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Second-order differential equations with random perturbations and small parameters

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We consider boundary-value problems for differential equations of second order containing a Brownian motion (random perturbation) and a small parameter and prove a special existence and uniqueness theorem for random solutions. We study the asymptotic behaviour of these solutions as the small parameter goes to zero and show the stochastic averaging theorem for such equations. We find the explicit limits for the solutions as the small parameter goes to zero.

Keywords: boundary-value problems; stochastic averaging method; Green functions

2010 *Mathematics subject classification:* Primary 60H10; 60J60; 58J37; 34B05; 34C29; 34D15; 34B27

1. Introduction

This paper is devoted to a limiting problem for differential equations containing stochastic terms and small parameters. Such problems reframe the asymptotic theory for ordinary differential equations. There are two general methods for studying these problems: Bogolubov's averaging principle [5] and Tikhonov's singularly perturbation method [22, 23]. In the first method the fast variables have no limit, but in the second method the fast variables go to some limit. Usually, to find this

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limit one has to replace the small parameter by zero in the equations, and the solution of the limiting problem gives the limit for the fast variables. The situation essentially becomes difficult if we consider the boundary problems. Therefore, let us consider, for example, an ordinary second-order differential equation with a small parameter on the interval $[0,1]$, i.e.

$$\ddot{x}^{\varepsilon}(t) = \varepsilon(px^{\varepsilon}(t) + f(t)), \tag{1.1}$$

where $p > 0$. In this paper we consider the following boundary conditions.

- First-order Neumann boundary conditions:

$$x(0) = x(1) = 0. \tag{1.2}$$

- Second-order Neumann boundary conditions:

$$\dot{x}(0) = \dot{x}(1) = 0. \tag{1.3}$$

- Periodic boundary conditions:

$$x(0) = x(1) \quad \text{and} \quad \dot{x}(0) = \dot{x}(1). \tag{1.4}$$

Note that if we replace the parameter ε in (1.1) by zero, we obtain the following limit problem:

$$\ddot{x} = 0.$$

Note that this equation with the first boundary condition has a unique solution $x = 0$. However, for the boundary conditions (1.3) or (1.4) this limiting problem has no unique solution, since any constant satisfies this equation, i.e. in this case the limiting problem does not give any information about the asymptotic (as $\varepsilon \to 0$) behaviour of the solution of (1.1). Therefore, the question of how to calculate this constant naturally arises. For the boundary conditions (1.3) or (1.4), one can show via the Green function method (see remark 3.3) that, for any $0 < t < 1$,

$$\lim_{\varepsilon \to 0} x^{\varepsilon}(t) = -\frac{1}{p}\int_0^1 f(s)\,\mathrm{d}s. \tag{1.5}$$

There exists a vast literature on stochastic differential equations with small parameters and their applications (for details see, for example, [11, 12, 16] and the references therein). It turns out that the small parameter methods for stochastic differential equations are useful for studying optimal stochastic control problems [2–4], stochastic volatility in financial markets [1, 10, 17, 24], statistical estimation in multi-scale stochastic systems [19–21] and many other applied problems. Usually, one considers the initial or terminal conditions for the stochastic differential equations. In this paper we propose to extend both these problems, i.e. we introduce a second-order random differential equation with boundary conditions in the form (1.2)–(1.4). For such models we study the stochastic averaging principle. This problem is well known for diffusion processes (see, for example, [11, 15]). To understand the averaging principle problem we start with the following example.

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EXAMPLE 1.1. On the interval $[0, \varepsilon]$ we consider (for some small parameter $0 < \varepsilon < 1$) the linear stochastic second-order differential equation with quickly oscillating and 'small' random force ('white noise'), i.e.

$$\ddot{y}^\varepsilon(u) = py^\varepsilon(u) + f\left(\frac{u}{\varepsilon}\right) + \delta\left(\frac{u}{\varepsilon}\right)\sqrt{\varepsilon}\dot{w}_u,$$

where $p > 0$ is some fixed parameter and δ is a $\mathbb{R} \to \mathbb{R}$ square integrable function, i.e.

$$\|\delta\|^2 = \int_0^1 \delta^2(t)\,\mathrm{d}t < \infty. \tag{1.6}$$

Moreover, $(w_t)_{t\geqslant 0}$ is a standard Brownian motion, and therefore $(\dot{w}_t)_{t\geqslant 0}$ is 'white noise'. By setting $t = u/\varepsilon$ and $x^\varepsilon(t) = y^\varepsilon(t\varepsilon)$ here, on the time interval $[0, 1]$ we obtain the following stochastic differential equation with small parameter:

$$\ddot{x}^\varepsilon(t) = \varepsilon^2(px^\varepsilon(t) + f(t)) + \varepsilon^2\delta(t)\dot{W}_t,$$

where $W_t = w_{t\varepsilon}/\sqrt{\varepsilon}$ is also a standard Brownian motion.

In this paper we consider a more general nonlinear case, i.e. an equation of the form

$$\mathrm{d}\dot{x}^\varepsilon(t) = \varepsilon^2(A(t, x^\varepsilon(t)) + f(t))\,\mathrm{d}t + \varepsilon^2\delta(t)\,\mathrm{d}W_t, \tag{1.7}$$

where A is some $\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ non-random function and $(W_t)_{0\leqslant t\leqslant 1}$ is the standard Brownian motion.

We shall study the problem of the asymptotic (as $\varepsilon \to 0$) behaviour of (1.7). First, we have to prove an existence and uniqueness theorem for solutions with the boundary conditions (1.2)–(1.4). To this end we make use of a uniform version of the implicit function theorem. Moreover, to study the asymptotic behaviour we propose an averaging method based on the Green function approach, which is usually used for deterministic differential equations (see, for example, [6]; [9, p. 31]; [13] and the references therein).

The paper is organized as follows. In § 2 we state the existence and uniqueness theorem. In § 3 we state averaging theorems. In § 4 we study the corresponding Green functions. In § 5 we give the principal proofs. In Appendix A we prove some technical results.

