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Homeomorphisms with the whole compacta being scrambled sets

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Abstract. A homeomorphism on a metric space (X, d) is completely scrambled if for each $x \neq y \in X$, $\limsup_{n\longrightarrow+\infty} d(f^n(x), f^n(y)) > 0$ and $\liminf_{n\longrightarrow+\infty} d(f^n(x), f^n(y)) = 0$. We study the basic properties of completely scrambled homeomorphisms on compacta and show that there are 'many' compacta admitting completely scrambled homeomorphisms, which include some countable compacta (we give a characterization), the Cantor set and continua of arbitrary dimension.

1. *Introduction*

All maps considered in the paper are continuous. By a *compactum* (respectively, *continuum*) we mean an infinite compact metric space (respectively, connected compactum) and by *dimension* we mean topological dimension.

The first mathematical treatment of chaotic behavior of a dynamical system appeared in the work of Li and Yorke in 1975 [**LY**]. Let (X, d) be a metric space and $f : X \longrightarrow X$ be a map. A subset S of X with $\mathrm{Card}(S) \geq 2$ is a *scrambled set* if for each pair of $x \neq y \in S$,

$$\limsup_{n\longrightarrow+\infty} d(f^n(x), f^n(y)) > 0 \tag{LY1}$$

and

$$\liminf_{n\longrightarrow+\infty} d(f^n(x), f^n(y)) = 0. \tag{LY2}$$

The map f is *chaotic in the sense of Li and Yorke* if S is uncountable, and f is *completely scrambled* if $S = X$. Hence, if X is uncountable and f is completely scrambled, then f is chaotic in the sense of Li and Yorke. Note that we say $f : X \longrightarrow X$ is *proximal* [**V**] if for each pair of $x, y \in X$, $\liminf_{n\longrightarrow+\infty} d(f^n(x), f^n(y)) = 0$. Thus, a completely scrambled map is proximal.

A natural question is: how 'large' can a scrambled set be? Li and Yorke proved that if a map of $[0, 1]$ has a periodic point of period three then it has an uncountable scrambled

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set. Misiurewicz [**Mi**] showed that there is a map of $[0, 1]$, which has a scrambled set of Lebesgue measure one and Kato [**K1**] showed that the same conclusion holds if we replace $[0, 1]$ by $[0, 1]^n$ for each $n \geq 2$. In [**M**] Mai showed that for each $n \geq 2$ there is a completely scrambled homeomorphism on $(0, 1)^n$ and he conjectured that there is no compactum admitting completely scrambled maps. Disproving the conjecture, we show that there are 'many' compacta admitting completely scrambled homeomorphisms, which include some countable compacta, the Cantor set and continua of arbitrary dimension. We study the basic properties of completely scrambled homeomorphisms on compacta and as a by-product we also show that there is a map g of $[0, 1]$ with a closed invariant scrambled set which is the Cantor set and is contained in the set of ω-limit points of g.

To end the section we make some remarks. Let (X, d) be a compactum. A homeomorphism of X is *expansive* if there exists $c > 0$ such that for each pair $x \neq y \in X$ there is $n = n(x, y) \in \mathbb{Z}$ with $d(f^n(x), f^n(y)) > c$. A homeomorphism f of X is *distal* (see [**F1**], [**F2**] and [**B**, p. 69]) if for each $x \neq y \in X$, $\liminf_{n \to +\infty} d(f^n(x), f^n(y)) > 0$. It is known that if f is expansive and $\dim(X) \geq 1$ then f is chaotic in the sense of Li and Yorke [**K2**]. By a result of Mǎné [**Ma**] if f is expansive then there are $x \neq y \in X$ with $\lim_{n \to +\infty} d(f^n(x), f^n(y)) = 0$. Thus, completely scrambled homeomorphisms are complementary to distal and expansive homeomorphisms. Moreover, if f is a completely scrambled homeomorphism on some compactum X, then

$$X \times X \setminus (\{(x, x) : x \in X\}) = \bigcup_{i=1}^{\infty} \left\{(x, y) : \limsup_{n \to +\infty} d(f^n(x), f^n(y)) > 1/i\right\}.$$

Hence in some sense a completely scrambled homeomorphism is 'expansive' for non-uniformly expansive constants.

2. *Basic properties of completely scrambled homeomorphisms*

First let us introduce some necessary notation. Let X be a compactum and f be a map of X. The set of periodic points, almost periodic points and ω-limit points of $x \in X$ will be denoted by $P(f)$, $AP(f)$ and $\omega(x, f)$, respectively. A non-empty closed invariant subset M of X is a *minimal set* of f if M contains no proper subset which is non-empty, closed and invariant. It is known that if $x \in AP(f)$, then $\omega(x, f)$ is a minimal set of f; if M is a minimal set of f, then $M \subset AP(f)$ [**BC**]. We say that f is (topologically) *transitive* if for each pair of non-empty open subsets U, V of X there is $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. It is known that f is transitive if and only if there exists $x \in X$ with $\omega(x, f) = X$. For $x \in X$ we set $\mathrm{Orb}(x, f) = \{x, f(x), \ldots\}$ and $\mathrm{Orb}_{\mathbb{Z}}(x, f) = \{\ldots, f^{-1}(x), x, f(x), \ldots\}$ if f is a homeomorphism. The set of all strictly increasing sequences of natural numbers will be denoted by $\mathcal{A}$.

PROPOSITION 2.1. *Let X be a compactum such that X admits a completely scrambled map f. Then there is an infinite closed invariant subset Z of X such that Z admits a completely scrambled homeomorphism.*

Proof. It is clear that f is injective (by (LY1)). Let $Z = \cap_{n=1}^{\infty} f^n(X)$. Then Z is a closed invariant subset of X. Moreover, $f|_Z : Z \longrightarrow Z$ is surjective. Hence $f|_Z$ is a homeomorphism.

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If $Z = \{p\}$, then for each $x \in X$, $\lim f^n(x) = p$, a contradiction of (LY1). Thus Z contains at least two points. At the same time, Z is not finite from (LY2). Hence Z admits a completely scrambled homeomorphism $f|_Z$. □

By Proposition 2.1 we only need to study completely scrambled homeomorphisms.

