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ON BOUNDS AND APPROXIMATING WEIGHTED DISTRIBUTIONS BY EXPONENTIAL DISTRIBUTIONS

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In this article, we obtain error bounds for exponential approximations to the classes of weighted residual and equilibrium lifetime distributions with monotone weight functions. These bounds are obtained for the class of distributions with increasing (decreasing) hazard rate and mean residual life functions.

1. INTRODUCTION

The application of weighted distributions in several areas of scientific studies is well known in the literature. These distributions have enjoyed applications in areas such as reliability studies, renewal theory, biometry, ecology, and wildlife population studies. For example, length-biased sampling is widely used for the collection and analysis of wildlife data (Patil and Rao [7]), fiber data (Daniels, [2]), or lifetime data (Zelen and Feinleib [8], Gupta and Keating [3], Oluyede [5,6]). In renewal theory, the residual lifetimes have a limiting distribution that is a weighted distribution with weight function that is equal to the reciprocal of the hazard or failure rate function. When data are unknowingly sampled from a weighted distribution as opposed to the parent distribution, the survival function, hazard function, and mean residual life function (MRLF) might be underestimated or overestimated depending on the weight function. For size-biased sampling, the analyst will usually give an overoptimistic estimate of the survival function and MRLFs. It is well known that the size-biased distribution of an increasing failure rate (IFR) distribution is always IFR. The converse is not true. Also, if the weight function is monotone increasing and concave, then the weighted distribution of an IFR distribution is an

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IFR distribution. Similarly, the size-biased distribution of a decreasing mean residual (DMRL) distribution has decreasing mean residual life.

The purpose of this article is to establish bounds and stability results on the distance among weighted residual lifetime distribution functions, weighted equilibrium lifetime distributions with monotone weight functions, and the exponential counterpart including the length-biased exponential distribution function in the class of life distributions with increasing or decreasing hazard rate and MRLFs. We obtain error bounds and inequalities for exponential approximations to the classes of weighted residual life distributions and weighted equilibrium distributions with monotone weight functions for the class of increasing or decreasing hazard rate distributions. Section 2 is concerned with some basic definitions and results. In Section 3 inequalities and stability results for repairable systems under weighted lifetime distributions are presented. Section 4 is concerned with inequalities for reliability measures for weighted residual lifetime distributions. In Section 5 some applications are presented.

2. SOME DEFINITIONS AND BASIC RESULTS

In this section we present some definitions and basic results. Let $\mathcal{F}$ be the set of absolutely continuous distribution functions satisfying

$$H(0) = 0, \qquad \lim_{x\to\infty} H(x) = 1, \qquad \sup\{x : H(x) < 1\} = \infty. \tag{1}$$

Note that if the mean of a random variable in $\mathcal{F}$ is finite, it is positive. Let X be a nonnegative random variable with distribution function F, survival function $\bar{F}$, and probability density function (p.d.f.) f. The weighted random variable X_W has a survival function given by

$$\bar{F}_W(x) = \bar{F}(x)(W(x) + T_F(x))/E_F(W(X)), \tag{2}$$

where $T_F(x) = \int_x^\infty (\bar{F}(u)W'(u)\,du)/\bar{F}(x)$ and $W'(u) = dW(u)/du$, assuming that $W(x)\bar{F}(x) \to 0$ as $x \to \infty$. The corresponding probability density function of the weighted random variable X_W is given by

$$f_W(x) = W(x)f(x)/\delta^*, \tag{3}$$

where δ^* is a normalizing constant. Note that $0 < \delta^* = E_F(W(X)) < \infty$. Patil and Rao [7] referred to (2) as a weighted distribution with weight function $W(x) \geq 0$.

The residual life at age t has a weighted distribution, with survival function given by

$$\bar{F}_t(x) = \bar{F}(x + t)/\bar{F}(t), \tag{4}$$

for $x \geq 0$. The weight function is $W(x) = f(x + t)/f(x)$, where $f(u) = dF(u)/du$ and the hazard function and MRLFs are $\lambda_{F_t}(x) = \lambda_F(x + t)$ and $\delta_{F_t}(x) = \delta_F(x + t)$,

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respectively. It is clear that if F is an IFR (DMRL) distribution, then F_t is an IFR (DMRL) distribution. The hazard function $\lambda_F(x)$ and MRLF $\delta_F(x)$ are given by $\lambda_F(x) = f(x)/\bar{F}(x)$ and $\delta_F(x) = \int_x^\infty \bar{F}(u)\,du/\bar{F}(x)$, respectively. The functions $\lambda_F(x)$, $\delta_F(x)$, and $\bar{F}(x)$ are equivalent (Gupta and Keating [3]). We now give some basic and important definitions.

