DYNAMIC RESPONSES OF TWO BEAMS CONNECTED BY A SPRING-MASS DEVICE

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ABSTRACT

This paper deals with the transverse free vibrations of a system in which two beams are coupled with a spring-mass device. The dynamics of this system are coupled through the motion of the mass. The entire system is modeled as two two-span beams and each span of the continuous beams is assumed to obey the Euler-Bernoulli beam theory. Considering the compatibility requirements across each spring connection position, the eigensolutions (natural frequencies and mode shapes) of this system can be obtained for different boundary conditions. Some numerical results and experimental validations are presented to demonstrate the method proposed in this article.

Keywords: Beam structures, Spring-mass system, Eigensolutions, Natural frequencies, Mode shapes.

1. INTRODUCTION

A lot of researches on the system dynamics of a beam carrying spring-mass devices have been studied extensively [1-9]. The natural frequencies of this system may deviate considerably from those of the beam itself. The solutions of this problem had been obtained by means of the analytical methods [1-3], or the numerical methods [4,5]. Gürgöze [6] used the Lagrange multiplier formalism to obtain the natural frequencies of a beam to which several spring-mass devices are attached. Wu and Chou [7] introduced a numerical technique to obtain the lower vibration modes of a uniform beam carrying spring-mass devices. Cha [8] discretized the governing equation for a linear elastica carrying a number of sprung masses by using the assumed-modes method. Bambill and Rossit [9] solved the forced vibrations of a beam carrying a spring-mass device.

Some of the researches studied the dynamic responses of two rods or beams with spring-mass devices. Kukla *et al*. [10] studied the dynamics of the longitudinal vibrations of two rods coupled by several translational springs by using Green's functional method. Gürgöze [11] used the assumed modes method to study the frequency equation of the longitudinally vibrating rods coupled by a double spring-mass system. Mermertas and Gürgöze [12] investigated a system with two clamped-free axially vibrating rods carrying tip masses and coupled by a double spring mass system. Inceoğlu and Gürgöze [13] extended the previous studies [12] by considering the longitudinal vibrations of rods coupled by several double spring-mass systems by using Green's function. Gürgöze *et al*. [14] treated the bending vibration of beams coupled by a double spring-mass system. Inceoğlu and Gürgöze [15] also considered similar problems in beams coupled by several double spring-mass systems.

This article presents a hybrid analytical/numerical method that permits an efficient computation of the eigensolutions for a system of two beams coupled by a spring-mass device. In this research, the entire system is modeled as two two-span beams and each span of the continuous beams is assumed to obey the Euler-Bernoulli beam theory. Considering the compatibility requirements across each spring connection position, the eigensolutions (natural frequencies and mode shapes) of this system can be obtained for different boundary conditions. Gürgöze *et al*. [14] solved this system and obtained the solutions from a 13×13 determinant matrix. Although it leaded to the exact solution, its scheme was highly numerically intensive, since the dimensional size of the characteristic determinant obtained is relatively high. However, by using the method proposed in this article, the characteristic equations for different boundary conditions are reduced to a determinant matrix of only 4×4.

2. THEORETICAL MODEL

Two Euler-Bernoulli beams of length L_1 and L_2 connected by a spring-mass system is considered as in Fig. 1. It is assumed that the spring is located at positions

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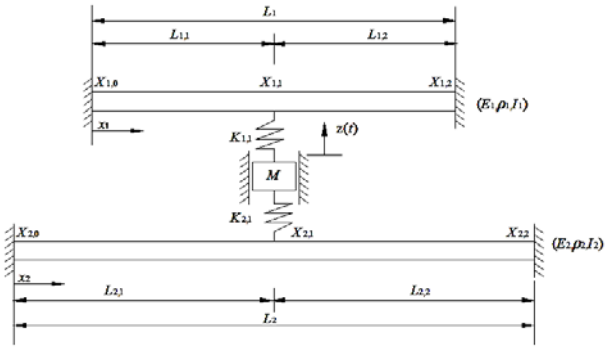


Fig. 1 Two beams with lengths L_1 and L_2 connected with a spring-mass devices; $L_{1,1}$, $L_{1,2}$ and $L_{2,1}$, $L_{2,2}$ are lengths of segments separated by spring connection positions $X_{1,1}$ and $X_{2,1}$; M is attached masses with displacements $z(t)$, respectively

$X_{1,1}$ and $X_{2,1}$ and with stiffness $K_{1,1}$ and $K_{2,1}$ respectively. $X_{1,0}$, $X_{2,0}$, $X_{1,2}$ and $X_{2,2}$ represent end-supported points; finally, $L_{1,1}$, $L_{1,2}$ and $L_{2,1}$, $L_{2,2}$ denote the lengths for each segment separated by the spring connection positions. The attached mass is M. The transverse displacements of the beams and the displacement of the mass are denoted by $y_1(x_1, t)$, $y_2(x_2, t)$, and $z(t)$ where x_1 and x_2 represent the space coordinates of beams respectively. By using the Euler-Bernoulli beam theory, the equation of motion, assumed to have a uniform cross section, is:

$$E_i I_i \frac{\partial^4 y_i(x_i, t)}{\partial x_i^4} + \rho_i A_i \frac{\partial^2 y_i(x_i, t)}{\partial t^2} = 0 , \quad i = 1, 2 \tag{1}$$

$$M \ddot{z}(t) + (K_{1,1} + K_{2,1}) z(t) = K_{1,1} y_1(X_{1,1}, t) + K_{2,1} y_2(X_{2,1}, t) , \tag{2}$$

where E_i is Young's modulus of the material, I_i is the moment of inertia of cross-sections of the beams, ρ_i is the density of material, A_i is the cross-sectional area of the beams respectively, and t is time. From Eq. (2), it is noted that the dynamics of the beams is coupled though the spring-mass device. The boundary conditions of the beams for a simply-supported case are:

$$y_i(0, t) = y_i(L_i, t) = 0 , \tag{3a}$$

$$y_i''(0, t) = y_i''(L_i, t) = 0 , \quad i = 1, 2 \tag{3b}$$

where the symbol (′) denotes the derivative with respect to the space coordinate x_i.

The "compatibility conditions" enforce continuities of the displacement field, the slope and the bending moment, respectively, across the spring connection position and can be expressed [16,17]:

$$y_i(X_{i,1}^-, t) = y_i(X_{i,1}^+, t) , \tag{4a}$$

$$y_i'(X_{i,1}^-, t) = y_i'(X_{i,1}^+, t) , \tag{4b}$$

$$y_i''(X_{i,1}^-, t) = y_i''(X_{i,1}^+, t) , \quad i = 1, 2 \tag{4c}$$

where the symbols $X_{i,1}^-$ and $X_{i,1}^+$ denote the locations immediately left and right of the spring connection positions $X_{i,1}$. Moreover, a discontinuity on the shear forces of the beam across the spring connection position exists and can be expressed [16,17] as:

$$y_i'''(X_{i,1}^-, t) = y_i'''(X_{i,1}^+, t) - \frac{K_{i,1}}{E_i I_i}\left(z(t) - y_i(X_{i,1}^-, t)\right) , \quad i = 1, 2 \tag{4d}$$

where $K_{i,1}$ is the spring stiffness connected to beam i.