PROPOSITION 2.2. *Let f be a completely scrambled homeomorphism of a compactum X. Then, f has a unique fixed point p and $P(f) = \{p\}$.*

Proof. Let $x \in X$. As $\underline{\lim}\, d(f^n(x), f^n(f(x))) = 0$, there exists $\{n_i\} \in \mathcal{A}$ such that

$$\lim f^{n_i}(x) \longrightarrow p = f(p) = \lim f^{n_i+1}(x).$$

Hence f has a fixed point. It is clear that $P(f) = \{p\}$. □

LEMMA 2.3. *Let f be a map of a compactum X. If there is $x \in X$ such that $\omega(x, f) = \{z\}$ for some $z \in X$, then z is a fixed point of f and $f^n(x) \longrightarrow z$.*

Proof. This is obvious. □

PROPOSITION 2.4. *Let f be a completely scrambled homeomorphism of a compactum X and p be the unique fixed point of f. Then for each $x \neq p \in X$ we have that $p \in \omega(x, f)$ and $\omega(x, f)$ is infinite. Consequently, $AP(f) = \{p\}$.*

Proof. Let $x \neq p$ and $Z = \omega(x, f)$. If $Z = \{p\}$, then $f^n(x) \longrightarrow p$ (Lemma 2.3), a contradiction. At the same time Z is infinite from (LY2). Hence $f|_Z : Z \longrightarrow Z$ is a completely scrambled homeomorphism. By Proposition 2.2, there is a fixed point p' of f in Z. As f has a unique fixed point, $p' = p$. That is $p \in \omega(x, f)$ for each $x \neq p \in X$.

Note that for each $x \in AP(f)$, $x \in \omega(x, f)$ and $\omega(x, f)$ is a minimal set. As $p \in \omega(x, f)$, we have $\omega(x, f) = \{p\}$. This implies that $AP(f) = \{p\}$. □

PROPOSITION 2.5. *Let f be a completely scrambled homeomorphism of a compactum X. Then for each $x \neq y \in X$, there is $\{n_i\} \in \mathcal{A}$ such that $f^{n_i}(x) \longrightarrow p$ and $f^{n_i}(y) \longrightarrow p$, where p is the unique fixed point of f.*

Proof. By (LY2), there exist $\{m_i\} \in \mathcal{A}$ and $z \in X$ such that

$$f^{m_i}(x) \longrightarrow z \longleftarrow f^{m_i}(y).$$

By Proposition 2.4, there exists $\{t_i\} \in \mathcal{A}$ with $f^{t_i}(z) \longrightarrow p$. Thus, Proposition 2.5 follows from the facts stated above. □

To show the next proposition we need one result from [**K**] (see also [**KS**]).

LEMMA 2.6. *Let X be a compactum and $f : X \longrightarrow X$ be a map. Then $\{x \in X : \overline{\mathrm{Orb}(x, f)} \neq X\}$ is a dense subset of X if f is not minimal.*

Denote by ω_0 the first limit ordinal number. We have the following.

PROPOSITION 2.7. *Let f be a completely scrambled homeomorphism of a compactum X. Then there are a countable limit ordinal number α and $x_\beta \in X$ with $1 \leq \beta \leq \alpha$ such that:*

(i) *$x_\alpha = p$ and $x_\beta \neq p$ for each $\beta < \alpha$; moreover, $x_\beta \in \omega(x_{\beta-1}, f)$ if β is not a limit ordinal number and $x_\beta \in \cap_{j<\beta}\omega(x_j, f)$ if β is a limit ordinal number;*

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(ii) $X \supsetneqq \omega(x_1, f) \supsetneqq \cdots \supsetneqq \omega(x_{\omega_0}, f) \supsetneqq \omega(x_{\omega_0+1}, f) \supsetneqq \cdots \supsetneqq \omega(x_\alpha, f) = \{p\}$ *and* $\cap_{i<\alpha}\omega(x_i, f) = \{p\}$.

Proof. As X is infinite there is $x_1 \in X \setminus \{p\}$ such that $\omega(x_1, f) \neq X$ (Lemma 2.6). By Proposition 2.4 $\omega(x_1, f)$ is infinite. Assume that we have defined x_i for all $i < \beta$ and some countable ordinal number β with $x_\beta \neq p$. If β is not a limit then by Lemma 2.6 we take $x_\beta \in \omega(x_{\beta-1}, f) \setminus \{p\}$ such that $\omega(x_{\beta-1}, f) \neq \omega(x_\beta, f)$ (by Proposition 2.4 $\omega(x_{\beta-1}, f)$ is infinite). If β is a limit then we have two cases. Namely, $\cap_{i<\beta}\omega(x_i, f) = \{p\}$ or $\cap_{i<\beta}\omega(x_i, f) \neq \{p\}$. In the first case let $x_\beta = p$, $\alpha = \beta$ and we stop. In the second case we take $x_\beta \in \cap_{i<\beta}\omega(x_i, f) \setminus \{p\}$. Let α be the least countable ordinal number such that $\omega(x_\alpha, f) = \{p\}$ (such an α must exist by the Baire–Hausdorff theorem). It is clear that $x_\alpha = p$ and α is a limit. Moreover, α and x_i, $i \leq \alpha$, satisfy (i) and (ii). □

In the following sections we will construct completely scrambled homeomorphisms suggested by Proposition 2.7.

3. *The construction of a completely scrambled homeomorphism*

In this section we will construct a completely scrambled homeomorphism of a countable compactum. We need the following definitions.

The notation of the derived set of X of order α can be found in Kuratowski's book [**K**, p. 261]. A point p of X is an *accumulation point* of the set X, if $p \in \overline{X \setminus p}$. The set X^d of accumulation points of X is called the *derived set* of X. The derived set of X of order α is defined by the conditions; $X^{(1)} = X^d$, $X^{(\alpha+1)} = (X^{(\alpha)})^d$ and $X^{(\lambda)} = \cap_{\alpha<\lambda}X^{(\alpha)}$ if λ is a limit ordinal number. If $X^{(\alpha)} \neq \emptyset$ and $X^{(\alpha+1)} = \emptyset$, then we put $d(X) = \alpha$. The ordinal $d(X)$ is called the *derived degree* of X. It is well known that a compactum X is a countable set if and only if $d(X)$ exists and it is a countable ordinal number. In this case, if $d(X) = \alpha$, then $X^{(\alpha)}$ is finite.