DEFINITION 1: *Let X and Y be two random variables with distribution functions F and G, respectively. We say $F <_{\text{st}} G$, stochastically ordered if $\bar{F}(x) \le \bar{G}(x)$ for $x \ge 0$ or, equivalently, for any increasing function $\Phi(x)$,*

$$E(\Phi(X)) \le E(\Phi(Y)). \tag{5}$$

DEFINITION 2: *A distribution function F is an increasing hazard rate (IHR) distribution if $\bar{F}(x+t)/\bar{F}(t)$ is decreasing in $0 < t < \infty$ for each $x \ge 0$. Similarly, a distribution function F is an decreasing hazard rate (DHR) distribution if $\bar{F}(x+t)/\bar{F}(t)$ is increasing in $0 < t < \infty$ for each $x \ge 0$. It is well known that IHR (DHR) implies DMRL (increasing mean residual life (IMRL)).*

Consider two renewal processes with life distribution function $F(x)$ and weighted distribution function $F_W(x)$, with weight function $W(x) > 0$. Let X_t denote the residual life of the unit functioning at time t. Then as $t \to \infty$, X_t has the limiting reliability function given by

$$\bar{F}_e(x) = \mu_F^{-1} \int_x^\infty \bar{F}(y)\,dy, \tag{6}$$

$x \ge 0$. The corresponding limiting p.d.f. is

$$f_e(x) = \bar{F}(x)/\mu_F, \tag{7}$$

$x \ge 0$. The weighted equilibrium reliability or survival function is

$$\bar{F}_{W_e}(x) = \mu_{F_W}^{-1} \int_x^\infty \bar{F}_W(y)\,dy, \tag{8}$$

where $\bar{F}_W(x)$ is given by (2). Note that $\bar{F}_{W_e}(x)$ can be expressed as

$$\bar{F}_{W_e}(x) = \mu_{F_W}^{-1} \bar{F}_W(x) \delta_{F_W}(x), \tag{9}$$

$x \ge 0$, and the corresponding hazard function is

$$\lambda_{F_{W_e}}(x) = f_{W_e}(x)/\bar{F}_{W_e}(x) = (\delta_{F_{W_e}}(x))^{-1}, \tag{10}$$

$x \ge 0$, where

$$\delta_{F_{W_e}}(x) = (\delta_F(x)\bar{F}(x))^{-1} \int_x^\infty \bar{F}(y)\delta_F(y)\,dy, \tag{11}$$

$x \ge 0$.

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3. INEQUALITIES FOR REPAIRABLE SYSTEMS UNDER WEIGHTED DISTRIBUTIONS

In this section we obtain useful inequalities for repairable systems under weighted distributions.

THEOREM 1: (Barlow, Marshall, and Proschan [1]): *If F has DMRL, then* $S_k(x) \leq S_k(0)e^{-x/\mu}, k = 1,2,\ldots,$ *and* $S_k(x) \geq \mu S_{k-1}(0)e^{-x/\mu} - \mu S_{k-1}(0) + S_k(0), k = 2,3,\ldots,$ *where*

$$S_k(x) = \begin{cases} \bar{F}(x) & \text{if } k = 0 \\ \displaystyle\int_0^\infty \bar{F}(x+t)t^{k-1}\,dt/(k-1)! & \text{if } k = 1,2,\ldots, \end{cases}$$

is a sequence of decreasing functions for which F possess moments of order J; that is, $\mu_k = E(X^k)$ *exists,* $k = 1,2,\ldots,J$.

We let $S_{-1}(x) = f(x)$ be the p.d.f. of F if it exists. Then $S_k(0) = \mu_k/k!$ and $S_k'(x) = -S_{k-1}(x), k = 0,1,2,\ldots,J$. The ratio $S_{k-1}(x)/S_k(x)$ is a hazard function of a distribution function with survival function $S_k(x)/S_k(0)$. The inequalities in Theorem 1 are reversed if *F* has IMRL.