2. Existence and uniqueness theorem

In this section we study the strong solutions of (1.7) with an arbitrary Brownian motion $(W_t)_{0\leqslant t\leqslant 1}$. A stochastic process $(x_t)_{0\leqslant t\leqslant 1}$ is called the *random solution* of (1.7) if it satisfies this equation almost surely, and for any $0 \leqslant t \leqslant 1$ the random variable x_t is measurable with respect to $\sigma\{W_u,\ 0 \leqslant u \leqslant 1\}$. Indeed, the solution may not be adapted with respect to the Wiener filtration.

The first problem is to find sufficient conditions for the existence of a unique strong solution. To this end we assume that the function $A(t, x)$ satisfies the following conditions.

(C1) There exists a positive constant p such that

$$A(t, x) = px + B(t, x), \tag{2.1}$$

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where the function $B(t,x)$ is bounded, i.e.

$$\beta^* = \sup_{0\leqslant t\leqslant 1}\sup_{x\in\mathbb{R}}|B(t,x)| < \infty.$$

(C2) The partial derivative $B'_x(t,x)$ is bounded by p, i.e.

$$\beta = \sup_{0\leqslant t\leqslant 1}\sup_{x\in\mathbb{R}}|B'_x(t,x)| < p, \tag{2.2}$$

and, moreover, it is uniformly continuous, i.e.

$$\lim_{\delta\to 0}\sup_{0\leqslant t\leqslant 1}\sup_{|x-y|\leqslant\delta}|B'_x(t,x)-B'_x(t,y)| = 0. \tag{2.3}$$

For example, we can take $B(t,x)=\beta^*\sin(\omega tx)$ for $0\leqslant\beta^*|\omega|<p$.

THEOREM 2.1. *Assume that conditions (C1) and (C2) hold. Then there exists a non-random parameter $\varepsilon_0>0$ such that, for all $0<\varepsilon\leqslant\varepsilon_0$, (1.7) with one of the boundary conditions (1.2)–(1.4) has a unique continuously differentiable almost surely random solution.*

The proof of this theorem is given in § 5.

REMARK 2.2. A well-known example of a stochastic process satisfying a special stochastic differential equation with boundary conditions (1.2) is the Brownian bridge (see, for example, [14]), which is defined as

$$Z_t = W_t - tW_1 = \int_0^1 G(t,s)\,\mathrm{d}W_s, \tag{2.4}$$

where

$$G(t,s) = (1-t)\mathbf{1}_{\{s\leqslant t\}} - t\mathbf{1}_{\{s>t\}}.$$

Taking into account that $t^{-1}(Z_t-W_t)=-W_1$ for any $0\leqslant t\leqslant 1$, we obtain

$$\int_0^t s^{-1}(Z_s-W_s)\,\mathrm{d}s = -tW_1 = Z_t - W_t,$$

i.e.

$$\mathrm{d}Z_t = t^{-1}(Z_t-W_t)\,\mathrm{d}t + \mathrm{d}W_t, \quad Z_0 = Z_1 = 0.$$

It is clear that the process (2.4) is not adapted to the Wiener filtration. But it should be noted that for any one boundary condition there exist backward stochastic differential equations with the adapted solutions (see, for example, [8]). In this paper we investigate second-order differential equations of the form (1.7) with the two boundary conditions, whose solutions, generally speaking, cannot be adapted to the filtration generated by the Wiener process.

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3. Averaging theorems

In this section we study the asymptotic (as $\varepsilon \to 0$) properties of the solutions of (1.7) with boundary conditions (1.2)–(1.4). To state our first averaging theorem we set up the following process:

$$\varkappa(t) = \int_0^1 \Upsilon(t,s)(B(s,0) + f(s))\,\mathrm{d}s + \int_0^1 \Upsilon(t,s)\delta(s)\,\mathrm{d}W_s, \tag{3.1}$$

where $\Upsilon(t,s) = -\min(t,s)(1-\max(t,s))$.

In the following we denote by $|\cdot|_*$ the uniform norm in $\boldsymbol{C}([0,1])$ and in $\boldsymbol{C}([0,1]\times[0,1])$, i.e. for any $x \in \boldsymbol{C}([0,1])$ and $G \in \boldsymbol{C}([0,1]\times[0,1])$,

$$|x|_* = \sup_{0\leqslant t\leqslant 1} |x(t)| \quad \text{and} \quad |G|_* = \sup_{0\leqslant t,s\leqslant 1} |G(t,s)|. \tag{3.2}$$

Moreover, for any $[0,1]\times[0,1] \to \mathbb{R}$ bounded function G, we set

$$|G|_{1,*} = \sup_{s\neq t,\ s,t\in[0,1]} |G(t,s)|. \tag{3.3}$$

THEOREM 3.1. *Assume that conditions (C1) and (C2) hold. Then the random solution of problem (1.7), (1.2) has the following limiting form:*

$$\mathbb{P}\text{-}\lim_{\varepsilon\to 0} |\varepsilon^{-2}x^\varepsilon - \varkappa|_* = 0.$$

Now, we introduce the function

$$B_0(x) = x + \frac{1}{p}\int_0^1 B(s,x)\,\mathrm{d}s \tag{3.4}$$

and we define the random variable (if it exists)

$$\zeta = B_0^{-1}(\eta), \tag{3.5}$$

where

$$\eta = -\frac{1}{p}\int_0^1 f(s)\,\mathrm{d}s - \frac{1}{p}\int_0^1 \delta(s)\,\mathrm{d}W_s.$$

THEOREM 3.2. *Assume that conditions (C1) and (C2) hold. Then the function (3.4) is invertible on $\mathbb{R}$. Moreover, the solutions of (1.7), (1.3) and (1.7), (1.4) satisfy the following property:*

$$\mathbb{P}\text{-}\lim_{\varepsilon\to 0} |x^\varepsilon - \zeta|_* = 0.$$

REMARK 3.3. It is easy to see that for $\delta \equiv 0$ and $B(t,x) \equiv 0$ we obtain the convergence (1.5).