In the above, the following quantities are introduced:

$$\bar{y}_i = \frac{y_i}{L_i} , \ \bar{x}_i = \frac{x_i}{L_i} , \ x_{i,1} = \frac{X_{i,1}}{L_i} , \quad \bar{z} = \frac{z}{L} , \ l_{ij} = \frac{L_{i,j}}{L_i} , \ \bar{t} = \frac{t}{\sqrt{L}} . \tag{5a ~ 5f}$$

where $L = L_1 + L_2$. Thus, Eqs. (1) and (2) can then be expressed in the non-dimensional forms as:

$$\frac{E_i I_i}{L_i^3} \frac{\partial^4 \bar{y}_i(\bar{x}_i, \bar{t})}{\partial \bar{x}_i^4} + \rho_i A_i \frac{L_i}{L} \frac{\partial^2 \bar{y}_i(\bar{x}_i, \bar{t})}{\partial \bar{t}^2} = 0 , \quad \bar{x}_i \neq x_{i,k} , \ i = 1, 2 \tag{6}$$

$$\ddot{\bar{z}}(\bar{t}) + \frac{(K_{1,1} + K_{2,1})L}{M} \bar{z}(\bar{t}) = \frac{K_{1,1} L_1}{M} \bar{y}_1(x_{1,1}, \bar{t}) + \frac{K_{2,1} L_2}{M} \bar{y}_2(x_{2,1}, \bar{t}) . \tag{7}$$

The non-dimensional "compatibility conditions" from Eqs. (4a) to (4d) are:

$$\bar{y}_i(x_{i,1}^-, \bar{t}) = \bar{y}_i(x_{i,1}^+, \bar{t}) , \tag{8a}$$

$$\bar{y}_i'(x_{i,1}^-, \bar{t}) = \bar{y}_i'(x_{i,1}^+, \bar{t}) , \tag{8b}$$

$$\bar{y}_i''(x_{i,1}^-, \bar{t}) = \bar{y}_i''(x_{i,1}^+, \bar{t}) , \tag{8c}$$

$$\bar{y}_i'''(x_{i,1}^-, \bar{t}) = \bar{y}_i'''(x_{i,1}^+, \bar{t}) - k_{i,1}\left[\frac{L}{L_i}\bar{z}(\bar{t}) - \bar{y}_i(x_{i,1}^-, \bar{t})\right] , \quad i = 1, 2 \tag{8d}$$

where $k_{i,1} = \dfrac{K_{i,1} L_i^3}{E_i I_i}$ is the non-dimensional spring constant.

3. FREE RESPONSES

The eigensolutions for the case of simply supported boundary conditions are derived. The solutions of the other boundary conditions can also be obtained easily through the similar procedure. By using the separable solutions $\bar{y}_i(\bar{x}_i, \bar{t}) = w_i(\bar{x}_i)\, e^{j\omega\bar{t}}$ and $\bar{z}(\bar{t}) = s\, e^{j\omega\bar{t}}$, Eqs. (6) and (7) lead to the following differential equations of the associated eigenvalue problem:

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$$w_i''''(\bar{x}_i) - \lambda_i^4 w_i(\bar{x}_i) = 0 \ , \ \bar{x} \neq x_i \ , \ i = 1, \ 2 \tag{9}$$

$$\left(-\omega^2 + \frac{(K_{1,1} + K_{2,1})L}{M} \right) s = \frac{K_{1,1} L_1}{M} w_1(x_{1,1}) + \frac{K_{2,1} L_2}{M} w_2(x_{2,1}) \ , \tag{10}$$

where $\lambda_i^4 = \dfrac{\rho_i A_i L_i^3 \omega^2}{E_i I_i} \dfrac{L_i}{L}$.

Thus, the relationship between λ_1 and λ_2 can be obtained as

$$\lambda_1^4 = \frac{\rho_1 A_1 L_1^3 \omega^2}{E_1 I_1} \frac{L_1}{L} \ , \tag{11a}$$

$$\lambda_2^4 = \frac{\rho_2 A_2 L_2^3 \omega^2}{E_2 I_2} \frac{L_2}{L} = \frac{E_1 I_1}{E_2 I_2} \frac{\rho_2 A_2 L_2^3}{\rho_1 A_1 L_1^3} \frac{L_2}{L_1} \quad \lambda_1^4 = \beta^4 \lambda_1^4 \ , \tag{11b}$$

where $\beta = \left(\dfrac{E_1 I_1}{E_2 I_2} \dfrac{\rho_2 A_2 L_2^3}{\rho_1 A_1 L_1^3} \dfrac{L_2}{L_1} \right)^{1/4}$. (11c)

From Eqs. (8a) to (8d), the corresponding compatibility conditions across each spring connection position lead to:

$$w_i(x_{i,1}^-) = w_i(x_{i,1}^+) \ , \tag{12a}$$

$$w_i'(x_{i,1}^-) = w_i'(x_{i,1}^+) \ , \tag{12b}$$

$$w_i''(x_{i,1}^-) = w_i''(x_{i,1}^+) \ , \tag{12c}$$

$$w_i'''(x_{i,1}^-) = w_i'''(x_{i,1}^+) - k_{i,1} \left[\frac{L}{L_i} s - w_i(x_{i,1}^-) \right] \ , \ i = 1, \ 2 \ . \tag{12d}$$

For the case of simply-supported beams, the corresponding boundary conditions of Eqs. (3a) and (3b) can thus be expressed as:

$$w_i(0) = 0 \ , \tag{13a}$$

$$w_i(1) = 0 \ , \tag{13b}$$

$$w_i''(0) = 0 \ , \tag{13c}$$

$$w_i''(1) = 0 \ . \tag{13d}$$

By dividing each beam into two segments separated by the spring connection positions (see Fig. 1), the general solution of Eq. (9), for each segment, is

$$\begin{aligned} w_{ij}(\bar{x}_i) = {} & A_{ij} \sin \lambda_i (\bar{x}_i - x_{i,j-1}) + B_{ij} \cos \lambda_i (\bar{x}_i - x_{i,j-1}) \\ & + C_{ij} \sinh \lambda_i (\bar{x}_i - x_{i,j-1}) + D_{ij} \cosh \lambda_i (\bar{x}_i - x_{i,j-1}) \\ & x_{i,j-1} < \bar{x}_i < x_{i,j} \ , \ i = 1, \ 2 \ , \ j = 1, \ 2 \end{aligned} \tag{14}$$

where A_{ij}, B_{ij}, C_{ij} and D_{ij} are constants associated with the j-th segment (j = 1, 2) for beam i (i = 1, 2). These constants in the second segment (A_{i2}, B_{i2}, C_{i2} and D_{i2}) are related to those in the first segment (A_{i1}, B_{i1}, C_{i1} and D_{i1}) through the compatibility conditions in Eqs. (12a) to (12d) and can be expressed as

$$\begin{Bmatrix} A_{12} \\ B_{12} \\ C_{12} \\ D_{12} \\ A_{22} \\ B_{22} \\ C_{22} \\ D_{22} \end{Bmatrix} = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{18} \\ \vdots & & & \\ t_{81} & \cdots\cdots & \cdots & t_{88} \end{bmatrix} \begin{Bmatrix} A_{11} \\ B_{11} \\ C_{11} \\ D_{11} \\ A_{21} \\ B_{21} \\ C_{21} \\ D_{21} \end{Bmatrix} = \underline{T}_{8\times 8} \begin{Bmatrix} A_{11} \\ B_{11} \\ C_{11} \\ D_{11} \\ A_{21} \\ B_{21} \\ C_{21} \\ D_{21} \end{Bmatrix} . \tag{15}$$

where $\underline{T}_{8\times 8}$ is the 8 × 8 transfer matrix which depends on the eigenvalue λ and the elements are derived in the Appendix A.

Through the application of Eq. (15), the eight constants in the first segments (A_{11}, B_{11}, C_{11}, D_{11}, A_{21}, B_{21}, C_{21} and D_{21}) of both beams can be mapped into those of the second segments, thereby, reducing the number of independent constants to eight. These eight remaining constants can be found through the satisfaction of boundary conditions.