THEOREM 2: *If* $\bar{F}_W(x)$ *is an IHR reliability function with increasing weight function, then*

$$\int_0^\infty |\bar{F}_W(x) - xe^{-x/\mu}|\,dx \leq 2\mu|\mu - \mu_2/2\mu^2|, \tag{12}$$

where $\bar{F}_W(x)$ *is given by (2).*

PROOF: Let $E = \{x|\bar{F}_W \leq xe^{-x/\mu}\}$. Then for $x \geq 0$,

$$\begin{aligned} \int_0^\infty |\bar{F}_W(x) - xe^{-x/\mu}|\,dx &\leq 2\int_E (xe^{-x/\mu} - \bar{F}_W(x))\,dx \\ &\leq 2\int_0^\infty (xe^{-x/\mu} - \bar{F}_W(x))\,dx \\ &\leq 2\left\{\int_0^\infty (xe^{-x/\mu} - \bar{F}(x))\,dx\right\} \\ &\leq 2\int_0^\infty (xe^{-x/\mu} - S_1(x)/\mu)\,dx \\ &= 2(\mu^2 - \mu_2/2\mu) \\ &= 2\mu(\mu - \mu_2/2\mu^2). \end{aligned} \tag{13}$$

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The first two inequalities are straightforward; the third inequality follows from the fact that $W(x)$ is increasing, so that $\bar{F}_W$ and $\bar{F}$ are stochastically ordered. The fourth inequality follows from Theorem 1. ■

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of operating times from a repairable system that starts functioning at time $t = 0$. The sequence of times $\{X_i\}_{i=1}^{\infty}$ forms a renewal-type stochastic point process. Following Kijima [4], if a system has virtual age $T_{m-1} = t$ immediately after the $(m-1)$th repair, then the length of the mth cycle X_m has the distribution

$$F_t(x) = P(X_m \leq x | T_{m-1} = t) = \{F(x+t) - F(t)\}/\bar{F}(t), \tag{14}$$

$x \geq 0$, where $\bar{F}(x) = 1 - F(x)$ is the reliability function of a new system. When $t = \sum_{i=1}^{j} X_i$, $j = 1,2,\ldots,m-1$, minimal repair is performed, keeping the virtual age intact, and when $t = 0$, we have perfect repair. The virtual age of the system is equal to its operating time for the case of minimal repair. The reliability function corresponding to (14) is given by

$$\bar{F}_t(x) = \bar{F}(x+t)/\bar{F}(t), \tag{15}$$

$x \geq 0$. The corresponding weighted reliability function is given by

$$\bar{F}_{W_t}(x) = \bar{F}_W(x+t)/\bar{F}_W(t), \tag{16}$$

$x \geq 0$, where $\bar{F}_W$ is given by (2). When $W(x) = x$, the weighted reliability function reduces to the length-biased reliability function given by

$$\bar{F}_{l_t}(x) = \bar{F}_l(x+t)/\bar{F}_l(t), \tag{17}$$

where $\bar{F}_l(t)$ is given by

$$\bar{F}_l(x) = \bar{F}(x)(x + \delta_F(x))/\mu, \tag{18}$$

where $\delta_F(x) = \int_x^{\infty}(\bar{F}(u)\,du)/\bar{F}(x)$.

THEOREM 3: *If $\bar{F}_{W_t}(x)$ is an IHR reliability function with increasing weight function, then*

$$\int_0^{\infty} |\bar{F}_{W_t}(x) - \bar{F}_l(x)|\,dx \leq 2\mu|1 + \mu - \mu_2/2\mu^2|, \tag{19}$$

where $\bar{F}_l(x) = \{1 + x/\mu\}e^{-x/\mu}$ is the length-biased exponential reliability function.