Let ω_0 be the first infinite ordinal number. We will construct a countable compactum $X \subset \mathbb{R}^2$ such that the derived degree of X is ω_0 and $X^{(\omega_0)}$ is a one point set, and then we define a homeomorphism on X and show that it is completely scrambled. We start with the construction of $X \subset \mathbb{R}^2$. Let

$$X_0 = \{(n_1, \ldots, n_k) : k \geq 1, n_i \in \{0\} \cup \mathbb{N}, 1 \leq i \leq k\},$$
$$X_1 = \{(n_1, \ldots, n_k, \infty) : k \geq 1, n_i \in \{0\} \cup \mathbb{N}, 1 \leq i \leq k\},$$
$$\cdots\cdots\cdots$$
$$X_j = \{(n_1, \ldots, n_k, \overbrace{\infty, \ldots, \infty}^{j}) : k \geq 1, n_i \in \{0\} \cup \mathbb{N}, 1 \leq i \leq k\},$$
$$\cdots\cdots\cdots$$

Set $X = \cup_{j=0}^{\infty} X_j \cup \{(\infty)\}$ and let

$$(\infty) = (\infty, \infty) = (\infty, \infty, \infty) = \cdots.$$

Let $p \in \mathbb{R}^2$ and l_i, $i \in \mathbb{N}$, be a line segment from p such that the length of l_i tends to zero and $l_i \cap l_j = \{p\}$ if $i \neq j$. Assume the other end point of l_i is P_i, $i \in \mathbb{N}$. We put an orientation on l_i such that $P_i > p$ for each $i \in \mathbb{N}$. Now we see $X \subset \cup_{i=1}^{\infty} l_i$ in the following way (Figure 1).

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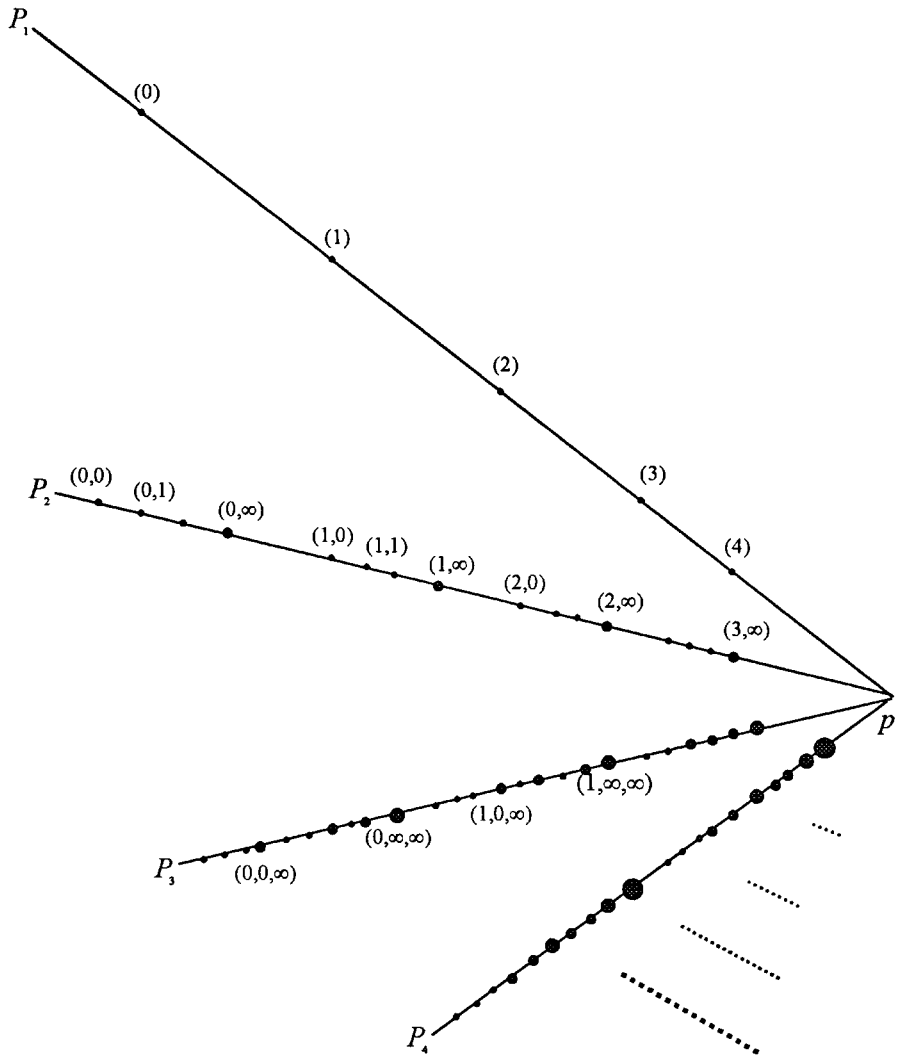


FIGURE 1.

(i) $p = (\infty) = (\infty, \infty) = \cdots$.

(ii) For each $i \in \{0\} \cup \mathbb{N} \cup \{\infty\}$, $(i) \in l_1$ with $(0) > (1) > (2) > \cdots$ and $\lim(i) = p$.

(iii) For each $n \geq 2$, let $I_{i,n} \subset l_n$ ($i \in \{0\} \cup \mathbb{N}$) be pairwise disjoint non-degenerate subintervals of l_n with:

(1) $\inf I_{i_1,n} > \sup I_{i_2,n}$ if $i_1 < i_2$;

(2) $\lim_i \text{length}(I_{i,n}) = 0$;

(3) $\lim_i d(p, I_{i,n}) = 0$, where d is the Euclid metric.

For each $n \geq 1$ and $i \in \{0\} \cup \mathbb{N}$, let $h_{n,i}$ be an orientation-preserving homeomorphism