For the case of simply-supported beams, the corresponding boundary conditions are shown in Eqs. (13a) to (13d). Beginning with those at the left support, Eqs. (14), (13a) and (13c), leads to

$$B_{11} = D_{11} = 0 \ . \tag{16a}$$

$$B_{21} = D_{21} = 0 \ . \tag{16b}$$

Satisfaction of the boundary conditions of Eq. (14) at the right supports, Eqs. (13b) and (13d) requires

$$\begin{aligned} & A_{12} \sin \lambda_1 l_{12} + B_{12} \cos \lambda_1 l_{12} \\ & \quad + C_{12} \sinh \lambda_1 l_{12} + D_{12} \cosh \lambda_1 l_{12} = 0 \ , \end{aligned} \tag{17a}$$

$$\begin{aligned} & -A_{12} \sin \lambda_1 l_{12} - B_{12} \cos \lambda_1 l_{12} + C_{12} \sinh \lambda_1 l_{12} \\ & \quad + D_{12} \cosh \lambda_1 l_{12} = 0 \ , \end{aligned} \tag{17b}$$

$$\begin{aligned} & A_{22} \sin \lambda_2 l_{22} + B_{22} \cos \lambda_2 l_{22} \\ & \quad + C_{22} \sinh \lambda_2 l_{12} + D_{22} \cosh \lambda_2 l_{22} = 0 \ , \end{aligned} \tag{17c}$$

$$\begin{aligned} & -A_{22} \sin \lambda_2 l_{22} - B_{22} \cos \lambda_2 l_{22} \\ & \quad + C_{22} \sinh \lambda_2 l_{12} + D_{22} \cosh \lambda_2 l_{22} = 0 \ . \end{aligned} \tag{17d}$$

The above equations (17a ~ 17d) can be expressed in matrix form as follows:

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$$\begin{Bmatrix}0\\0\\0\\0\end{Bmatrix}=\begin{bmatrix}\sin\lambda_1 l_{12} & \cos\lambda_1 l_{12} & \sinh\lambda_1 l_{12} & \cosh\lambda_1 l_{12} & 0 & 0 & 0 & 0\\ -\sin\lambda_1 l_{12} & -\cos\lambda_1 l_{12} & \sinh\lambda_1 l_{12} & \cosh\lambda_1 l_{12} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \sin\lambda_2 l_{22} & \cos\lambda_2 l_{22} & \sinh\lambda_2 l_{22} & \cosh\lambda_2 l_{22}\\ 0 & 0 & 0 & 0 & -\sin\lambda_2 l_{22} & -\cos\lambda_2 l_{22} & \sinh\lambda_2 l_{22} & \cosh\lambda_2 l_{22}\end{bmatrix}\begin{Bmatrix}A_{12}\\B_{12}\\C_{12}\\D_{12}\\A_{22}\\B_{22}\\C_{22}\\D_{22}\end{Bmatrix}$$

$$=\underline{\mathrm{B}}_{4\times 8}\begin{Bmatrix}A_{12}\\B_{12}\\C_{12}\\D_{12}\\A_{22}\\B_{22}\\C_{22}\\D_{22}\end{Bmatrix}, \tag{18a}$$

where $\underline{\mathrm{B}}_{4\times 8}=\begin{bmatrix}\sin\lambda_1 l_{12} & \cos\lambda_1 l_{12} & \sinh\lambda_1 l_{12} & \cosh\lambda_1 l_{12} & 0 & 0 & 0 & 0\\ -\sin\lambda_1 l_{12} & -\cos\lambda_1 l_{12} & \sinh\lambda_1 l_{12} & \cosh\lambda_1 l_{12} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \sin\lambda_2 l_{22} & \cos\lambda_2 l_{22} & \sinh\lambda_2 l_{22} & \cosh\lambda_2 l_{22}\\ 0 & 0 & 0 & 0 & -\sin\lambda_2 l_{22} & -\cos\lambda_2 l_{22} & \sinh\lambda_2 l_{22} & \cosh\lambda_2 l_{22}\end{bmatrix}$. (18b)

Substitution of Eq. (15) into Eq. (18a) and use of Eqs. (16a,b) leads to

$$\begin{Bmatrix}0\\0\\0\\0\end{Bmatrix}=\underline{\mathrm{B}}_{4\times 8}\begin{Bmatrix}A_{12}\\B_{12}\\C_{12}\\D_{12}\\A_{22}\\B_{22}\\C_{22}\\D_{22}\end{Bmatrix}=\underline{\mathrm{B}}_{4\times 8}\,\underline{\mathrm{T}}_{8\times 8}\begin{Bmatrix}A_{11}\\B_{11}\\C_{11}\\D_{11}\\A_{21}\\B_{21}\\C_{21}\\D_{21}\end{Bmatrix}=\underline{\mathrm{R}}_{4\times 8}\begin{Bmatrix}A_{11}\\B_{11}\\C_{11}\\D_{11}\\A_{21}\\B_{21}\\C_{21}\\D_{21}\end{Bmatrix}=\begin{bmatrix}r_{11} & r_{12} & \cdots & r_{18}\\ \vdots & & & \\ r_{41} & r_{42} & \cdots & r_{48}\end{bmatrix}\begin{Bmatrix}A_{11}\\0\\C_{11}\\0\\A_{21}\\0\\C_{21}\\0\end{Bmatrix}, \tag{19a}$$

where $\underline{\mathrm{R}}_{4\times 8}=\underline{\mathrm{B}}_{4\times 8}\,\underline{\mathrm{T}}_{8\times 8}=\begin{bmatrix}r_{11} & r_{12} & \cdots & r_{18}\\ \vdots & & & \\ r_{41} & r_{42} & \cdots & r_{48}\end{bmatrix}$. (19b)

Thus, the existence of non-trivial solutions of Eq. (19a) requires:

$$\det\begin{vmatrix}r_{11}(\omega) & r_{13}(\omega) & r_{15}(\omega) & r_{17}(\omega)\\ r_{21}(\omega) & r_{23}(\omega) & r_{25}(\omega) & r_{27}(\omega)\\ r_{31}(\omega) & r_{33}(\omega) & r_{35}(\omega) & r_{37}(\omega)\\ r_{41}(\omega) & r_{43}(\omega) & r_{45}(\omega) & r_{47}(\omega)\end{vmatrix}=0\,. \tag{20}$$

This determinant provides the single (characteristic) equation for the solution of the eigenvalue ω_n. The coefficients of the eigenfunctions, $w_{i,n}(x)$, are obtained by back substitution into Eqs. (19a), (15) and then Eq. (14).

For the cases of other usually used boundary conditions, through the similar procedure, the following relations can be obtained.

a). **Two fixed-free supported beams**: The existence of non-trivial solutions for the constants A_{11}, B_{11}, C_{11}, D_{11}, A_{21}, B_{21}, C_{21} and D_{21} requires

$$\det\begin{vmatrix}r_{11}-r_{13} & r_{12}-r_{14} & r_{15}-r_{17} & r_{16}-r_{18}\\ r_{21}-r_{23} & r_{22}-r_{24} & r_{25}-r_{27} & r_{26}-r_{28}\\ r_{31}-r_{33} & r_{32}-r_{34} & r_{35}-r_{37} & r_{36}-r_{38}\\ r_{41}-r_{43} & r_{42}-r_{44} & r_{45}-r_{47} & r_{46}-r_{48}\end{vmatrix}=0\,. \tag{21}$$

The matrix $\underline{\mathrm{B}}_{4\times 8}$ in Eq. (18a) now becomes

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$$\underline{\mathrm{B}}_{4\times 8}=\begin{bmatrix} -\cos\lambda_1 l_{12} & \sin\lambda_1 l_{12} & \cosh\lambda_1 l_{12} & \sinh\lambda_1 l_{12} & 0 & 0 & 0 & 0 \\ -\sin\lambda_1 l_{12} & -\cos\lambda_1 l_{12} & \sinh\lambda_1 l_{12} & \cosh\lambda_1 l_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\cos\lambda_2 l_{22} & \sin\lambda_2 l_{22} & \cosh\lambda_2 l_{22} & \sinh\lambda_2 l_{22} \\ 0 & 0 & 0 & 0 & -\sin\lambda_2 l_{22} & -\cos\lambda_2 l_{22} & \sinh\lambda_2 l_{22} & \cosh\lambda_2 l_{22} \end{bmatrix}. \tag{22}$$

b). **Two fixed-fixed supported beams**: The existence of non-trivial solutions is the same as Eq. (21) but the matrix $\underline{\mathrm{B}}_{4\times 8}$ in Eq. (18a) now becomes