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PROOF: Let $A = \{x | \bar{F}_{W_t} \leq \{1 + x/\mu\}e^{-x/\mu}\}$. Then for $x \geq 0$, we have

$$\begin{aligned}
\int_0^\infty |\bar{F}_{W_t}(x) - \{1 + x/\mu\}e^{-x/\mu}|\, dx &\leq 2\int_A (\{1 + x/\mu\}e^{-x/\mu} - \bar{F}_{W_t}(x))\, dx \\
&\leq 2\int_0^\infty (\{1 + x/\mu\}e^{-x/\mu} - \bar{F}_{W_t}(x))\, dx \\
&\leq 2\left\{\int_0^\infty (\{1 + x/\mu\}e^{-x/\mu} - \bar{F}_W(x + t))\, dx\right\} \\
&\leq 2\left\{\int_0^\infty (\{1 + x/\mu\}e^{-x/\mu} - \bar{F}(x + t))\, dx\right\} \\
&\leq 2\int_0^\infty (\{1 + x/\mu\}e^{-x/\mu} - S_1(x + t)/\mu)\, dx \\
&= 2(\mu + \mu^2 - \mu_2/2\mu) \\
&= 2\mu(1 + \mu - \mu_2/2\mu^2). \qquad \textbf{(20)}
\end{aligned}$$

The first two inequalities are straightforward; the third inequality follows from the fact that $W(x)$ is increasing, so that $\bar{F}_W$ and $\bar{F}$ are stochastically ordered. The fourth and fifth inequalities follow from Theorem 1. ■

THEOREM 4: *If $\bar{F}_{W_t}(x)$ is an IHR reliability function with increasing weight function, then*

$$\int_0^\infty |\bar{F}_{W_t}(x) - xe^{-x/\mu}|\, dx \leq 2\mu|1 - \mu_2/2\mu^2|. \qquad \textbf{(21)}$$

PROOF: Let $B = \{x | \bar{F}_{W_t} \leq xe^{-x/\mu}\}$. Then for $x \geq 0$,

$$\begin{aligned}
\int_0^\infty |\bar{F}_{W_t}(x) - xe^{-x/\mu}|\, dx &\leq 2\int_B (xe^{-x/\mu} - \bar{F}_{W_t}(x))\, dx \\
&\leq 2\int_0^\infty (xe^{-x/\mu} - \bar{F}_W(x + t))\, dx \\
&\leq 2\int_0^\infty (xe^{-x/\mu} - \bar{F}(x + t))\, dx \\
&\leq 2\int_0^\infty (xe^{-x/\mu} - S_1(x + t)/\mu)\, dx \\
&= 2(\mu - \mu_2/2\mu) \\
&= 2\mu(1 - \mu_2/2\mu^2). \qquad \textbf{(22)}
\end{aligned}$$

The first two inequalities are straightforward; the third inequality follows from the fact that $W(x)$ is increasing, so that $\bar{F}_{W_t}$ and $\bar{F}$ are stochastically ordered. ■

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THEOREM 5: *If $\bar{F}_{W_t}(x)$ is a DHR reliability function, then*

$$\int_0^\infty |\bar{F}_{W_t}(x) - xe^{-x/\mu}|\,dx \geq 2\mu e^{-\epsilon/\mu} \max\{0, e^{-t/\mu} - \epsilon\mu^{-1} - 1\}, \tag{23}$$

provided $W(x)$ is an increasing weight function.

PROOF: Let $\bar{F}_{W_t}$ be a DHR survival function, then there exist $\epsilon \geq \mu$ such that $\bar{F}_{W_t} \leq xe^{-x/\mu}$ or $\bar{F}_{W_t} \geq xe^{-x/\mu}$ as $x \leq \epsilon$ or $x \geq \epsilon$. Now

$$\begin{aligned}
\int_0^\infty |\bar{F}_{W_t}(x) - xe^{-x/\mu}|\,dx &= 2\int_\epsilon^\infty (\bar{F}_{W_t}(x) - xe^{-x/\mu})\,dx \\
&\geq 2\int_\epsilon^\infty (\bar{F}_W(x+t) - xe^{-x/\mu})\,dx \\
&\geq 2\int_\epsilon^\infty (\bar{F}(x+t) - xe^{-x/\mu})\,dx \\
&= 2\{S_1(\epsilon + t) - (\epsilon e^{-\epsilon/\mu} + \mu e^{-\epsilon/\mu})\} \\
&\geq 2\mu S_0(\epsilon + t) - (\epsilon e^{-\epsilon/\mu} + \mu e^{-\epsilon/\mu}) \\
&= 2\{\mu e^{(-\epsilon+t)/\mu} - (\epsilon e^{-\epsilon/\mu} + \mu e^{-\epsilon/\mu})\} \\
&= 2\mu e^{-\epsilon/\mu}(e^{-t/\mu} - \epsilon\mu^{-1} - 1).
\end{aligned} \tag{24}$$