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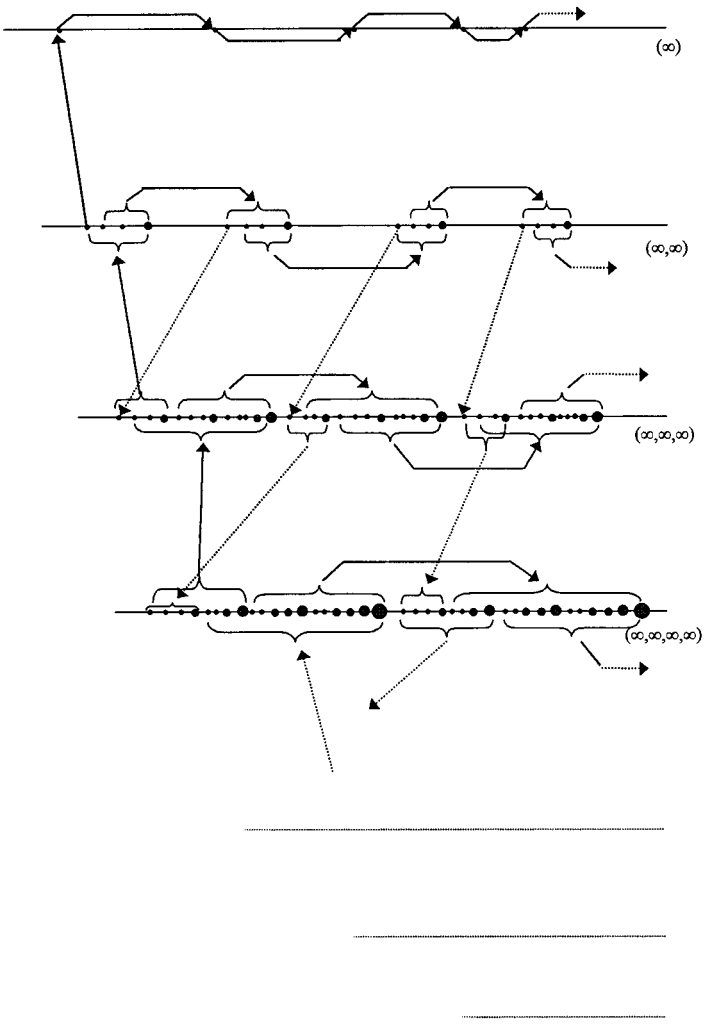


FIGURE 2.

from l_n onto $I_{i,n+1}$. Then

$$I_{i,n+1} \cap X = \{h_{n,i}(y) = (i, y) : y \in l_n \cap X\}.$$

Note that $(\underbrace{\infty, \ldots, \infty}_{n}) \in l_n$.

In such a way X is a compactum with the properties that:

(a) the derived degree of X is ω_0 and $X^{(\omega_0)} = \{p\}$;
(b) X_0 is the set of isolated points of X and $X^{(j)} = (\cup_{l=j}^{\infty} X_j) \cup \{\infty\}$ for each $j \geq 1$.

We will define $g : X \longrightarrow X$. First we define $g : X_0 \longrightarrow X_0$ as follows (see Figure 2).

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Note that $n_i \in \{0\} \cup \mathbb{N}$ for $1 \le i \le k$ and $k \in \mathbb{N}$.

$$(n_1) \longrightarrow (n_1 + 1), \tag{P_0}$$
$$(n_1, n_2) \longrightarrow (n_1 + 1, n_2 - 1), \quad n_2 \ge 1, \tag{P_1}$$
$$(0, 0) \longrightarrow (0), \tag{P_2}$$
$$(n_1, n_2, n_3) \longrightarrow (n_1 + 1, n_2 - 1, n_3), \quad n_2 \ge 2, \tag{P_3}$$
$$(n_1, 1, n_3) \longrightarrow (n_1 + 1, 0, n_3 + 1), \tag{P_4}$$
$$(0, 0, n_3) \longrightarrow (0, n_3), \tag{P_5}$$
$$(n_1 + 1, 0) \longrightarrow (n_1, 0, 0), \tag{P_6}$$
$$(n_1, n_2, n_3, \ldots, n_k) \longrightarrow (n_1 + 1, n_2 - 1, n_3, \ldots, n_k), \quad n_2 \ge 2, k \ge 4, \tag{P_7}$$
$$(n_1, 1, n_3, n_4, \ldots, n_k) \longrightarrow (n_1 + 1, 0, n_3 + 1, n_4, \ldots, n_k), \quad k \ge 4, \tag{P_8}$$
$$(0, 0, n_3, n_4, \ldots, n_k) \longrightarrow (0, n_3, n_4, \ldots, n_k), \quad n_3 \ge 1, k \ge 4, \tag{P_9}$$
$$(0, 0, 0, n_4, \ldots, n_k) \longrightarrow (0, 0, n_4 + 1, \ldots, n_k), \quad k \ge 4, \tag{P_{10}}$$
$$(n_1 + 1, 0, n_4, \ldots, n_k) \longrightarrow (n_1, 0, 0, n_4, \ldots, n_k), \quad k \ge 4. \tag{P_{11}}$$

For each $j \ge 1$ and $(n_1, \ldots, n_k, \overbrace{\infty, \ldots, \infty}^{j}) \in X_j$ we set

$$g(n_1, \ldots, n_k, \overbrace{\infty, \ldots, \infty}^{j}) = (m_1, \ldots, m_l, \overbrace{\infty, \ldots, \infty}^{j})$$

if $g(n_1, \ldots, n_k) = (m_1, \ldots, m_l)$. Furthermore, we put $g(\infty) = (\infty)$.

Thus we have defined $g : X \longrightarrow X$. It turns out that g has the following nice property (the proof can be found in Appendix A).

THEOREM 3.1. *Let* $x_0 = (0)$. *Then* $X_0 = \mathrm{Orb}_{\mathbb{Z}}(x_0, g)$.

Moreover, we have the following.

THEOREM 3.2. $g : X \longrightarrow X$ *is a homeomorphism.*

Proof. As X_0 is infinite and $X_0 = \mathrm{Orb}_{\mathbb{Z}}(x_0, g)$, $g : X_0 \longrightarrow X_0$ is a bijection. Hence for each $j \ge 1$, $g : X_j \longrightarrow X_j$ is a bijection. Thus $g : X \longrightarrow X$ is a bijection. As X is a compact metric space it remains to show that g is continuous. To do this, we assume that x_i $(i \in \mathbb{N})$, $x \in X$ with $\lim x_i = x$ and we want to show that $\lim g(x_i) = g(x)$.

There are two possibilities.

(A) $x = (\infty)$. For each $y = (m_1, \ldots, m_k, \overbrace{\infty, \ldots, \infty}^{j}) \in X$ set $\alpha(y) = m_1 + k + j$ $(\alpha((\infty)) = \infty)$. Notice that:

(i) if $x_i \in X$, $i \in \mathbb{N}$, then $\lim x_i = (\infty)$ if and only if $\lim \alpha(x_i) = \infty$;

(ii) $\alpha(g(y)) \ge \alpha(y) - 2$.

As $\lim x_i = (\infty)$, we have $\lim \alpha(x_i) = \infty$, which yields $\lim \alpha(g(x_i)) = \infty$. Thus $\lim g(x_i) = (\infty)$.