REMARK 3.4. Note that theorems 2.1, 3.1 and 3.2 also hold for any finite-dimensional state space.

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4. Properties of the Green functions

Let $p > 0$ be an arbitrary fixed constant. We make use of the following differential equation:

$$\ddot{u} - \varepsilon^2 p u = 0. \tag{4.1}$$

One can check directly that for problem (4.1), (1.2) the Green function is defined as

$$G_{1,\varepsilon}(t,s) = -\frac{\exp\{-\varepsilon\sqrt{p}|t-s|\}g_{1,\varepsilon}(t,s)g_{2,\varepsilon}(t,s)}{2\varepsilon\sqrt{p}(1-\exp\{-2\varepsilon\sqrt{p}\})}, \tag{4.2}$$

where

$$g_{1,\varepsilon}(t,s) = 1-\exp\{-2\varepsilon\sqrt{p}\min(t,s)\}, \quad g_{2,\varepsilon}(t,s) = 1-\exp\{-2\varepsilon\sqrt{p}(1-\max(t,s))\}.$$

Proposition 4.1. *The function (4.2) satisfies the following limit properties:*

$$\lim_{\varepsilon\to 0} |G_{1,\varepsilon} - \Upsilon|_* = 0$$

and

$$\lim_{\varepsilon\to 0} \left|\frac{\partial}{\partial t}(G_{1,\varepsilon} - \Upsilon)\right|_{1,*} = 0,$$

where the norms $|\cdot|_*$ and $|\cdot|_{1,*}$ are defined in (3.2) and (3.3).

For the problem (4.1), (1.3) the Green function is

$$G_{2,\varepsilon}(t,s) = -\frac{\exp\{-\varepsilon\sqrt{p}|t-s|\}g_{3,\varepsilon}(t,s)g_{4,\varepsilon}(t,s)}{2\varepsilon\sqrt{p}(1-\exp\{-2\varepsilon\sqrt{p}\})}, \tag{4.3}$$

where

$$g_{3,\varepsilon}(t,s) = 1+\exp\{-2\varepsilon\sqrt{p}\min(t,s)\}, \qquad g_{4,\varepsilon}(t,s) = 1+\exp\{-2\varepsilon\sqrt{p}(\max(t,s))\}.$$

Moreover, for the problem (4.1), (1.4) the Green function has the form

$$G_{3,\varepsilon}(t,s) = -\frac{\exp\{-\varepsilon\sqrt{p}|t-s|\} + \exp\{-\varepsilon\sqrt{p}(1-|t-s|)\}}{2\varepsilon\sqrt{p}(1-\exp\{-\varepsilon\sqrt{p}\})}. \tag{4.4}$$

Proposition 4.2. *The functions (4.3) and (4.4) satisfy the following limit properties:*

$$\lim_{\varepsilon\to 0} \max_{2\leqslant i\leqslant 3} \left|\varepsilon^2 G_{i,\varepsilon} + \frac{1}{p}\right|_* = 0 \tag{4.5}$$

and

$$\lim_{\varepsilon\to 0} \varepsilon^2 \max_{2\leqslant i\leqslant 3} \left|\frac{\partial}{\partial t} G_{i,\varepsilon}\right|_{1,*} = 0. \tag{4.6}$$

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5. Proofs

5.1. Proof of theorem 2.1

We define for $1 \leqslant i \leqslant 3$ the following $\mathcal{C}[0,1] \to \mathcal{C}[0,1]$ operators:

$$\Psi_i(\varepsilon, x)(t) = x(t) - \int_0^1 \tilde{G}_{i,\varepsilon}(t,s) B(s, x(s))\, \mathrm{d}s, \tag{5.1}$$

where $\tilde{G}_{i,\varepsilon}(t,s) = \varepsilon^2 G_{i,\varepsilon}(t,s)$ and the corresponding Green functions $G_{i,\varepsilon}$ are defined in (4.2)–(4.4), respectively, and the function $B(t,x)$ is given in condition (C1).

To prove this theorem we have to check the conditions of theorem A.3 (see Appendix A). Note that

$$\Psi_1(0,x)(t) = x(t)$$

and

$$\Psi_2(0,x)(t) = \Psi_3(0,x)(t) = x(t) + \frac{1}{p}\int_0^1 B(s, x(s))\, \mathrm{d}s. \tag{5.2}$$

Therefore, conditions (AP1) and (AP2) are obvious for $i = 1$. Let us check these conditions for $i = 2$ and $i = 3$. To this end, for any function $\phi \in \mathcal{C}[0,1]$, we introduce the $\mathbb{R} \to \mathbb{R}$ function

$$T_\phi(v) = -\frac{1}{p}\int_0^1 B(s, \phi(s) + v)\, \mathrm{d}s.$$

It is clear that

$$T'_\phi(v) = -\frac{1}{p}\int_0^1 B'_x(s, \phi(s) + v)\, \mathrm{d}s$$

and, by the inequality (2.2),

$$\sup_{\phi \in \mathcal{C}[0,1]} \sup_{v \in \mathbb{R}} |T'_\phi(v)| < 1.$$

Therefore, for any $\phi \in \mathcal{C}[0,1]$, the equation

$$v = T_\phi(v) \tag{5.3}$$

has a unique solution. This implies directly that in this case, for any $\phi \in \mathcal{C}[0,1]$, the inverse function in condition (AP1) is given by

$$x_0^\phi(t) = \phi(t) + v_\phi,$$

where v_ϕ is the solution of (5.3).