The first inequality is trivial. The second inequality follows from the fact that $W(x)$ is increasing, so that $\bar{F}_W(y) \geq \bar{F}(y)$ for all $y \geq 0$. The last inequality follows from Theorem 1 and the fact that

$$S_k(x) \geq \mu S_{k-1}(x),$$

for all $x \geq 0$, $k \geq 1$, where

$$S_k(x) = \int_x^\infty S_{k-1}(y)\,dy. \qquad \blacksquare$$

4. SOME BOUNDS AND INEQUALITIES FOR WEIGHTED RESIDUAL DISTRIBUTIONS

In this section we derive bounds and inequalities for weighted residual lifetime distributions. Bounds and stability results on the distance between the weighted residual lifetime distribution functions and the size-biased exponential counterparts are established. These results are given in the context of lifetime distributions with monotone hazard and MRLFs.

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THEOREM 6: *Let $\bar{F}_{W_e}(x)$ be an IHR weighted equilibrium reliability function with increasing weight function. Then*

$$\int_0^\infty |\bar{F}_{W_e}(x) - (1 + x/\mu)e^{-x/\mu}|\, dx \le 2\mu|1 + \mu - \mu_2/2\mu^2|. \tag{25}$$

PROOF: Let $A = \{x|\bar{F}_{W_e} \le (1 + x/\mu)e^{-x/\mu}\}$. Then we have for $x > 0$,

$$\begin{aligned}
\int_0^\infty |\bar{F}_{W_e}(x) - (1 + x/\mu)e^{-x/\mu}|\, dx &\le 2\int_A ((1 + x/\mu)e^{-x/\mu} - \bar{F}_{W_e}(x))\, dx \\
&\le 2\int_0^\infty ((1 + x/\mu)e^{-x/\mu} - \bar{F}_e(x))\, dx \\
&= 2\int_0^\infty ((1 + x/\mu)e^{-x/\mu} - S_1(x)/\mu)\, dx \\
&= 2(\mu^2 + \mu - S_2(0)/\mu) \\
&= 2(\mu^2 + \mu - \mu_2/2\mu) \\
&= 2\mu(1 + \mu - \mu_2/2\mu^2). \qquad (26)
\end{aligned}$$

The first inequality is trivial and the second inequality is due to the stochastic order between $\bar{F}_{W_e}(x)$ and $\bar{F}_e(x)$ when $W(x)$ is increasing in $x \ge 0$. ■

THEOREM 7: *Let $\bar{F}_{W_e}$ be the weighted equilibrium reliability function with DHR. If the weight function $W(x)$ is increasing, then*

$$\int_0^\infty |\bar{F}_{W_e}(x) - (1 + x/\mu)e^{-x/\mu}|\, dx \ge 2e^{-\eta/\mu} \max\{0, |-\eta\mu - 1|\},$$

for $\eta \ge \mu$.

PROOF: Let $\bar{F}_{W_e}$ be a DHR reliability function, then there exist $\eta \ge \mu$ such that $\bar{F}_{W_e}(x) \le (1 + x/\mu)e^{-x/\mu}$ or $\bar{F}_{W_e}(x) \ge (1 + x/\mu)e^{-x/\mu}$ as $x \le \eta$ or $x \ge \eta$. Now

$$\begin{aligned}
\int_0^\infty |\bar{F}_{W_e}(x) - (1 + x/\mu)e^{-x/\mu}|\, dx &= 2\int_\eta^\infty (\bar{F}_{W_e}(x) - (1 + x/\mu)e^{-x/\mu})\, dx \\
&\ge 2\int_\eta^\infty (\bar{F}_e(x) - (1 + x/\mu)e^{-x/\mu})\, dx \\
&= (2/\mu)\int_\eta^\infty (S_1(x) - \mu(1 + x/\mu)e^{-x/\mu})\, dx \\
&= (2/\mu)(S_2(\eta) - \{\mu e^{-\eta/\mu} + \mu^2\eta e^{-\eta/\mu} + \mu e^{-\eta/\mu}\}) \\
&\ge 2S_1(\eta) - 2e^{-\eta/\mu}(\mu + \mu\eta + 1) \\
&\ge 2\mu S_0(\eta) - 2e^{-\eta/\mu}(\mu + \mu\eta + 1) \\
&= 2e^{-\eta/\mu}(-\eta\mu - 1). \qquad (27)
\end{aligned}$$