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(B) $x = (n_1, n_2, \ldots, n_k, \overbrace{\infty, \ldots, \infty}^{j})$, $k, j \in \mathbb{N}$. Without loss of generality we assume that

$$x_i = (n_1, n_2, \ldots, n_k, n^i_{k+1}, \ldots, n^i_{k+l_i}, \overbrace{\infty, \ldots, \infty}^{j-l_i})$$

with $n^i_{k+1} \geq 2$. It is not difficult to check that $\lim x_i = x$ if and only if $\lim n^i_{k+1} = \infty$. We should distinguish three cases.

(i) $k = 1$. In this case

$$x = (n_1, \overbrace{\infty, \ldots, \infty}^{j}) \quad \text{and} \quad x_i = (n_1, n^i_2, \ldots, n^i_{1+l_i}, \overbrace{\infty, \ldots, \infty}^{j-l_i}).$$

As $n^i_2 \geq 2$, we have

$$g(x_i) = (n_1 + 1, n^i_2 - 1, n^i_3, \ldots, n^i_{1+l_i}, \overbrace{\infty, \ldots, \infty}^{j-l_i})$$

(by the definition of g). Since $\lim(n^i_2 - 1) = \infty$, it follows that

$$\lim g(x_i) = (n_1 + 1, \overbrace{\infty, \ldots, \infty}^{j}) = g(x).$$

(ii) $k = 2$. In this case

$$x = (n_1, n_2, \overbrace{\infty, \ldots, \infty}^{j}) \quad \text{and} \quad x_i = (n_1, n_2, n^i_3, \ldots, n^i_{2+l_i}, \overbrace{\infty, \ldots, \infty}^{j-l_i}).$$

Let $g((n_1, n_2)) = (m_1, \ldots, m_u)$, $u \in \mathbb{N}$. Then

$$g(x_i) = (m_1, \ldots, m_u, \widetilde{n^i_3}, n^i_4, \ldots, n^i_{2+l_i}, \overbrace{\infty, \ldots, \infty}^{j-l_i})$$

with

$$\widetilde{n^i_3} = \begin{cases} n^i_3 + 1, & \text{if } n_2 = 1, \\ n^i_3, & \text{otherwise} \end{cases}$$

(by the definition of g).

As $\lim n^i_3 = \infty$, we have $\lim \widetilde{n^i_3} = \infty$. Hence

$$\lim g(x_i) = (m_1, \ldots, m_u, \overbrace{\infty, \ldots, \infty}^{j}) = g(x).$$

(iii) $k \geq 3$. In this case

$$x = (n_1, \ldots, n_k, \overbrace{\infty, \ldots, \infty}^{j}),$$

$$x_i = (n_1, \ldots, n_k, n^i_{k+1}, \ldots, n^i_{k+l_i}, \overbrace{\infty, \ldots, \infty}^{j-l_i}).$$

Let $g((n_1, \ldots, n_k)) = (m_1, \ldots, m_u)$, $u \in \mathbb{N}$. Then

$$g(x_i) = (m_1, \ldots, m_u, \widetilde{n^i_{k+1}}, n^i_{k+2}, \ldots, n^i_{k+l_i}, \overbrace{\infty, \ldots, \infty}^{j-l_i})$$

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with

$$\widetilde{n^i_{k+1}} = \begin{cases} n^i_{k+1} + 1, & \text{if } (n_1, \ldots, n_k) = (0, 0, 0) \\ n^i_{k+1}, & \text{otherwise} \end{cases}$$

(by the definition of g).

As $\lim n^i_{k+1} = \infty$, we have $\lim \widetilde{n^i_{k+1}} = \infty$. Hence

$$\lim g(x_i) = (m_1, \ldots, m_u, \overbrace{\infty, \ldots, \infty}^{j}) = g(x). \qquad \square$$

THEOREM 3.3. *Let $f = g^{-1}$. Then $f : X \longrightarrow X$ is a completely scrambled homeomorphism. Moreover, we have that (set $p = (\infty)$).*

HY1. *For each $x \in X \setminus \{p\}$ and each $\{m_i\} \in \mathcal{A}$ with $f^{m_i}(x) \longrightarrow z \neq p$, we have that $f^{m_i}(y) \longrightarrow w \neq z$ for each $y \in X \setminus \{p, x\}$.*

HY2. *There exists $\{n_i\} \in \mathcal{A}$ such that for each $x \in X$, $f^{n_i}(x) \longrightarrow p$, and $\lim f^{-n}(x) = p$ for each $x \in X$.*

Proof. We show that f satisfies HY1 and HY2, which clearly implies that f is completely scrambled.

Let $\{n_i\} \in \mathcal{A}$ such that $f^{n_i}((0)) \longrightarrow p$. It is clear that $f^{n_i}(x) \longrightarrow p$ for each $x \in X_0$. Now let $x = (n_1, \ldots, n_k, \overbrace{\infty, \ldots, \infty}^{j}) \in X_j$ with $k, j \geq 1$. As $f^{n_i}((n_1, \ldots, n_k)) \longrightarrow p$, we have $f^{n_i}(x) \longrightarrow p$. It is clear that $\lim f^{-n}(x) = \lim g^n(x) = p$ for each $x \in X$. Hence HY2 holds.

By Theorem 3.1, we have that for each $j \geq 0$ and each $x \in X_j$, $\omega(x, f) = X^{(j+1)}$. Assume that $x \in X \setminus \{p\}$ and $\{m_i\} \in \mathcal{A}$ with $f^{m_i}(x) \longrightarrow z \neq p$. Assume further that $y \neq p \in X$.

There are several cases.

Case 1. There is $j \geq 0$ such that $x, y \in X_j$. In this case there exists $n \in \mathbb{Z}$ such that $y = f^n(x)$ (Theorem 3.1). Then $f^{m_i}(y) \longrightarrow f^n(z)$. As $z \neq p$, $f^n(z) \neq z$.

Case 2. There are $j_1 \neq j_2$ such that $x \in X_{j_1}$ and $y \in X_{j_2}$. In this case assume that

$$x = (k_1, \ldots, k_{t_1}, \overbrace{\infty, \ldots, \infty}^{j_1}) \quad \text{and} \quad y = (p_1, \ldots, p_{t_2}, \overbrace{\infty, \ldots, \infty}^{j_2}).$$

Let

$$y' = (k_1, \ldots, k_{t_1}, \overbrace{\infty, \ldots, \infty}^{j_2}) \quad \text{and} \quad z = (q_1, \ldots, q_{t_3}, \overbrace{\infty, \ldots, \infty}^{j_3}).$$

Then $j_3 > j_1$ and

$$f^{m_i}(y') \longrightarrow w' = (q_1, \ldots, q_{t_3}, \overbrace{\infty, \ldots, \infty}^{j_2+j_3-j_1}).$$

It is clear that $w' \neq p$ and $f^{m_i}(y) \longrightarrow w = f^s(w') \in X_{j_2+j_3-j_1}$ for some $s \in \mathbb{Z}$. Thus, $w \neq p$ and $w \neq z$. Hence HY1 holds. $\square$

Remark. In the rest of the paper the notation X, f and $\{n_i\}$ will be used.