$$\underline{\mathrm{B}}_{4\times 8}=\begin{bmatrix} \sin\lambda_1 l_{12} & \cos\lambda_1 l_{12} & \sinh\lambda_1 l_{12} & \cosh\lambda_1 l_{12} & 0 & 0 & 0 & 0 \\ \cos\lambda_1 l_{12} & -\sin\lambda_1 l_{12} & \cosh\lambda_1 l_{12} & \sinh\lambda_1 l_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin\lambda_2 l_{22} & \cos\lambda_2 l_{22} & \sinh\lambda_2 l_{22} & \cosh\lambda_2 l_{22} \\ 0 & 0 & 0 & 0 & \cos\lambda_2 l_{22} & -\sin\lambda_2 l_{22} & \cosh\lambda_2 l_{22} & \sinh\lambda_2 l_{22} \end{bmatrix}. \tag{23}$$

c). **Two beams with left ends fixed and right ends attached to a mass**: The boundary conditions on the right ends are

$$E_i\, I_i\, y_i''(L_i,\, t)=M_{si}\, \ddot{y}_i(L_i,\, t)\ ,\quad i=1,2\ , \tag{24}$$

where M_{si} is the attached mass and Eq. (24) can be transferred as

$$w_i''(1)=-\frac{M_{si}\,\omega^2}{E_i\, I_i}\frac{L_i^3}{L}w_i(1)\ ,\quad i=1,\ 2\ . \tag{25}$$

Equation (25) leads to

$$\begin{aligned}
&\left(-\cos\lambda_1 l_{12}+\frac{M_{s1}}{\rho_1 A_1 L_1}\lambda_1\sin\lambda_1 l_{12}\right)A_{12}+\left(\sin\lambda_1 l_{12}+\frac{M_{s1}}{\rho_1 A_1 L_1}\lambda_1\cos\lambda_1 l_{12}\right)B_{12}\\
&\quad+\left(\cosh\lambda_1 l_{12}+\frac{M_{s1}}{\rho_1 A_1 L_1}\lambda_1\sinh\lambda_1 l_{12}\right)C_{12}+\left(\sinh\lambda_1 l_{12}+\frac{M_{s1}}{\rho_1 A_1 L_1}\lambda_1\cosh\lambda_1 l_{12}\right)D_{12}=0\\
&\left(-\cos\lambda_2 l_{22}+\frac{M_{s2}}{\rho_2 A_2 L_2}\lambda_2\sin\lambda_2 l_{22}\right)A_{22}+\left(\sin\lambda_2 l_{22}+\frac{M_{s2}}{\rho_2 A_2 L_2}\lambda_2\cos\lambda_2 l_{22}\right)B_{22}\\
&\quad+\left(\cosh\lambda_2 l_{22}+\frac{M_{s2}}{\rho_2 A_2 L_2}\lambda_2\sinh\lambda_2 l_{22}\right)C_{22}+\left(\sinh\lambda_2 l_{22}+\frac{M_{s2}}{\rho_2 A_2 L_2}\lambda_2\cosh\lambda_2 l_{22}\right)D_{22}=0\ .
\end{aligned}$$

Thus, matrix $\underline{\mathrm{B}}_{2\times 4}$ now becomes

$$\underline{\mathrm{B}}_{2\times 4}=\begin{bmatrix} -\sin\lambda_1 l_{12} & -\cos\lambda_1 l_{12} & \sinh\lambda_1 l_{12} & \cosh\lambda_1 l_{12} \\ -\cos\lambda_1 l_{12}+m_{s1}\lambda_1\sin\lambda_1 l_{12} & \sin\lambda_1 l_{12}+m_{s1}\lambda_1\cos\lambda_1 l_{12} & \cosh\lambda_1 l_{12}+m_{s1}\lambda_1\sinh\lambda_1 l_{12} & \sinh\lambda_1 l_{12}+m_{s1}\lambda_1\cosh\lambda_1 l_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\sin\lambda_2 l_{22} & -\cos\lambda_2 l_{22} & \sinh\lambda_2 l_{22} & \cosh\lambda_2 l_{22} \\ -\cos\lambda_2 l_{22}+m_{s2}\lambda_2\sin\lambda_2 l_{22} & \sin\lambda_2 l_{22}+m_{s2}\lambda_2\cos\lambda_2 l_{22} & \cosh\lambda_2 l_{22}+m_{s2}\lambda_2\sinh\lambda_2 l_{22} & \sinh\lambda_2 l_{22}+m_{s2}\lambda_2\cosh\lambda_2 l_{22} \end{bmatrix} \tag{26}$$

where $m_{s1}=\dfrac{M_{s1}}{\rho_1 A_1 L_1}$ and $m_{s2}=\dfrac{M_{s2}}{\rho_2 A_2 L_2}$.

The existence of non-trivial solutions is also the same as Eq. (21) and the matrix $\underline{\mathrm{B}}_{4\times 8}$ is in Eq. (26).

4. NUMERICAL RESULTS AND DISCUSSION

The proposed methods in this article are used to solve the free responses of two beams coupled with a spring-mass system. The method for obtaining the free responses (eigenvalues) is that of finding the non-trivial solutions of the determinants in Eqs. (20) and (21) for various boundary conditions. These are nonlinear algebraic equations which can be solved by using the standard Newton-Raphson iterations or other numerical methods.

In order to show the method used in this article, some numerical examples are presented and compared with the existing literature. First is the case of two

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cantilever beams with tip masses coupled with a spring-mass system as in reference [14]. The system parameters are: Lengths $L_1 = L_2 = 0.4$m, spring connection positions $L_{1,1} = L_{2,1} = 0.2$m, cross sections of beams are square with width and height $b = h = 0.0095$m, tip masses $M_1 = M_2 = 0.5658$Kg, attached mass $M = 0.2829$Kg, spring stiffness $K_{1,1} = K_{2,1} = 2.1943 \times 10^6$N/m, material density $\rho_1 = \rho_2 = 7{,}840$Kg/m^3 and Young's modulus $E_1 = E_2 = 2.069 \times 10^{11}$N/m^2. These parameters match the non-dimensional parameters as in reference [14]: $x_{1,1} = x_{2,1} = 0.5$, $k_{1,1} = k_{2,1} = 1000.0$, $m_1 = m_2 = 2.0$, $m = 1.0$ and $K_r = 1.0$. Table 1 shows the comparisons of the non-dimensional frequency parameters obtained in this article and the reference [14]. It is observed that the results are almost the same.

Figure 2 shows the other case for two cantilever beams coupled with a spring-mass system. The system parameters are: Lengths $L_1 = 0.68$m, $L_2 = 0.48$m, spring connection positions $L_{2,1} = L_{2,2} = 0.24$m, $L_{1,1} = 0.24$m, $L_{1,2} = 0.44$m, beams cross sections are square with width $b = h = 0.0095$m, attached mass $M = 0.1684$Kg, spring stiffness $K_{1,1} = 1.0 \times 10^5$N/m, $K_{2,1} = 1.65 \times 10^5$, material density $\rho_1 = \rho_2 = 7{,}840$Kg/m^3 and Young's modulus $E_1 = E_2 = 2.069 \times 10^{11}$N/m^2. The lowest natural frequencies of the two bare cantilever beams are 17.06Hz and 34.25Hz. The calculated analytical lowest four eigenvalues of this system are: 2.0046, 3.1297, 4.8913 and 6.0070 and their corresponding natural frequencies are: 19.49Hz, 47.50Hz, 116.01Hz and 174.98Hz. By modal testing, the measured transfer function is shown in Fig. 3. Table 2 shows the comparisons of experimental measured and analytical calculated natural frequencies. It is observed that all the analytical results are higher than those of the experimental data. The reason is that the fixed ends of cantilever beams in the experiment are not as rigid as the case in the analytical ideal condition. From Table 2, the errors between analysis and experiments are within 5% and are acceptable.

Figure 4 shows the mode shapes for the previous example. From this figure, it is observed that the first mode and the second mode look similar to the lowest mode shapes of the bare cantilever beams. However, the first mode shape of two beams is in phase and the second mode shape of two beams is out of phase. The third and the fourth modes of this system also have similar characteristics.