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The first inequality is due to the fact that $W(x)$ is increasing in x, so that $\bar{F}_{W_e}(x)$ and $\bar{F}_e(x)$ are stochastically ordered. The last two inequalities are due to the fact that $S_k(x) \geq \mu S_{k-1}(x)$ for all $x \geq 0$, $k = 1,2,\ldots$, where $S_k(x) = \int_x^\infty S_{k-1}(y)\,dy$. ■

THEOREM 8: *Let $\bar{F}$ and $\bar{F}_{W_e}$ be the parent and equilibrium weighted survival functions, respectively. Suppose $W(x)$ is increasing and $\bar{F}$ has DMRL; then*

$$\bar{F}_{W_e}(x) \geq \max\left\{0, 2\int_x^\infty \{W(y)e^{-y/\mu}\,dy/\alpha\} + e^{-x/\mu}\{2\mu_2/\alpha\mu - 2\mu/\alpha - \mu_2/2\mu\}\right\}, \tag{28}$$

where $0 < \alpha = E(W(X)) < \infty$.

PROOF: Note that

$$\begin{aligned}
\bar{F}_{W_e}(x) &= (\mu_{F_W})^{-1}\int_x^\infty \bar{F}_W(u)\,du \\
&= \alpha^{-1}(\mu_{F_W})^{-1}\int_x^\infty \left\{W(u)\bar{F}(u) + \int_u^\infty \bar{F}(y)W'(y)\,dy\right\} du \\
&\geq \alpha^{-1}(\mu_{F_W})^{-1}\int_x^\infty \left\{W(u)S_2(u)/S_2(0) + \int_x^\infty S_2(y)\,dy/S_2(0)\right\} du \\
&= (\alpha S_2(0)\mu_{F_W})^{-1}\int_x^\infty \{W(u)S_2(u) + S_3(u)\}\,du \\
&\geq (\alpha S_2(0)\mu_{F_W})^{-1}\int_x^\infty \{W(u)[\mu S_1(0)e^{-u/\mu} - \mu S_1(u)] + \mu S_2(0)e^{-u/\mu} - \mu S_2(0)\}\,du \\
&= 2(\alpha\mu)^{-1}(\mu_{F_W})^{-1}\int_x^\infty \{e^{-u/\mu}[W(u)\mu + \mu_2/2] - \mu - \mu_2/2\}\,du \\
&= (\mu_{F_W})^{-1}\int_x^\infty e^{-u/\mu}\{2W(u)/\alpha + \mu_2/\alpha\mu - 2/\alpha - \mu_2/\alpha\mu\}\,du.
\end{aligned} \tag{29}$$

The inequalities follow from the fact that $W(x)$ is increasing and from the application of Theorem 1. The final result follows immediately. ■

5. APPLICATIONS

Let F and F_W be the original and weighted distribution functions, respectively. We give some examples of the results presented in earlier sections.

Example 1: Let

$$f(x;\alpha,\beta) = \begin{cases} \dfrac{x^{\alpha-1}\beta^\alpha}{\Gamma(\alpha)}\,e^{-x/\beta} & \text{if } x > 0,\ \alpha > 0,\ \beta > 0 \\ 0 & \text{otherwise.} \end{cases}$$

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Then $\mu = \alpha\beta$ and $\mu_2 = \alpha(\alpha+1)\beta^2$ and the hazard rate is increasing for $\alpha \geq 1$ and decreasing for $\alpha \leq 1$. If $W(x) = x$, then the weighted p.d.f. is given by

$$f_W(x;\alpha,\beta) = \begin{cases} \dfrac{x^\alpha \beta^{\alpha+1}}{\Gamma(\alpha+1)} e^{-x/\beta} & \text{if } x > 0,\ \alpha > 0,\ \beta > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Applying Theorem 3, for $\alpha > 1$ we have

$$\int_0^\infty |\bar{F}_{W_t}(x) - (1 + x/\alpha\beta)e^{-x/\alpha\beta}|\, dx \leq 2\alpha\beta|1 + \alpha\beta - (\alpha(\alpha+1)\beta^2/2\alpha^2\beta^2)|$$
$$= \beta|2\alpha^2\beta + \alpha - 1|. \tag{30}$$