For condition (AP2), note that the Fréchet derivative of the function (5.2) is given by the following linear $\mathcal{C}[0,1] \to \mathcal{C}[0,1]$ operator:

$$D_x(h)(t) = h(t) + \frac{1}{p}\int_0^1 B'_x(s, x(s)) h(s)\, \mathrm{d}s.$$

Taking into account the inequality (2.1), we can check directly that this operator is an isomorphism and that

$$D_x^{-1}(h)(t) = h(t) - \left(\int_0^1 B'_x(s, x(s)) h(s)\, \mathrm{d}s\right)\left(p + \int_0^1 B'_x(s, x(s))\, \mathrm{d}s\right)^{-1}.$$

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Now, we calculate directly that, for any $x \in \boldsymbol{C}[0,1]$,

$$\|D_x^{-1}\| = \sup_{h \in \boldsymbol{C}[0,1],\ |h|_*=1} |D_x^{-1}(h)| \leqslant 1 + \frac{\beta}{p-\beta}.$$

This implies condition (AP2).

Moreover, the boundedness of the function $B(t,x)$ given in condition (C1) implies condition (AP3) for all $1 \leqslant i \leqslant 3$. Condition (C2) enables conditions (AP4) and (AP5). Therefore, by making use of theorem A.3 for the random functions

$$\phi_{i,\varepsilon}(t) = \int_0^1 \tilde{G}_{i,\varepsilon}(t,s) f(s)\,\mathrm{d}s + \int_0^1 \tilde{G}_{i,\varepsilon}(t,s)\delta(s)\,\mathrm{d}W_s,$$

we obtain that there exists some non-random parameter $\varepsilon_* > 0$ such that for any $0 \leqslant \varepsilon \leqslant \varepsilon_*$ and for any $1 \leqslant i \leqslant 3$ there exist random functions $x_i^\varepsilon \in \mathcal{C}[0,1]$ for which

$$\Psi_{i,\varepsilon}(\varepsilon, x_i^\varepsilon) = \phi_{i,\varepsilon},$$

i.e.

$$x_i^\varepsilon(t) = \int_0^1 \tilde{G}_{i,\varepsilon}(t,s) B(s, x_i^\varepsilon(s))\,\mathrm{d}s + \int_0^1 \tilde{G}_{i,\varepsilon}(t,s) f(s)\,\mathrm{d}s + \int_0^1 \tilde{G}_{i,\varepsilon}(t,s)\delta(s)\,\mathrm{d}W_s. \tag{5.4}$$

By lemma A.2 this function satisfies the stochastic differential equation (1.7) with ith boundary condition in (1.2)–(1.4).

Now we show that this solution is unique. To this end note that by lemma A.2 we obtain that the uniqueness of (1.7) is equivalent to the uniqueness of (5.4). Moreover, by taking into account the asymptotic properties (4.1), (4.5) and the inequality (2.2), we can find some parameters $0 < \varepsilon_0 \leqslant \varepsilon_*$ and $0 < \theta < 1$ such that

$$\max_{1 \leqslant i \leqslant 3} \sup_{0 < \varepsilon \leqslant \varepsilon_0} \sup_{0 \leqslant t,s \leqslant 1} \sup_{x,y \in \mathbb{R}} |\tilde{G}_{i,\varepsilon}(t,s)| \frac{|B(s,x) - B(s,y)|}{|x-y|} \leqslant \theta.$$

This immediately implies that (5.4) has a unique solution. This completes the proof.

5.2. Proof of theorem 3.1

First, we set

$$\bar{x}^\varepsilon(t) = \frac{1}{\varepsilon^2} x_1^\varepsilon(t) - \varkappa(t) \quad \text{and} \quad \bar{G}_\varepsilon(t,s) = G_{1,\varepsilon}(t,s) - \Upsilon(t,s).$$

By making use of the representation (5.4) for $i = 1$, we obtain

$$\bar{x}^\varepsilon(t) = \int_0^1 (G_{1,\varepsilon}(t,s) B(s, x_1^\varepsilon(s)) - \Upsilon(t,s) B(s,0))\,\mathrm{d}s + \varsigma_t^\varepsilon, \tag{5.5}$$

where

$$\varsigma_t^\varepsilon = \int_0^1 \bar{G}_{1,\varepsilon}(t,s) f(s)\,\mathrm{d}s + \eta_t^\varepsilon$$

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and

$$\eta_t^\varepsilon = \int_0^1 \bar{G}_\varepsilon(t,s)\delta(s)\,\mathrm{d}W_s.$$

It is clear that the property (4.1) yields

$$\lim_{\varepsilon\to 0}\sup_{0\leqslant t\leqslant 1}\int_0^1 \bar{G}_{1,\varepsilon}(t,s)f(s)\,\mathrm{d}s = 0.$$

Furthermore, taking into account that

$$\Delta\bar{G}_\varepsilon(t,t) = \Delta\bar{G}_\varepsilon(t,t) - \Delta\bar{G}_\varepsilon(t-0,t) = 0,$$

lemma A.1 yields

$$\eta_t^\varepsilon = \eta_0^\varepsilon + \int_0^t \bar{D}_\varepsilon(s)\,\mathrm{d}s,$$

where

$$\bar{D}_\varepsilon(s) = \int_0^1 \bar{G}_{1,\varepsilon}(t,s)\delta(s)\,\mathrm{d}W_s \quad \text{and} \quad \bar{G}_{1,\varepsilon}(t,s) = \frac{\partial}{\partial t}\bar{G}_\varepsilon(t,s).$$

First, note that, in view of (4.1),

$$\mathbb{E}(\eta_0^\varepsilon)^2 = \int_0^1 \bar{G}_\varepsilon^2(0,s)\delta^2(s)\,\mathrm{d}s \leqslant |\bar{G}_\varepsilon|_*^2\|\delta\|^2 \to 0 \quad \text{as } \varepsilon\to 0.$$

Moreover,

$$\sup_{0\leqslant s\leqslant 1}\mathbb{E}(\bar{D}_\varepsilon(s))^2 \leqslant |\bar{G}_{1,\varepsilon}|_{1,*}^2,$$

and therefore, in view of (4.1),

$$\lim_{\varepsilon\to 0}\sup_{0\leqslant s\leqslant 1}\mathbb{E}(\bar{D}_\varepsilon(s))^2 = 0.$$