5. CONCLUSIONS

This study deals with the transverse free vibrations of a system in which two beams are coupled by a spring-mass device. The dynamics of this system are coupled through the motions of the attached mass. A method to determine the eigensolutions of this system has been proposed in this article. By considering the compatibility requirements across the spring connection position, the eigensolutions (natural frequencies and mode shapes) can be obtained easily for different boundary conditions. The characteristic equation can be reduced to a determinant matrix of only 4×4 instead of 13×13 which is in reference [14]. The characteristic equations encompasses a function of the eigenvalues, the location of the spring connection positions, the ratio of the stiffness of the springs, the ratio of the lengths of the beams, the ratio of the sectional properties of the beams and the suspended mass. Some numerical results and experimental validations are presented to demonstrate the method proposed in this article.

Table 1 Lowest eight non-dimensional natural frequencies for two cantilever beams carrying a spring-mass system, parameters are: lengths $L_1 = L_2 = 0.4$m, spring connection positions $L_{1,1} = L_{2,1} = 0.2$m, width and height $b = h = 0.0095$m, tip masses $M_1 = M_2 = 0.5658$Kg, attached mass $M = 0.2829$Kg, spring stiffness $K_{1,1} = K_{2,1} = 2.1943 \times 10^6$N/m, material density $\rho_1 = \rho_2 = 7{,}840$Kg/m^3 and Young's modulus $E_1 = E_2 = 2.069 \times 10^{11}$N/m^2

Lowest eight eigenvalues	In reference [14]	From this article	Corresponding natural frequencies (rad/sec)
λ_1	1.07026286	1.07026290	100.8806
λ_2	1.55878983	1.55878981	213.9941
λ_3	3.30959152	3.30959149	964.6625
λ_4	6.33425479	6.33425478	3533.6025
λ_5	6.81983036	6.81983042	4096.1305
λ_6	7.46938324	7.46938323	4913.5584
λ_7	7.92595594	7.92595591	5532.6085
λ_8	10.66673582	10.66673583	10020.5074

Table 2 Comparisons of experimental and analytical results

Lowest four Natural frequencies	from this article (Hz)	measured Natural frequencies (Hz)	Error %
First frequency	19.49	19.25	1.24
Second frequency	47.50	47.25	0.52
Third frequency	116.01	111.25	4.28
Fourth frequency	174.98	167.50	4.46

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Fig. 2 System of two cantilever-beams connected with a spring-mass devices; beam lengths L_1 = 0.68m, L_2 = 0.48m, spring connection positions $L_{2,1} = L_{2,2}$ = 0.24m, $L_{1,1}$ = 0.24m, $L_{1,2}$ = 0.44m, beam cross sections are square with width $b = h$ = 0.0095m, attached mass M = 0.1684Kg, spring stiffness $K_{1,1} = 1.0 \times 10^5$N/m, $K_{2,1} = 1.65 \times 10^5$N/m

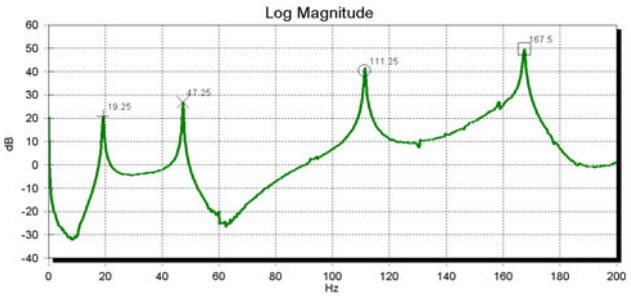
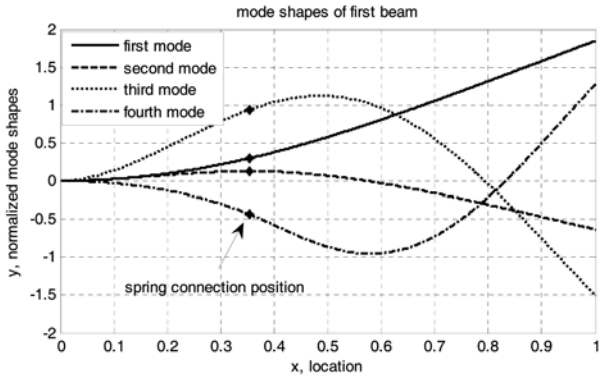
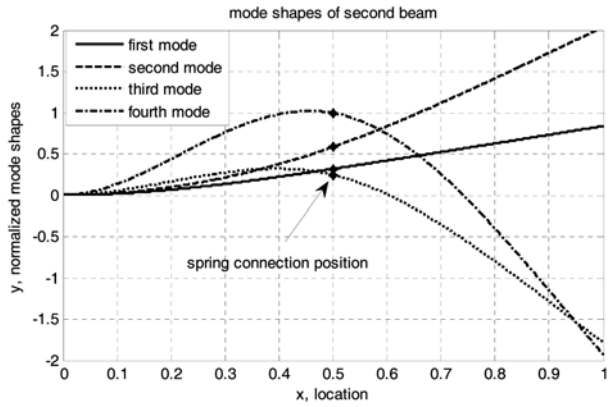


Fig. 3 Measured transfer function



4(a) Mode shapes of the first beam



4(b) Mode shapes of the second beam

Fig. 4 Lowest four mode shapes

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NOMENCLATURES

L_i : length of beam i
$L_{i,j}$: length of j-th segment for beam i
L : total length, $L = L_1 + L_2$
$X_{i,1}$: spring located position at beam i
$K_{i,1}$: spring stiffness connected on beam i
M : attached mass
x_i : space coordinate of beam i
$\overline{x}_i$: non-dimensional space coordinate of beam i, $\overline{x}_i = \dfrac{x_i}{L_i}$
t : time
$\overline{t}$: transformation parameter for time, $\overline{t} = \dfrac{t}{\sqrt{L}}$
$y_i(x_i, t)$: transverse displacement of the beam i
$\overline{y}_i$: non-dimensional transverse displacement of the beam i, $\overline{y}_i = \dfrac{y_i}{L_i}$
$z(t)$: displacement of the attached mass
$\overline{z}$: non-dimensional displacement of the attached mass, $\overline{z} = \dfrac{z}{L}$
E_i : Young's modulus of beam i
I_i : area cross-sectional moment of inertia of beam i
ρ_i : density of beam i
A_i : cross-sectional area of beam i
λ_i : eigenvalues of the system

APPENDIX A
TRANSFER MATRIX DERIVATION

The compatibility conditions across the spring connection positions are represented in Eqs. (12a) to (12d). The motions of the attached mass in Eq. (10) can be expressed as:

$$s = \frac{1}{\dfrac{L}{L_1}\left(1+\dfrac{K_{2,1}}{K_{1,1}}\right)-\dfrac{\omega^2}{\omega_1^2}} (A_{11}\sin\lambda_1 l_{11} + B_{11}\cos\lambda_1 l_{11} + C_{11}\sinh\lambda_1 l_{11} + D_{11}\cosh\lambda_1 l_{11}) + \frac{1}{\dfrac{L}{L_2}\left(1+\dfrac{K_{1,1}}{K_{2,1}}\right)-\dfrac{\omega^2}{\omega_2^2}} (A_{21}\sin\lambda_2 l_{21} + B_{21}\cos\lambda_2 l_{21} + C_{21}\sinh\lambda_2 l_{21} + D_{21}\cosh\lambda_2 l_{21})$$

where $\omega_1^2 = \dfrac{K_{1,1} L_1}{M}$ and $\omega_2^2 = \dfrac{K_{2,1} L_2}{M}$.