With $\beta = \frac{1}{4}$ and $\alpha \geq 1$,

$$C(\alpha) = \int_0^\infty |\bar{F}_{W_t}(x) - (1 + 4x/\alpha)e^{-4x/\alpha}|\, dx$$
$$\leq (\alpha^2 + \alpha - 1)/4. \tag{31}$$

Similarly, applying Theorem 2, for $\alpha \geq 1$ and with $\beta = \frac{1}{4}$ we have

$$C^*(\alpha) = \int_0^\infty |\bar{F}_W(x) - xe^{-4x/\alpha}|\, dx$$
$$\leq |\alpha^2 - \alpha - 1|/4. \tag{32}$$

In particular, if $\alpha = \frac{3}{2}$, we get

$$\int_0^\infty |\bar{F}_W(x) - xe^{-x}|\, dx \leq 0.0625.$$

Example 2 (Shifted exponential distribution): Consider the survival function given by

$$\bar{F}(x;\theta,\epsilon) = \begin{cases} e^{-(x-\theta)/(1-2\epsilon)^{1/2}} & \text{if } x > \theta,\ \theta = 1 - (1-2\epsilon)^{1/2} \\ 1 & \text{otherwise.} \end{cases}$$

Then $\mu = 1$ and $\mu_2 = 2(1-\epsilon)$. Clearly, the hazard rate $\lambda_F(x)$ is increasing, so that

$$D(\epsilon) = \int_0^\infty |\bar{F}_W(x) - e^{-(x-\theta)/(1-2\epsilon)^{1/2}}|\, dx$$
$$\leq 2\mu \left| 1 - \frac{\mu_2}{2\mu^2} \right|$$
$$= 2\left| 1 - \frac{2(1-\epsilon)}{2} \right|. \tag{33}$$

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With $\epsilon = 0.03125$, we have

$$D(0.03125) = \int_0^\infty |\bar{F}_W(x) - e^{-(x-\theta)/(1-2\epsilon)^{1/2}}|\,dx$$
$$\leq 0.0625. \tag{34}$$

Example 3 (Exponential distribution): The exponential distribution is one of the most useful models in reliability and survival analysis. The p.d.f. is given by

$$f(x;\theta) = \begin{cases} (1/\theta)e^{-(x/\theta)} & \text{if } x > 0, \theta > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding weighted reliability function with $W(x) = x$ is given by

$$\bar{F}_W(x;\theta) = \begin{cases} \{(x+\theta)/\theta\}e^{-(x/\theta)} & \text{if } x > 0, \theta > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Applying Theorem 6, we have

$$K(\alpha) = \int_0^\infty |\bar{F}_{W_e}(x) - (1+x/\alpha\beta)e^{-x/\alpha\beta}|\,dx$$
$$\leq 2\alpha\beta|1+\alpha\beta-(\alpha(\alpha+1)\beta^2/2\alpha^2\beta^2)|$$
$$= \beta|2\alpha^2\beta+\alpha-1|. \tag{35}$$

With $\alpha = 1$ and $\beta = 0.125$, we have

$$K(1) = \int_0^\infty |\bar{F}_{W_e}(x) - (1+4x)e^{-4x}|\,dx$$
$$\leq 0.03125. \tag{36}$$

Example 4: Consider the following reliability function:

$$\bar{F}_n(x) = \begin{cases} (1 - x/(n+1))^n & \text{if } 0 < x < n+1 \\ 0 & x \geq n+1. \end{cases}$$

Then $\mu = 1$ and $\mu_2 = 2(n+1)/(n+2)$. Clearly, the hazard rate $\lambda_{F_n}(x) = n/(n+2-x)$ is increasing on $(0, n+1)$, so that

$$E(n) = \int_0^\infty |\bar{F}_W(x) - (1 - x/(n+1))^n|\,dx$$
$$\leq 2\mu|1 - \mu_2/2\mu^2|$$
$$= 2|1 - 2((n+1)/(n+2))/2|$$
$$= 2/(n+2). \tag{37}$$

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With $n = 100$, we have

$$E(100) = \int_0^\infty |\bar{F}_W(x) - (1 - x/(n+1))^n| \, dx$$
$$\leq 0.0196. \tag{38}$$

Similarly, $E(1000) \leq 0.001996$. Clearly, $E(n) \to 0$ as $n \to \infty$.

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