(Tx_n, x_n)| = \|T\|.$$

By taking an appropriate subsequence, if necessary, we may suppose that

$$\lim_{n\to\infty} (Tx_n, x_n) = \lambda,$$

where $|\lambda| = \|T\| > 0$; since T is a compact operator, we may also suppose that (Tx_n) is convergent, say to y. Since $\|Tx_n\| \leqslant \|T\|$, it follows that $\|y\| \leqslant \|T\| = |\lambda|$.

We now have

$$\begin{aligned}\|Tx_n - \lambda x_n\|^2 &= (Tx_n, Tx_n) - \lambda(x_n, Tx_n) - \bar{\lambda}(Tx_n, x_n) + \lambda\bar{\lambda}(x_n, x_n)\\ &\to \|y\|^2 - \lambda\bar{\lambda} - \bar{\lambda}\lambda + \lambda\bar{\lambda} \quad (n \to \infty)\\ &= \|y\|^2 - |\lambda|^2\\ &\leqslant 0.\end{aligned}$$

Since the left-hand side is non-negative for all n, we must have $Tx_n - \lambda x_n \to 0$ $(n \to \infty)$ and $\|y\| = |\lambda| > 0$; since $Tx_n \to y$ and $\lambda \neq 0$, we deduce that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \lambda^{-1} Tx_n = \lambda^{-1} y,$$

whence

$$Ty = \lambda \lim_{n\to\infty} Tx_n = \lambda y,$$

i.e. λ is a non-zero eigenvalue of T, as required.

The spectral theorem for compact normal operators can now be obtained in the same way as for compact Hermitian operators (Taylor (2), pp. 335–337), by means of Theorem 3 and arguments analogous to those of § 2 above.

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