Thus,

$$\mathbb{P}\text{-}\lim_{\varepsilon\to 0}|\eta^\varepsilon|_* = 0 \quad \text{and} \quad \mathbb{P}\text{-}\lim_{\varepsilon\to 0}|\varsigma^\varepsilon|_* = 0.$$

Now we rewrite the equality (5.5) as

$$\bar{x}^\varepsilon(t) = \int_0^1 \Upsilon(t,s)(B(s,x_1^\varepsilon(s)) - B(s,0))\,\mathrm{d}s + \int_0^1 \bar{G}_\varepsilon(t,s)B(s,x_1^\varepsilon(s))\,\mathrm{d}s + \varsigma_t^\varepsilon. \tag{5.6}$$

Here, taking into account conditions (C1) and (C2), we obtain

$$\begin{aligned}|\bar{x}^\varepsilon|_* &\leqslant \beta|\Upsilon|_*|x_1^\varepsilon|_* + \beta^*|\bar{G}_\varepsilon|_* + |\varsigma^\varepsilon|_*\\ &\leqslant \varepsilon^2\beta|\Upsilon|_*|\bar{x}^\varepsilon|_* + \varepsilon^2\beta|\Upsilon|_*|\varkappa|_* + \beta^*|\bar{G}_\varepsilon|_* + |\varsigma^\varepsilon|_*.\end{aligned}$$

Therefore, for sufficiently small ε we get

$$|\bar{x}^\varepsilon|_* \leqslant \frac{1}{1-\varepsilon^2\beta|\Upsilon|_*}(\varepsilon^2\beta|\Upsilon|_*|\varkappa|_* + \beta^*|\bar{G}_\varepsilon|_* + |\varsigma^\varepsilon|_*) \to 0 \quad \text{as } \varepsilon\to 0.$$

This completes the proof.

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5.3. Proof of theorem 3.2

First, note that the boundedness of the function $B(s,x)$ in condition (C1) implies

$$\lim_{x\to+\infty} B_0(x) = +\infty \quad \text{and} \quad \lim_{x\to-\infty} B_0(x) = -\infty.$$

Moreover, by the inequality (2.2) we get

$$\boldsymbol{b}_* = \inf_{x\in\mathbb{R}} |B_0'(x)| > 0. \tag{5.7}$$

These properties imply that for any $a \in \mathbb{R}$ the equation

$$B_0(x) = a$$

has a unique solution, i.e. the function B_0 defined in (3.4) is invertible on $\mathbb{R}$. Moreover, the representation (5.4) implies

$$B_0(x^\varepsilon(t)) = \eta + u^\varepsilon(t) + v^\varepsilon(t), \tag{5.8}$$

where

$$u^\varepsilon(t) = \int_0^1 V^\varepsilon(t,s)(B(s,x^\varepsilon(s)) + f(s))\,\mathrm{d}s$$

and

$$v^\varepsilon(t) = \int_0^1 V^\varepsilon(t,s)\delta(s)\,\mathrm{d}W_s \quad \text{with } V^\varepsilon(t,s) = \tilde{G}_\varepsilon(t,s) + \frac{1}{p}.$$

The limit (4.5) implies directly that

$$\mathbb{P}\text{-}\lim_{\varepsilon\to0} \sup_{0\leqslant t\leqslant1} |u^\varepsilon(t)| = 0. \tag{5.9}$$

Now, we show that

$$\mathbb{P}\text{-}\lim_{\varepsilon\to0} \sup_{0\leqslant t\leqslant1} |v^\varepsilon(t)| = 0. \tag{5.10}$$

Indeed, by lemma A.1, the process $v^\varepsilon(t)$ is almost surely continuously differentiable, and, taking into account that $\Delta V^\varepsilon(t,t) = 0$, we find that

$$v^\varepsilon(t) = v^\varepsilon(0) + \int_0^t \check{D}_\varepsilon(s)\,\mathrm{d}s,$$

where

$$\check{D}_\varepsilon(s) = \int_0^1 \frac{\partial}{\partial s} V^\varepsilon(s,\vartheta)\delta(\vartheta)\,\mathrm{d}W_\vartheta = \int_0^1 \frac{\partial}{\partial s}\tilde{G}_\varepsilon(s,\vartheta)\delta(\vartheta)\,\mathrm{d}W_\vartheta.$$

Recall that $\tilde{G}_\varepsilon$ is equal to $\tilde{G}_{2,\varepsilon}$ or $\tilde{G}_{3,\varepsilon}$. So, by (4.5)

$$\mathbb{E}(v^\varepsilon(0))^2 = \int_0^1 (V^\varepsilon(0,s))^2\mathrm{d}s \to 0 \quad \text{as } \varepsilon\to0,$$

and by (4.6)

$$\sup_{0\leqslant s\leqslant1} \mathbb{E}(\check{D}_\varepsilon(s))^2 = \sup_{0\leqslant s\leqslant1} \int_0^1 \left(\frac{\partial}{\partial s}\tilde{G}_\varepsilon(s,\vartheta)\right)^2 \mathrm{d}\vartheta \to 0 \quad \text{as } \varepsilon\to0.$$

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This immediately implies (5.10). Therefore, taking into account that

$$B_0(\zeta) = \eta$$

and

$$|x - y| \leqslant \frac{1}{\boldsymbol{b}_*}|B_0(x) - B_0(y)|,$$

we obtain theorem 3.2.

6. Conclusion

In this paper we introduced and studied the second-order differential equations with Brownian motion and a small parameter with boundary conditions. We proved an existence and uniqueness theorem for random solutions and proved the average principle as the small parameter goes to zero. By the Green function method we obtained the explicit limit forms for solutions.

Appendix A.