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4. *Compacta admitting completely scrambled homeomorphisms*

In this section we shall show that if X is a countable compactum, then X admits a completely scrambled homeomorphism if and only if X satisfies the following condition.

Condition L. $d(X)$ is a limit ordinal and $X^{(d(X))}$ is a set consisting of one point.

Moreover, we will show that many compacta admit completely scrambled homeomorphisms, which include the Cantor set and continua of arbitrary dimension. As a by-product we show that there is a continuous map g of $[0, 1]$ such that there is an invariant scrambled set for g which is the Cantor set and it is contained in the set of ω-limit points of g. We need the following lemma (see [**K3**] for a proof).

LEMMA 4.1. *Let Y and Z be countable compacta. Then Y is homeomorphic to Z if and only if $d(Y) = d(Z)$ and $Y^{(d(Y))}$ is homeomorphic to $Z^{(d(Z))}$.*

In [**K3**] the authors proved that a countable compactum Y admits an expansive homeomorphism if and only if $d(Y)$ is not a limit ordinal. Contrary to this situation we have the following theorem.

THEOREM 4.2. *Let Y be a countable compactum. Then Y admits a completely scrambled homeomorphism if and only if Y satisfies Condition L. Moreover, if Y admits a completely scrambled homeomorphism then there is a completely scrambled homeomorphism of Y satisfying HY2 for $\{n_i\}$.*

Proof. Assume first that Y admits a completely scrambled homeomorphism g. Suppose that $\alpha = d(Y)$ is not a limit ordinal. Then $Z = Y^{(\alpha-1)}$ is an invariant subset of g and Z^d is finite. Then, $Z^d = \{p\}$ and p is a fixed point of g, as Z^d is invariant and g is completely scrambled. Hence for each $x \in Z$, $p = \lim g^n(x)$. Clearly, this implies that there are $x \neq y \in Z$ with $\lim_n d(g^n(x), g^n(y)) = 0$, a contradiction. Thus, α is a limit ordinal. It is easy to see that Y^α is a set consisting of one point. To sum up, Y satisfies Condition L.

Now assume that Y satisfies Condition L. If $d(Y) = \omega_0$, then Y admits a completely scrambled homeomorphism g as Y is homeomorphic to the space X (Lemma 4.1). By Theorem 3.3, g satisfies HY2 for $\{n_i\}$.

Suppose that $\alpha > \omega_0$ is a limit ordinal number and, for each limit ordinal number $\beta < \alpha$, if Y satisfies Condition L then Y admits a completely scrambled homeomorphism g satisfying HY2 for $\{n_i\}$. There are two cases.

Case 1. There are limit ordinal $\beta_i < \alpha$ such that $\lim_i \beta_i = \alpha$. In this case assume that Y_{β_i} satisfies Condition L, $i \in \mathbb{N}$. Let g_{β_i} be a completely scrambled homeomorphism on Y_{β_i} and p_{β_i} be the unique fixed point of g_{β_i}. Assume further that $Y_{\beta_i} \subset \mathbb{R}^2$ with $p = p_{\beta_i}$, $Y_{\beta_i} \cap Y_{\beta_j} = \{p\}$ $(i \neq j)$ and $\operatorname{diam}(Y_{\beta_i}) \longrightarrow 0$. Let $Z = \cup_{i=1}^{\infty} Y_{\beta_i}$. Then Z satisfies Condition L with $d(Z) = \alpha$. Now we define $g : Z \longrightarrow Z$ as follows: $g|_{Y_{\beta_i}} = g_{\beta_i}$ for each $i \geq 1$. Then g is a homeomorphism and g is completely scrambled. Moreover, g satisfies HY2 for $\{n_i\}$ and (LY1). As Z is homeomorphic to Y, we finish the proof in Case 1.

Case 2. There is a limit ordinal number $\beta < \alpha$ and $n \in \mathbb{N}$ such that $\alpha = \beta + n\omega_0$. In this case let Y_1 be a compactum with Condition L $(d(Y_1) = \beta)$ and let $g_1 : Y_1 \longrightarrow Y_1$ be a completely scrambled homeomorphism on Y_1 satisfying HY2 for $\{n_i\}$. Let X be

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the compactum defined in Theorem 3.3. Set $Z = Y_1 \times \overbrace{X \times \cdots \times X}^{n}$. Then Z satisfies Condition L with $d(Z) = \alpha$. Define $g : Z \longrightarrow Z$ such that $g = g_1 \times f \times \cdots \times f$, i.e. $g(y, x_1, \ldots, x_n) = (g_1(y), f(x_1), \ldots, f(x_n))$ for each $(y, x_1, \ldots, x_n) \in Z$. Then it is easy to check that g is a completely scrambled homeomorphism, and g satisfies HY2 for $\{n_i\}$. As Z is homeomorphic to Y, we finish the proof in Case 2 and hence the proof of the theorem. □

Let Y and Z be compacta, and let $y_0 \in Y$ and $z_0 \in Z$. Collapsing $\{y_0\} \times Z$ in $Y \times Z$ we get a compactum denoted by YZ_1. Collapsing $(\{y_0\} \times Z) \cup (Y \times \{z_0\})$ we get a compactum denoted by YZ_2. Note that if Z is a continuum then YZ_2 is also a continuum (see [**N**]).

THEOREM 4.3. *If Y admits a completely scrambled homeomorphism satisfying HY2, then for each compactum Z, both YZ_1 and YZ_2 admit completely scrambled homeomorphims satisfying HY2. Consequently, the Cantor set and continua of arbitrary dimension admit completely scrambled homeomorphisms. Moreover, if compacta Y_j admit completely scrambled homeomorphisms satisfying HY2 for some $\{m_i\} \in \mathcal{A}$ for each $j \geq 1$, then $\prod_{j=1}^{\infty} Y_j$ (with product topology) admits completely scrambled homeomorphisms satisfying HY2 for $\{m_i\}$. We remark that the proof of Theorem 4.2 in Case 1 provides another method to construct completely scrambled homeomorphisms.*

Proof. Assume that Y admits a completely scrambled homeomorphism r satisfying HY2, and $y_0 \in Y$ is the unique fixed point of r. Define $g : Y \times Z \longrightarrow Y \times Z$ such that $g = r \times \mathrm{Id}_Z$. As $g(\{y_0\} \times Z) = \{y_0\} \times Z$, g induces a homeomorphism g_1 on YZ_1. It is easy to check that g_1 is a completely scrambled homeomorphism satisfying HY2.