From Eqs. (12a) and (12c), the following can be obtained as:

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$$B_{i2}+D_{i2}=A_{i1}\sin\lambda_i\,l_{i1}+B_{i1}\cos\lambda_i\,l_{i1}+C_{i1}\sinh\lambda_i\,l_{i1}+D_{i1}\cosh\lambda_i\,l_{i1}\ , \tag{A1}$$

$$-B_{i2}+D_{i2}=-A_{i1}\sin\lambda_i\,l_{i1}-B_{i1}\cos\lambda_i\,l_{i1}+C_{i1}\sinh\lambda_i\,l_{i1}+D_{i1}\cosh\lambda_i\,l_{i1}\ , \tag{A2}$$

Similarly, Eqs. (12b) and (12d) lead to:

$$A_{i2}+C_{i2}=A_{i1}\cos\lambda_i\,l_{i1}-B_{i1}\sin\lambda_i\,l_{i1}+C_{i1}\cosh\lambda_i\,l_{i1}+D_{i1}\sinh\lambda_i\,l_{i1}\ . \tag{A3}$$

$$\begin{aligned}-A_{i2}+C_{i2}=&-A_{i1}\cos\lambda_i\,l_{i1}+B_{i1}\sin\lambda_i\,l_{i1}+C_{i1}\cosh\lambda_i\,l_{i1}+D_{i1}\sinh\lambda_i\,l_{i1}\\&+\frac{k_{i,1}}{\lambda_i^3}\left[\frac{L}{L_i}\left(\frac{1}{\dfrac{L}{L_1}\left(1+\dfrac{K_{2,1}}{K_{1,1}}\right)-\dfrac{\omega^2}{\omega_1^2}}(A_{11}\sin\lambda_1\,l_{11}+B_{11}\cos\lambda_1\,l_{11}+C_{11}\sinh\lambda_1\,l_{11}+D_{11}\cosh\lambda_1\,l_{11})\right.\right.\\&\left.\left.+\frac{1}{\dfrac{L}{L_2}\left(1+\dfrac{K_{1,1}}{K_{2,1}}\right)-\dfrac{\omega^2}{\omega_2^2}}(A_{21}\sin\lambda_2\,l_{21}+B_{21}\cos\lambda_2\,l_{21}+C_{21}\sinh\lambda_2\,l_{21}+D_{21}\cosh\lambda_2\,l_{21})\right)\right.\\&\left.-(A_{i1}\sin\lambda_i\,l_{i1}+B_{i1}\cos\lambda_i\,l_{i1}+C_{i1}\sinh\lambda_i\,l_{i1}+D_{i1}\cosh\lambda_i\,l_{i1})\right]\ .\end{aligned} \tag{A4}$$

From Eqs. (A1) and (A2)

$$B_{i2}=A_{i1}\sin\lambda_i\,l_{i1}+B_{i1}\cos\lambda_i\,l_{i1}\ .$$

$$D_{i2}=C_{i1}\sinh\lambda_i\,l_{i1}+D_{i1}\cosh\lambda_i\,l_{i1}\ .$$

From Eq. (A4)

$$\begin{aligned}-A_{12}+C_{12}=&A_{11}\left[-\cos\lambda_1\,l_{11}+\frac{k_{1,1}}{\lambda_1^3}\left[\left(\frac{1}{\left(1+\dfrac{K_{2,1}}{K_{1,1}}\right)-\dfrac{L_1}{L}\dfrac{\omega^2}{\omega_1^2}}\right)-1\right]\sin\lambda_1\,l_{11}\right]\\&+B_{11}\left[\sin\lambda_1\,l_{11}+\frac{k_{1,1}}{\lambda_1^3}\left[\left(\frac{1}{\left(1+\dfrac{K_{2,1}}{K_{1,1}}\right)-\dfrac{L_1}{L}\dfrac{\omega^2}{\omega_1^2}}\right)-1\right]\cos\lambda_1\,l_{11}\right]\\&+C_{11}\left[\cosh\lambda_1\,l_{11}+\frac{k_{1,1}}{\lambda_1^3}\left[\left(\frac{1}{\left(1+\dfrac{K_{2,1}}{K_{1,1}}\right)-\dfrac{L_1}{L}\dfrac{\omega^2}{\omega_1^2}}\right)-1\right]\sinh\lambda_1\,l_{11}\right]\\&+D_{11}\left[\sinh\lambda_1\,l_{11}+\frac{k_{1,1}}{\lambda_1^3}\left[\left(\frac{1}{\left(1+\dfrac{K_{2,1}}{K_{1,1}}\right)-\dfrac{L_1}{L}\dfrac{\omega^2}{\omega_1^2}}\right)-1\right]\cosh\lambda_1\,l_{11}\right]\\&+A_{21}\frac{k_{1,1}}{\lambda_1^3}\ \frac{1}{\dfrac{L_1}{L_2}\left(1+\dfrac{K_{1,1}}{K_{2,1}}\right)-\dfrac{L_1}{L}\dfrac{\omega^2}{\omega_2^2}}\sin\lambda_2\,l_{21}+B_{21}\frac{k_{1,1}}{\lambda_1^3}\ \frac{1}{\dfrac{L_1}{L_2}\left(1+\dfrac{K_{1,1}}{K_{2,1}}\right)-\dfrac{L_1}{L}\dfrac{\omega^2}{\omega_2^2}}\cos\lambda_2\,l_{21}\\&+C_{21}\frac{k_{1,1}}{\lambda_1^3}\ \frac{1}{\dfrac{L_1}{L_2}\left(1+\dfrac{K_{1,1}}{K_{2,1}}\right)-\dfrac{L_1}{L}\dfrac{\omega^2}{\omega_2^2}}\sinh\lambda_2\,l_{21}+D_{21}\frac{k_{1,1}}{\lambda_1^3}\ \frac{1}{\dfrac{L_1}{L_2}\left(1+\dfrac{K_{1,1}}{K_{2,1}}\right)-\dfrac{L_1}{L}\dfrac{\omega^2}{\omega_2^2}}\cosh\lambda_2\,l_{21}\ .\end{aligned} \tag{A5}$$

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From Eq. (A3)

$$A_{12} + C_{12} = A_{11}\cos\lambda_1 l_{11} - B_{11}\sin\lambda_1 l_{11} + C_{11}\cosh\lambda_1 l_{11} + D_{11}\sinh\lambda_1 l_{11} \ . \tag{A6}$$

Solving for Eqs. (A1) to (A4), the constants A_{12}, B_{12}, C_{12}, D_{12}, A_{22}, B_{22}, C_{22} and D_{22} are obtained as:

$$C_{12} = A_{11}\left[\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\left[\left(\frac{1}{\left(1+\frac{K_{2,1}}{K_{1,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_1^2}}\right)-1\right]\sin\lambda_1 l_{11}\right] + B_{11}\left[\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\left[\left(\frac{1}{\left(1+\frac{K_{2,1}}{K_{1,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_1^2}}\right)-1\right]\cos\lambda_1 l_{11}\right]$$

$$+C_{11}\left[\cosh\lambda_1 l_{11} + \frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\left[\left(\frac{1}{\left(1+\frac{K_{2,1}}{K_{1,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_1^2}}\right)-1\right]\sinh\lambda_1 l_{11}\right]$$

$$+D_{11}\left[\sinh\lambda_1 l_{11} + \frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\left[\left(\frac{1}{\left(1+\frac{K_{2,1}}{K_{1,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_1^2}}\right)-1\right]\cosh\lambda_1 l_{11}\right]$$

$$+A_{21}\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\ \frac{1}{\frac{L_1}{L_2}\left(1+\frac{K_{1,1}}{K_{2,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_2^2}}\sin\lambda_2 l_{21} + B_{21}\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\ \frac{1}{\frac{L_1}{L_2}\left(1+\frac{K_{1,1}}{K_{2,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_2^2}}\cos\lambda_2 l_{21}$$

$$+C_{21}\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\frac{1}{\frac{L_1}{L_2}\left(1+\frac{K_{1,1}}{K_{2,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_2^2}}\ \sinh\lambda_2 l_{21} + D_{21}\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\ \frac{1}{\frac{L_1}{L_2}\left(1+\frac{K_{1,1}}{K_{2,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_2^2}}\cosh\lambda_2 l_{21}\ ,$$

$$A_{12} = A_{11}\left[\cos\lambda_1 l_{11} - -\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\left[\left(\frac{1}{\left(1+\frac{K_{2,1}}{K_{1,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_1^2}}\right)-1\right]\sin\lambda_1 l_{11}\right] + B_{11}\left[-\sin\lambda_1 l_{11} - -\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\left[\left(\frac{1}{\left(1+\frac{K_{2,1}}{K_{1,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_1^2}}\right)-1\right]\cos\lambda_1 l_{11}\right]$$