A.1. Technical lemmas

LEMMA A.1. *Let $G(t,s)$ be a function having the form*

$$G(t,s) = \begin{cases} \sum_{j=1}^{m} g_{1j}(t)h_{1j}(s) & \text{if } t \geqslant s; \\ \sum_{j=1}^{m} g_{2j}(t)h_{2j}(s) & \text{if } t < s. \end{cases}$$

Assume that the functions g_{1j} and g_{2j} belong to $\boldsymbol{C}^2[0,1]$. Then the stochastic process

$$\xi_t = \int_0^1 G(t,s)\delta(s)\,\mathrm{d}W_s$$

admits the following Itô differential:

$$\mathrm{d}\xi_t = D(t)\,\mathrm{d}t + \Delta G(t,t)\delta(t)\,\mathrm{d}W_t, \tag{A 1}$$

where

$$D(t) = \int_0^1 G_1(t,s)\delta(s)\,\mathrm{d}W_s \quad \text{with } G_1(t,s) = \frac{\partial}{\partial t}G(t,s)$$

and

$$\Delta G(t,t) = G(t,t) - G(t-0,t) = \sum_{j=1}^{m} g_{1j}(t)h_{1j}(t) - \sum_{j=1}^{m} g_{2j}(t)h_{2j}(t).$$

Proof. First, note that

$$\xi_t = \int_0^t G(t,s)\delta(s)\,\mathrm{d}W_s + \int_t^1 G(t,s)\delta(s)\,\mathrm{d}W_s.$$

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Taking into account here the definition of the function G, we get

$$\begin{aligned}\xi_t &= \sum_{j=1}^{m} g_{1j}(t)\int_0^t h_{1j}(s)\,\mathrm{d}W_s + \sum_{j=1}^{m} g_{2j}(t)\int_t^1 h_{2j}(s)\,\mathrm{d}W_s \\ &= \sum_{j=1}^{m} g_{1j}(t)\int_0^t h_{1j}(s)\,\mathrm{d}W_s + \sum_{j=1}^{m} g_{2j}(t)\int_0^1 h_{2j}(s)\,\mathrm{d}W_s \\ &\quad - \sum_{j=1}^{m} g_{2j}(t)\int_0^t h_{2j}(s)\,\mathrm{d}W_s.\end{aligned}$$

Now, the lemma follows directly from the Itô formula. □

Now we make use of the following conditions for the Green functions.

(H1) The $[0,1]\times[0,1]\to\mathbb{R}$ function $G(\cdot,\cdot)$ is continuous and has the form

$$G(t,s) = \begin{cases} g_{11}(t)h_{11}(s) + g_{12}(t)h_{12}(s) & \text{if } t \geqslant s, \\ g_{21}(t)h_{21}(s) + g_{22}(t)h_{22}(s) & \text{if } t < s, \end{cases} \tag{A 2}$$

where the functions $g_{11}(\cdot)$, $g_{12}(\cdot)$, $g_{21}(\cdot)$ and $g_{22}(\cdot)$ are twice continuously differentiable such that

$$\frac{\partial^2}{\partial t^2}G(t,s) - p(t)G(t,s) = 0 \tag{A 3}$$

for any $t \neq s$ from $[0,1]$.

(H2) The first partial derivative of the function G has the jump

$$\frac{\partial}{\partial t}G(t,t) - \frac{\partial}{\partial t}G(t-0,t) = 1.$$

Lemma A.2. *Assume that the function G satisfies conditions (H1) and (H2) and that φ is an arbitrary function from $\boldsymbol{C}[0,1]$. Then, almost surely, the stochastic process*

$$\xi_t = \int_0^1 G(t,s)\varphi(s)\,\mathrm{d}s + \int_0^1 G(t,s)\delta(s)\,\mathrm{d}W_s$$

has a continuous derivative and satisfies the following second-order stochastic equation:

$$\ddot{\xi}_t = p(t)\xi_t + \varphi(t) + \delta(t)\dot{W}_t, \tag{A 4}$$

i.e. the derivative $\dot{\xi}_t$ satisfies the following stochastic differential equation:

$$\mathrm{d}\dot{\xi}_t = p(t)\xi_t\mathrm{d}t + \varphi(t)\,\mathrm{d}t + \delta(t)\,\mathrm{d}W_t.$$

Proof. First, we set

$$\boldsymbol{a}_t = \int_0^1 G(t,s)\varphi(s)\,\mathrm{d}s \quad \text{and} \quad \boldsymbol{b}_t = \int_0^1 G(t,s)\delta(s)\,\mathrm{d}W_s. \tag{A 5}$$

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It is easy to check directly that

$$\ddot{\boldsymbol{a}}_t = p(t)\boldsymbol{a}_t + \varphi(t).$$

Moreover, note that condition (H1) implies $\Delta G(t,t) = 0$. Therefore, by lemma A.1 we obtain

$$\boldsymbol{b}_t = \int_0^t D(s)\,\mathrm{d}s,$$

where the process $D(t)$ is defined in (A 1). From here we obtain that

$$\dot{\boldsymbol{b}}_t = D(t) = \int_0^1 G_1(t,s)\delta(s)\,\mathrm{d}W_s.$$

Therefore, by lemma A.1 we find

$$\mathrm{d}\dot{\boldsymbol{b}}_t = D_1(t)\,\mathrm{d}t + \Delta G_1(t,t)\delta(t)\,\mathrm{d}W_t,$$

where

$$D_1(t) = \int_0^1 \frac{\partial}{\partial t} G_1(t,s)\delta(s)\,\mathrm{d}W_s = \int_0^1 \frac{\partial^2}{\partial t^2} G(t,s)\delta(s)\,\mathrm{d}W_s.$$