Since $g(Y \times \{z_0\})$ is invariant, g induces a homeomorphism g_2 on YZ_2. It is easy to check that g_2 is also a completely scrambled homeomorphism satisfying HY2. Note that if Z is a continuum then YZ_2 is a continuum, too.

Letting $Y = X$ and Z be a Cantor set or a continuum with a given dimension, we get that YZ_2 is a Cantor set or a continuum of the same dimension.

Now we assume that Y_j admits a completely scrambled homeomorphism g_j satisfying HY2 for some $\{m_i\} \in \mathcal{A}$ and each $j \geq 1$. Let $g : \prod_{j=1}^{\infty} Y_j \longrightarrow \prod_{j=1}^{\infty} Y_j$ such that

$$g(x_1, x_2, \ldots) = (g_1(x_1), g_2(x_2), \ldots),$$

for each $(x_1, x_2, \ldots) \in \prod_{j=1}^{\infty} Y_j$. Then g is the completely scrambled homeomorphism we need. □

By a *Cantor set* we mean a compactum which is homeomorphic to the Cantor middle-third set. As a by-product we have the following.

COROLLARY 4.4. *There is a continuous map g of $[0, 1]$ such that there is an invariant scrambled set $S \subset \cup_{x \in [0,1]} \omega(x, g)$ for g which is a Cantor set and $g|_S$ is a homeomorphism.*

Proof. Let C be a Cantor set contained in $[0, 1]$ and $g : C \longrightarrow C$ be the completely scrambled homeomorphism defined in Theorem 4.3. Extending g to a continuous map $g : [0, 1] \longrightarrow [0, 1]$ and letting $S = \cup_{x \in C} \omega(x, g)$, we get that S is also a Cantor set and g is the continuous map we need. □

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Remark. Note that all the completely scrambled homeomorphisms we construct in this paper have the property that each of them has only one recurrent point. We do not know if there is a transitive, completely scrambled homeomorphism†.

By [**XY**] or [**I**], a topologically weakly mixing map is chaotic in the sense of Li and Yorke and it is unknown if such a map can be completely scrambled. As the completely scrambled homeomorphisms we constructed have zero topological entropy, it is also unknown to us if there are completely scrambled homeomorphisms with positive entropy‡. Nevertheless, we conjecture that completely scrambled examples with mixing properties should exist, i.e. mixing properties do not imply the existence of proper asymptotic pairs.

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A. *Appendix*

Here we give the proof of Theorem 3.1.

Proof of Theorem 3.1. We divide the proof into the following steps.

Step A. For each $l \in \mathbb{N}$, $m, k \in \{0\} \cup \mathbb{N}$ and $n_1, \ldots, n_k \in \{0\} \cup \mathbb{N}$, there exists $n \in \mathbb{N}$ such that

$$g^n(0, l, m, n_1, \ldots, n_k) = (0, l + m + 1, n_1, \ldots, n_k).$$

Proof of Step A. If $l = 1$ then

$$\begin{aligned}(0, 1, m, n_1, \ldots, n_k) &\xrightarrow[P_4,P_8]{} (1, 0, m + 1, n_1, \ldots, n_k)\\ &\xrightarrow[P_{11}]{} (0, 0, 0, m + 1, n_1, \ldots, n_k)\\ &\xrightarrow[P_{10}]{} (0, 0, m + 2, n_1, \ldots, n_k)\\ &\xrightarrow[P_9]{} (0, m + 2, n_1, \ldots, n_k).\end{aligned}$$

Now assume that Step A holds for $1 \le l \le u$. For $l = u + 1 \ge 2$,

$$\begin{aligned}(0, l, m, n_1, \ldots, n_k) &\xrightarrow[P_3,P_7]{g^{l-1}} (l - 1, 1, m, n_1, \ldots, n_k)\\ &\xrightarrow[P_4,P_8]{} (l, 0, m + 1, n_1, \ldots, n_k)\\ &\xrightarrow[P_{11}]{g^l} (\overbrace{0, \ldots, 0}^{l+2}, m + 1, n_1, \ldots, n_k)\\ &\xrightarrow[P_{10}]{} (\overbrace{0, 0, 1, 0, \ldots, 0}^{l+1}, m + 1, n_1, \ldots, n_k)\end{aligned}$$

† After submitting the paper, we found that one can show that a certain transitive system in [**KW**] is completely scrambled.

‡ Blanchard *et al* [**BHR**] have just proved that a system with positive entropy contains proper asymptotic pairs. Thus, this question has a negative answer.

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$$\xrightarrow[\mathrm{P}_9]{} (\overbrace{0, 1, 0, \ldots, 0}^{l}, m+1, n_1, \ldots, n_k)$$
$$\xrightarrow[\text{induction}]{} (0, l-1, m+1, n_1, \ldots, n_k)$$
$$\xrightarrow[\text{induction}]{} (0, l+m+1, n_1, \ldots, n_k).$$

Step B. For each $m \in \{0\} \cup \mathbb{N}$, there is $n \in \mathbb{N}$ such that $g^n(m, 0) = (0)$.

Proof of Step B. If $m = 0$, then $g(0, 0) = (0)$ (P_2); if $m = 1$, then

$$(1, 0) \xrightarrow[\mathrm{P}_6]{} (0, 0, 0) \xrightarrow[\mathrm{P}_5]{} (0, 0) \xrightarrow[P_2]{} (0).$$

Assume now that Step B holds for $2 \le m \le u$. For $m = u + 1$,

$$(m, 0) \xrightarrow[\mathrm{P}_6, \mathrm{P}_{11}]{g^m} (\overbrace{0, 0, \ldots, 0}^{m+2}) \xrightarrow[\mathrm{P}_{10}]{} (\overbrace{0, 0, 1, 0, \ldots, 0}^{m+1})$$
$$\xrightarrow[\mathrm{P}_9]{} (\overbrace{0, 1, 0, \ldots, 0}^{m}) \xrightarrow[\text{Step A}]{g^k} (0, m-1)$$
$$\xrightarrow[\mathrm{P}_1]{g^{m-1}} (m-1, 0) \xrightarrow{\text{induction}} (0).$$

Step C. For each $k \ge 2$, $n_1, n_2, \ldots, n_k \in \{0\} \cup \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $g^m(n_1, \ldots, n_k) = (0)$.