$$+C_{11}\left[-\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\left[\left(\frac{1}{\left(1+\frac{K_{2,1}}{K_{1,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_1^2}}\right)-1\right]\sinh\lambda_1 l_{11}\right] + D_{11}\left[-\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\left[\left(\frac{1}{\left(1+\frac{K_{2,1}}{K_{1,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_1^2}}\right)-1\right]\cosh\lambda_1 l_{11}\right]$$

$$-A_{21}\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\ \frac{1}{\frac{L_1}{L_2}\left(1+\frac{K_{1,1}}{K_{2,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_2^2}}\sin\lambda_2 l_{21} - B_{21}\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\ \frac{1}{\frac{L_1}{L_2}\left(1+\frac{K_{1,1}}{K_{2,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_2^2}}\cos\lambda_2 l_{21}$$

$$-C_{21}\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\ \frac{1}{\frac{L_1}{L_2}\left(1+\frac{K_{1,1}}{K_{2,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_2^2}}\sinh\lambda_2 l_{21} - D_{21}\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\ \frac{1}{\frac{L_1}{L_2}\left(1+\frac{K_{1,1}}{K_{2,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_2^2}}\cosh\lambda_2 l_{21}\ ,$$

$$B_{12} = A_{11}\sin\lambda_1 l_{11} + B_{11}\cos\lambda_1 l_{11}\ ,$$

$$D_{12} = C_{11}\sinh\lambda_1 l_{11} + D_{11}\cosh\lambda_1 l_{11}\ ,$$

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$$C_{22}=A_{21}\left[\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\left[\left(\frac{1}{\left(1+\frac{K_{1,1}}{K_{2,1}}\right)-\frac{L_2}{L}\frac{\omega^2}{\omega_2^2}}\right)-1\right]\sin\lambda_2 l_{21}\right]+B_{21}\left[\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\left[\left(\frac{1}{\left(1+\frac{K_{1,1}}{K_{2,1}}\right)-\frac{L_2}{L}\frac{\omega^2}{\omega_2^2}}\right)-1\right]\cos\lambda_2 l_{21}\right]$$

$$+C_{21}\left[\cosh\lambda_2 l_{21}+\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\left[\left(\frac{1}{\left(1+\frac{K_{1,1}}{K_{2,1}}\right)-\frac{L_2}{L}\frac{\omega^2}{\omega_2^2}}\right)-1\right]\sinh\lambda_2 l_{21}\right]+D_{21}\left[\sinh\lambda_2 l_{21}+\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\left[\left(\frac{1}{\left(1+\frac{K_{1,1}}{K_{2,1}}\right)-\frac{L_2}{L}\frac{\omega^2}{\omega_2^2}}\right)-1\right]\cosh\lambda_2 l_{21}\right]$$

$$+A_{11}\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\frac{L_1/L_2}{\left(1+\frac{K_{2,1}}{K_{1,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_1^2}}\sin\lambda_1 l_{11}+B_{11}\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\frac{L_1/L_2}{\left(1+\frac{K_{2,1}}{K_{1,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_1^2}}\cos\lambda_1 l_{11}$$

$$+C_{11}\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\frac{L_1/L_2}{\left(1+\frac{K_{2,1}}{K_{1,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_1^2}}\sinh\lambda_1 l_{11}+D_{11}\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\frac{L_1/L_2}{\left(1+\frac{K_{2,1}}{K_{1,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_1^2}}\cosh\lambda_1 l_{11}\ ,$$

$$A_{22}=A_{21}\left[\cos\lambda_2 l_{21}--\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\left[\left(\frac{1}{\left(1+\frac{K_{1,1}}{K_{2,1}}\right)-\frac{L_2}{L}\frac{\omega^2}{\omega_2^2}}\right)-1\right]\sin\lambda_2 l_{21}\right]+B_{21}\left[-\sin\lambda_2 l_{21}--\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\left[\left(\frac{1}{\left(1+\frac{K_{1,1}}{K_{2,1}}\right)-\frac{L_2}{L}\frac{\omega^2}{\omega_2^2}}\right)-1\right]\cos\lambda_2 l_{21}\right]$$

$$+C_{21}\left[-\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\left[\left(\frac{1}{\left(1+\frac{K_{1,1}}{K_{2,1}}\right)-\frac{L_2}{L}\frac{\omega^2}{\omega_2^2}}\right)-1\right]\sinh\lambda_2 l_{21}\right]+D_{21}\left[-\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\left[\left(\frac{1}{\left(1+\frac{K_{1,1}}{K_{2,1}}\right)-\frac{L_2}{L}\frac{\omega^2}{\omega_2^2}}\right)-1\right]\cosh\lambda_2 l_{21}\right]$$

$$-A_{11}\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\frac{L_1/L_2}{\left(1+\frac{K_{2,1}}{K_{1,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_1^2}}\sin\lambda_1 l_{11}-B_{11}\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\frac{L_1/L_2}{\left(1+\frac{K_{2,1}}{K_{1,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_1^2}}\cos\lambda_1 l_{11}$$

$$-C_{11}\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\frac{L_1/L_2}{\left(1+\frac{K_{2,1}}{K_{1,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_1^2}}\sinh\lambda_1 l_{11}-D_{11}\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\frac{L_1/L_2}{\left(1+\frac{K_{2,1}}{K_{1,1}}\right)-\frac{L_1}{L}\frac{\omega^2}{\omega_1^2}}\cosh\lambda_1 l_{11}\ ,$$

$$B_{22}=A_{21}\sin\lambda_2 l_{21}+B_{21}\cos\lambda_2 l_{21}\ ,$$

$$D_{22}=C_{21}\sinh\lambda_2 l_{21}+D_{21}\cosh\lambda_2 l_{21}\ ,$$

and which can be expressed in matrix form as:

$$\begin{Bmatrix}A_{12}\\B_{12}\\C_{12}\\D_{12}\\A_{22}\\B_{22}\\C_{22}\\D_{22}\end{Bmatrix}=\begin{bmatrix}t_{11}&t_{12}&\cdots&t_{18}\\\vdots&&&\\t_{81}&\cdots\cdots\cdots\cdots&&t_{88}\end{bmatrix}\begin{Bmatrix}A_{11}\\B_{11}\\C_{11}\\D_{11}\\A_{21}\\B_{21}\\C_{21}\\D_{21}\end{Bmatrix}=\underline{\mathrm{T}}_{8\times8}\begin{Bmatrix}A_{11}\\B_{11}\\C_{11}\\D_{11}\\A_{21}\\B_{21}\\C_{21}\\D_{21}\end{Bmatrix}\ .$$

Here, $\underline{\mathrm{T}}_{8\times8}$ is a transfer matrix composed of the elements

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$$t_{11}=\cos\lambda_1 l_{11}-\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\left[\left(\frac{1}{(1+K_r)-\frac{m}{k_{1,1}}\lambda_1^4}\right)-1\right]\sin\lambda_1 l_{11}\ ,\quad t_{12}=-\sin\lambda_1 l_{11}-\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\left[\left(\frac{1}{(1+K_r)-\frac{m}{k_{1,1}}\lambda_1^4}\right)-1\right]\cos\lambda_1 l_{11}\ ,$$

$$t_{13}=-\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\left[\left(\frac{1}{(1+K_r)-\frac{m}{k_{1,1}}\lambda_1^4}\right)-1\right]\sinh\lambda_1 l_{11}\ ,\quad t_{14}=-\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\left[\left(\frac{1}{(1+K_r)-\frac{m}{k_{1,1}}\lambda_1^4}\right)-1\right]\cosh\lambda_1 l_{11}\ ,$$

$$t_{15}=-\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\ \frac{L_r}{\left(1+\frac{1}{K_r}\right)-\frac{m}{k_{2,1}}\lambda_2^4\frac{1}{M_r}}\sin\lambda_2 l_{21}\ ,\quad t_{16}=-\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\ \frac{L_r}{\left(1+\frac{1}{K_r}\right)-\frac{m}{k_{2,1}}\lambda_2^4\frac{1}{M_r}}\cos\lambda_2 l_{21}\ ,$$