Now, in view of (A 3) and the definition (A 5), we obtain that

$$D_1(t) = p(t)\boldsymbol{b}(t).$$

Therefore, condition (H2) implies this lemma. □

A.2. Green functions

In this section we consider the second-order $\boldsymbol{C}^2[0,1] \to \boldsymbol{C}^2[0,1]$ linear differential operator L defined by

$$L(x)(t) = \ddot{x}(t) + p(t)\dot{x}(t) + q(t)x(t),$$

with boundary conditions

$$\alpha_1 x(0) + \beta_1 \dot{x}(0) = 0 \quad \text{and} \quad \alpha_2 x(1) + \beta_2 \dot{x}(1) = 0, \tag{A 6}$$

where α_1, β_1, α_2 and β_2 are some fixed constants. We shall also consider the following mixed conditions:

$$x(0) = x(1) \quad \text{and} \quad \dot{x}(0) = \dot{x}(1). \tag{A 7}$$

The operator L is called *regular* if the equation

$$L(x) = 0 \tag{A 8}$$

with the boundary conditions (A 6) has only trivial solution $x \equiv 0$. If the operator L is regular, then the problem

$$L(x) = f$$

with the boundary conditions (A 6) may be written as

$$x(t) = \int_0^1 G(t,s)f(s)\,\mathrm{d}s$$

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(see, for example, [18, p. 39]). Here the $[0,1]^2 \to \mathbb{R}$ function G is called the *Green function* generated by the operator L and the boundary conditions (A 6). The function G has the following properties.

(1) $G(t,s)$ is continuous with respect to t and s.

(2) For $t \neq s$ the function $G(t,s)$ is twice continuously differentiable with respect to t and s. Moreover, for $t \neq s$,

$$\frac{\partial^2}{\partial t^2}G(t,s) + p(t)\frac{\partial}{\partial t}G(t,s) + q(t)G(t,s) = 0$$

and

$$\alpha_1 G(0,s) + \beta_1 \frac{\partial}{\partial t}G(0,s) = 0 \quad \text{and} \quad \alpha_1 G(1,s) + \beta_1 \frac{\partial}{\partial t}G(1,s) = 0,$$

or

$$G(0,s) = G(1,s) \quad \text{and} \quad \frac{\partial}{\partial t}G(0,s) = \frac{\partial}{\partial t}G(1,s).$$

(3) The function G is symmetric, i.e.

$$G(t,s) = G(s,t).$$

(4) The partial derivative with respect to t has a unitary jump, i.e.

$$\frac{\partial}{\partial t}G(t+0,t) - \frac{\partial}{\partial t}G(t-0,t) = 1.$$

A.3. Uniform implicit function theorem

Let F be a Banach space and let $\Psi\colon [0,1] \times F \to F$ be a continuous function. In this section we study the following equation:

$$\Psi(\varepsilon, x) = \phi \quad \text{for } \phi \in F. \tag{A 9}$$

(AP1) Equation (A 9) for $\varepsilon = 0$ has a unique solution x_0^ϕ for any $\phi \in F$.

(AP2) For any $\phi \in F$, the partial Fréchet derivative

$$\Upsilon(\phi) = \frac{\partial}{\partial x}\Psi(0, x_0^\phi)$$

is an $F \to F$ isomorphism and

$$\sup_{\phi \in F} |\Upsilon^{-1}(\phi)| < \infty. \tag{A 10}$$

(AP3) The function Ψ is uniform continuous in ε, i.e.

$$\lim_{\varepsilon \to 0} \sup_{x \in F} |\Psi(\varepsilon, x) - \Psi(0, x)| = 0.$$

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(AP4) The function Ψ is uniformly differentiable in the Fréchet sense with respect to x, i.e.

$$\lim_{\delta\to 0}\sup_{0<\varepsilon\leqslant 1}\sup_{|x-y|\leqslant\delta}\left|\Psi(\varepsilon,y)-\Psi(\varepsilon,x)-\frac{\partial}{\partial x}\Psi(\varepsilon,x)(y-x)\right|\frac{1}{|x-y|}=0.$$

(AP5) The partial Fréchet derivative of $\partial\Psi/\partial x$ is uniformly continuous, i.e.

$$\lim_{\varepsilon\to 0}\limsup_{\delta\to 0}\sup_{|x-y|<\delta}\left|\frac{\partial}{\partial x}\Psi(\varepsilon,x)-\frac{\partial}{\partial x}\Psi(0,y)\right|=0.$$

THEOREM A.3. *Assume that conditions (AP1)–(AP5) hold. Then there exist* $\varepsilon_0\in(0,1)$ *and* $\delta>0$ *such that for any* $0<\varepsilon\leqslant\varepsilon_0$ *and any* $\phi\in F$ *there exists a unique* $x_\varepsilon^\phi\in\boldsymbol{B}(x_0^\phi,\delta)$ *satisfying (A 9), i.e.*

$$\Psi(\varepsilon,x_\varepsilon^\phi)=\phi,$$

where $\boldsymbol{B}(a,\delta)=\{x\in F\colon|x-a|\leqslant\delta\}$.

Proof. Using the method in [7, p. 57] we introduce the following special function:

$$F(\varepsilon,z)=z-\Upsilon^{-1}(\phi)(\Psi(\varepsilon,z)-\phi).$$

Note that conditions (AP2), (AP4) and (H5) imply that there exist $\varepsilon_*>0$, $\delta>0$ and $0<\theta<1$ such that for any $0\leqslant\varepsilon\leqslant\varepsilon_0$ and x, y from $\boldsymbol{B}(x_0^\phi,\delta)$ we obtain

$$|F(\varepsilon,x)-F(\varepsilon,y)|\leqslant\theta|x-y|.$$

Now, we define the following approximating sequence:

$$z_n^\varepsilon=F(\varepsilon,z_{n-1}^\varepsilon)\quad\text{for } n\geqslant 1$$

and $z_0^\varepsilon=x_0^\phi$. First, note that

$$|z_1-x_0^\phi|=|z_1-z_0|=|\Upsilon^{-1}(\phi)(\Psi(\varepsilon,x_0^\phi)-\Psi(0,x_0^\phi))|,$$

and by the inequality (A 10) and condition (AP3) we can choose $0<\varepsilon_0\leqslant\varepsilon_*$ such that, for all $0<\varepsilon\leqslant\varepsilon_0$,

$$|z_1^\varepsilon-z_0^\varepsilon|\leqslant(1-\theta)\delta<\delta,$$

i.e. $z_1^\varepsilon\in\boldsymbol{B}(z_0,\delta)$. Now, by induction, we assume that $z_j^\varepsilon\in\boldsymbol{B}(z_0,\delta)$ for all $1\leqslant j\leqslant n-1$ with $n\geqslant 2$. Let us check that in this case $z_n^\varepsilon\in\boldsymbol{B}(z_0,\delta)$ as well. Indeed, taking our condition into account directly implies

$$|z_n^\varepsilon-z_{n-1}^\varepsilon|\leqslant\theta|z_{n-1}^\varepsilon-z_{n-2}^\varepsilon|\leqslant\theta^{n-1}|z_1^\varepsilon-z_0^\varepsilon|.$$