Proof of Step C. If $k = 2$, then Step C follows from Step B and P_1. Hence we assume that $k \ge 3$. Assume now that Step C holds for $2 \le k \le u$. For $k = u + 1$ by P_7 and P_8 we may assume that $n_2 = 0$. Thus

$$(n_1, 0, n_3, \ldots, n_k) \xrightarrow[\mathrm{P}_{11}, \mathrm{P}_6]{g^{n_1}} (\overbrace{0, \ldots, 0}^{n_1+2}, n_3, \ldots, n_k).$$

There are several cases.

Case 1. $n_1 = 0$. In this case

$$(0, 0, n_3, \ldots, n_k) \longrightarrow \begin{cases} (0, n_3), & \text{if } k = 3, \ (\mathrm{P}_5), \\ (0, 0, n_4 + 1, \ldots, n_k), & \text{if } n_3 = 0, k \ge 4, \ (\mathrm{P}_{10}, \mathrm{P}_9), \\ (0, n_3, \ldots, n_k), & \text{if } n_3 \ge 1, k \ge 4, \ (\mathrm{P}_9). \end{cases}$$

Case 2. $n_1 = 1$. In this case

$$(0, 0, 0, n_3, \ldots, n_k) \xrightarrow{\mathrm{P}_{10}} (0, 0, n_3 + 1, \ldots, n_k) \xrightarrow{\text{Case 1}} (0, n_3 + 1, \ldots, n_k).$$

Case 3. $n_2 \ge 2$. In this case

$$(\overbrace{0, \ldots, 0}^{n_1+2}, n_3, \ldots, n_k) \xrightarrow[\mathrm{P}_{10}, \mathrm{P}_9]{g^2} (\overbrace{0, 1, 0, \ldots, 0}^{n_1}, n_3, \ldots, n_k)$$
$$\xrightarrow{\text{Step A}} (0, n_1 - 1, n_3, \ldots, n_k)$$
$$\xrightarrow{\text{Step A}} (0, n_1 + n_3, n_4, \ldots, n_k).$$

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In all cases we may use induction. Hence, Step C holds.

Step D. $g : X_0 \longrightarrow X_0$ is an injection.

Proof. Let $A_k = \{(n_1, n_2, \ldots, n_k) : n_i \in \{0\} \cup \mathbb{N}, 1 \le i \le k\}$, $k \in \mathbb{N}$.

$$\begin{aligned} A_{11} &= \{(n_1 + 1) : n_1 \in \{0\} \cup N\}, \\ A_{12} &= \{(0)\}, \\ A_{21} &= \{(n_1 + 1, n_2 - 1) : n_2 \ge 1)\}, \\ A_{22} &= \{(0, n_3) : n_3 \in \{0\} \cup N\}. \end{aligned}$$

If $k \ge 3$ we set

$$\begin{aligned} A_{k1} &= \{(n_1 + 1, n_2 - 1, n_3, \ldots, n_k) : n_2 \ge 2\}, \\ A_{k2} &= \{(n_1 + 1, 0, n_3 + 1, n_4, \ldots, n_k)\}, \\ A_{k3} &= \{(0, n_3, \ldots, n_k, n_{k+1}) : n_3 \ge 1\}, \\ A_{k4} &= \{(0, 0, n_4 + 1, \ldots, n_{k+1})\}, \\ A_{k5} &= \{(n_1, 0, 0, n_4, \ldots, n_k)\}. \end{aligned}$$

Notice that:

(1) $A_1 = A_{11} \cup A_{12}$, $A_{11} \cap A_{12} = \phi$;
(2) $A_2 = A_{21} \cup A_{22}$, $A_{21} \cap A_{22} = \phi$;
(3) $A_k = \cup_{i=1}^{5} A_{ki}$, $k \ge 3$;
(4) $A_{ki} \cap A_{kj} = \emptyset$, $1 \le i < j \le 5$, $k \ge 3$.

To prove g is injective, we only need to show that for each $x_k = (n_1, n_2, \ldots, n_k) \in A_k$, $k \in \mathbb{N}$,

$$\mathrm{Card}(g^{-1}(x_k)) = 1. \qquad (*)$$

If $k = 1, 2$, $(*)$ is obvious.

For $k \ge 3$, let $y_j = (m_1, m_2, \ldots, m_j)$, $j \in \mathbb{N}$ with $g(y_j) = x_k$. It is clear that $j \in \{k - 1, k, k + 1\}$. There are three cases.

Case 1. $j = k$. If $n_2 \ge 1$, we have $m_2 \ge 2$ and $x_k = (m_1 + 1, m_2 - 1, m_3, \ldots, m_j)$ (by P_3, P_7). Namely, $x_k \in A_{k1}$. If $n_2 = 0$, we have $m_2 = 1$ and $x_k = (m_1 + 1, 0, m_3 + 1, m_4, \ldots, m_j)$ (by P_4, P_8). Namely, $x_k \in A_{k2}$.

Case 2. $j = k + 1$. If $n_2 \ge 1$, we have $m_1 = m_2 = 0$ and $x_k = (0, m_3, \ldots, m_j)$ (by P_9). Namely, $x_k \in A_{k3}$. If $n_2 = 0$, we have $m_1 = m_2 = m_3 = 0$ and $x_k = (0, 0, m_4 + 1, m_5, \ldots, m_j)$ (by P_{10}). Namely, $x_k \in A_{k4}$.

Case 3. $j = k - 1$. We have $m_2 = 0$ and $x_k = (m_1, 0, 0, m_3, \ldots, m_j)$ (by P_6, P_{11}). Namely, $x_k \in A_{k5}$.

Since $A_k = \cup_{l=1}^{5} A_{kl}$, $A_{ki} \cap A_{kj} = \phi$, $1 \le i < j \le 5$, $k \ge 3$, by what we have proved above, we get that y_j is uniquely determined by x_k. That is, $\mathrm{Card}(g^{-1}(x_k)) = 1$ and Step D holds.

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Note that for each $n_1 \geq 0$, $g^{n_1}(0) = (n_1)$. By Steps C and D, we have that $X_0 = \mathrm{Orb}_{\mathbb{Z}}(x_0, g)$. This ends the proof of Theorem 3.1. □

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