$$t_{17}=-\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\ \frac{L_r}{\left(1+\frac{1}{K_r}\right)-\frac{m}{k_{2,1}}\lambda_2^4\frac{1}{M_r}}\sinh\lambda_2 l_{21}\ ,\quad t_{18}=-\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\ \frac{L_r}{\left(1+\frac{1}{K_r}\right)-\frac{m}{k_{2,1}}\lambda_2^4\frac{1}{M_r}}\cosh\lambda_2 l_{21}\ ,$$

$$t_{21}=\sin\lambda_1 l_{11}\ ,\quad t_{22}=\cos\lambda_1 l_{11}\ ,\quad t_{23}=t_{24}=t_{25}=t_{26}=t_{27}=t_{28}=0\ ,$$

$$t_{31}=\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\left[\left(\frac{1}{(1+K_r)-\frac{m}{k_{1,1}}\lambda_1^4}\right)-1\right]\sin\lambda_1 l_{11}\ ,\quad t_{32}=\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\left[\left(\frac{1}{(1+K_r)-\frac{m}{k_{1,1}}\lambda_1^4}\right)-1\right]\cos\lambda_1 l_{11}\ ,$$

$$t_{33}=\cosh\lambda_1 l_{11}+\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\left[\left(\frac{1}{(1+K_r)-\frac{m}{k_{1,1}}\lambda_1^4}\right)-1\right]\sinh\lambda_1 l_{11}\ ,$$

$$t_{34}=\sinh\lambda_1 l_{11}+\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\left[\left(\frac{1}{(1+K_r)-\frac{m}{k_{1,1}}\lambda_1^4}\right)-1\right]\cosh\lambda_1 l_{11}\ ,$$

$$t_{35}=\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\ \frac{L_r}{\left(1+\frac{1}{K_r}\right)-\frac{m}{k_{2,1}}\lambda_2^4\frac{1}{M_r}}\sin\lambda_2 l_{21}\ ,\quad t_{36}=\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\ \frac{L_r}{\left(1+\frac{1}{K_r}\right)-\frac{m}{k_{2,1}}\lambda_2^4\frac{1}{M_r}}\cos\lambda_2 l_{21}\ ,$$

$$t_{37}=\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\ \frac{L_r}{\left(1+\frac{1}{K_r}\right)-\frac{m}{k_{2,1}}\lambda_2^4\frac{1}{M_r}}\sinh\lambda_2 l_{21}\ ,\quad t_{38}=\frac{1}{2}\frac{k_{1,1}}{\lambda_1^3}\ \frac{L_r}{\left(1+\frac{1}{K_r}\right)-\frac{m}{k_{2,1}}\lambda_2^4\frac{1}{M_r}}\cosh\lambda_2 l_{21}\ ,$$

$$t_{41}=0\ ,\quad t_{42}=0\ ,\quad t_{43}=\sinh\lambda_1 l_{11}\ ,\quad t_{44}=\cosh\lambda_1 l_{11}\ ,\quad t_{45}=t_{46}=t_{47}=t_{48}=0\ ,$$

$$t_{51}=-\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\ \frac{1/L_r}{(1+K_r)-\frac{m}{k_{1,1}}\lambda_1^4}\sin\lambda_1 l_{11}\ ,\quad t_{52}=-\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\ \frac{1/L_r}{(1+K_r)-\frac{m}{k_{1,1}}\lambda_1^4}\cos\lambda_1 l_{11}\ ,$$

$$t_{53}=-\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\ \frac{1/L_r}{(1+K_r)-\frac{m}{k_{1,1}}\lambda_1^4}\sinh\lambda_1 l_{11}\ ,\quad t_{54}=-\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\ \frac{1/L_r}{(1+K_r)-\frac{m}{k_{1,1}}\lambda_1^4}\cosh\lambda_1 l_{11}\ ,$$

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$$t_{55}=\cos\lambda_2 l_{21}-\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\left[\left(\frac{1}{\left(1+\frac{1}{K_r}\right)-\frac{m}{k_{2,1}}\lambda_2^4\frac{1}{M_r}}\right)-1\right]\sin\lambda_2 l_{21}\ ,$$

$$t_{56}=-\sin\lambda_2 l_{21}-\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\left[\left(\frac{1}{\left(1+\frac{1}{K_r}\right)-\frac{m}{k_{2,1}}\lambda_2^4\frac{1}{M_r}}\right)-1\right]\cos\lambda_2 l_{21}\ ,$$

$$t_{57}=-\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\left[\left(\frac{1}{\left(1+\frac{1}{K_r}\right)-\frac{m}{k_{2,1}}\lambda_2^4\frac{1}{M_r}}\right)-1\right]\sinh\lambda_2 l_{21}\ ,\quad t_{58}=-\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\left[\left(\frac{1}{\left(1+\frac{1}{K_r}\right)-\frac{m}{k_{2,1}}\lambda_2^4\frac{1}{M_r}}\right)-1\right]\cosh\lambda_2 l_{21}\ ,$$

$$t_{61}=t_{62}=t_{63}=t_{64}=t_{67}=t_{68}=0\ ,\quad t_{65}=\sin\lambda_2 l_{21}\ ,\quad t_{66}=\cos\lambda_2 l_{21}\ ,$$

$$t_{71}=\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\ \frac{1/L_r}{(1+K_r)-\frac{m}{k_{1,1}}\lambda_1^4}\sin\lambda_1 l_{11}\ ,\quad t_{72}=\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\ \frac{1/L_r}{(1+K_r)-\frac{m}{k_{1,1}}\lambda_1^4}\cos\lambda_1 l_{11}\ ,$$

$$t_{73}=\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\ \frac{1/L_r}{(1+K_r)-\frac{m}{k_{1,1}}\lambda_1^4}\sinh\lambda_1 l_{11}\ ,\quad t_{74}=\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\ \frac{1/L_r}{(1+K_r)-\frac{m}{k_{1,1}}\lambda_1^4}\cosh\lambda_1 l_{11}\ ,$$

$$t_{75}=\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\left[\left(\frac{1}{\left(1+\frac{1}{K_r}\right)-\frac{m}{k_{2,1}}\lambda_2^4\frac{1}{M_r}}\right)-1\right]\sin\lambda_2 l_{21}\ ,\quad t_{76}=\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\left[\left(\frac{1}{\left(1+\frac{1}{K_r}\right)-\frac{m}{k_{2,1}}\lambda_2^4\frac{1}{M_r}}\right)-1\right]\cos\lambda_2 l_{21}\ ,$$

$$t_{77}=\cosh\lambda_2 l_{21}+\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\left[\left(\frac{1}{\left(1+\frac{1}{K_r}\right)-\frac{m}{k_{2,1}}\lambda_2^4\frac{1}{M_r}}\right)-1\right]\sinh\lambda_2 l_{21}\ ,$$

$$t_{78}=\sinh\lambda_2 l_{21}+\frac{1}{2}\frac{k_{2,1}}{\lambda_2^3}\left[\left(\frac{1}{\left(1+\frac{1}{K_r}\right)-\frac{m}{k_{2,1}}\lambda_2^4\frac{1}{M_r}}\right)-1\right]\cosh\lambda_2 l_{21}\ ,$$

$$t_{81}=t_{82}=t_{83}=t_{84}=t_{85}=t_{86}=0\ ,\quad t_{87}=\sinh\lambda_2 l_{21}\ ,\quad t_{88}=\cosh\lambda_2 l_{21}\ ,$$

where $K_r\equiv\frac{K_{2,1}}{K_{1,1}}$, $m\equiv\frac{M}{\rho_1 A_1 L_1}$, $L_r\equiv\frac{L_2}{L_1}$, $M_r\equiv\frac{\rho_2 A_2 L_2}{\rho_1 A_1 L_1}$ and $k_{i,1}=\frac{K_{i,1}L_i^3}{E_i I_i}$.

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