We get

$$|z_n^\varepsilon-x_0^\phi|=|z_n^\varepsilon-z_0^\varepsilon|\leqslant\sum_{j=1}^n|z_j^\varepsilon-z_{j-1}^\varepsilon|\leqslant\sum_{j=1}^n\theta^{j-1}|z_1^\varepsilon-z_0^\varepsilon|\leqslant\delta.$$

Thus, $z_n^\varepsilon\in\boldsymbol{B}(z_0,\delta)$ for all $n\geqslant 2$. Therefore, for any $n\geqslant 1$,

$$|z_n^\varepsilon-z_{n-1}^\varepsilon|\leqslant\theta|z_{n-1}^\varepsilon-z_{n-2}^\varepsilon|,$$

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i.e. for any $0 \leqslant \varepsilon \leqslant \varepsilon_0$ there exists the limit

$$\lim_{n\to\infty} z_n^\varepsilon = z_*^\varepsilon,$$

which is the unique solution of the equation $z = F(\varepsilon, z)$ in the ball $\boldsymbol{B}(z_0, \delta)$ for $\varepsilon \leqslant \varepsilon_0$. This immediately implies theorem A.3. □

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References

1 Y. Aït-Sahalia, P. A. Mykland and L. Zhang. How often to sample a continuous-time process in the presence of market microstructure noise. *Rev. Fin. Stud.* **18** (2005), 351–416.
2 O. Alvarez and M. Bardi. Viscosity solutions methods for singular perturbations in deterministic and stochastic control. *SIAM J. Control Optim.* **40** (2001), 1159–1188.
3 Z. Artzein and V. Gaitsgory. The value function of singularly perturbed control systems. *Appl. Math. Optim.* **41** (2000), 425–445.
4 A. Bensoussan. *Perturbation methods in optimal control* (New York: Wiley/Gauthier Villars, 1988).
5 N. N. Bogolubov and Yu. A. Mitropol′skii. *Asymptotic methods in the theory of nonlinear oscillation* (Moscow: Fizmatgiz, 1963; in Russian).
6 V. F. Butuzov, A. B. Vasil′eva and M. V. Fedoryuk. Asymptotic methods in the theory of ordinary differential equations. In *Mathematical analysis*, Progress in Mathematics, vol. 8, pp. 1–82 (New York: Springer, 1970).
7 H. Cartan. *Cours de calcul différentiel* (Paris: Hermann, 1990).
8 N. El Karoui and L. Mazliak. *Backward stochastic differential equations*. Pitman Research Notes in Mathematics Series, vol. 364 (Harlow: Longman, 1997).
9 M. V. Fedoryuk. *Asymptotic analysis: linear ordinary differential equations* (Berlin: Springer, 1993).
10 J.-P. Fouque, G. Papanicolaou and K. Sicar. *Derivatives in financial markets with stochastic volatility* (Cambridge University Press, 2000).
11 M. I. Freidlin and A. D. Wentzell. *Random perturbations of dynamical systems* (New York: Springer, 1986).
12 Yu. Kabanov and S. Pergamenshchikov. *Two scale stochastic systems: asymptotic analysis and control*. Applications of Mathematics: Stochastic Modeling and Applied Probability, vol. 49 (Berlin: Springer, 2003).
13 M. Kamenskii, P. Nistri and M. Quincampoix. Sliding mode control of uncertain systems: a singular perturbation approach. *IMA J. Math. Control Informat.* **19** (2002), 377–398.
14 I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus* (Berlin: Springer, 1991).
15 R. Khasminskii and N. Krylov. On averaging principle for diffusion processes with nullrecurrent fast component. *Stoch. Process. Applic.* **93** (2001), 229–240.
16 H. Kushner. *Weak convergence methods and singularly perturbed stochastic control and filtering* (New York: IEEE Press, 1986).

https://doi.org/10.1017/S0308210516000354 Published online by Cambridge University Press

17 D. Lamberton and B. Lapeyre. *Introduction to stochastic calculus applied to finance* (London: Chapman and Hall, 1996).

18 M. A. Naimark. *Linear differential operators* (English transl. E. R. Dawson, ed. W. N. Everitt) (London: Harrap, 1968).

19 A. Papavasiliou, G. A. Pavliotis and A. M. Stuart. Maximum likelihood drift estimation for multi-scale diffusions. *Stoch. Process. Applic.* **119** (2009), 3173–3210.

20 G. A. Pavliotis and A. M. Stuart. Parameter estimation for multiscale diffusions. *J. Statist. Phys.* **127** (2007), 741–781.

21 G. A. Pavliotis and A. M. Stuart. *Multiscale methods: averaging and harmonization*. Texts in Applied Mathematics, vol. 53 (New York: Springer, 2008).

22 A. N. Tikhonov. On systems of differential equations containing parameters. *Mat. Sb.* **27** (1950), 147–156.

23 A. N. Tikhonov. Systems of differential equations containing a small parameters in the derivatives. *Mat. Sb.* **31** (1952), 575–586.

24 L. Zhang, P. A. Mykland and Y. Aït-Sahalia. A tale of two time scales: determining integrated volatility with noisy high-frequency data. *J. Am. Stat. Assoc.* **100** (2005), 1394–1411.

https://doi.org/10.1017/S0308210516000354 Published online by Cambridge University Press