An instructive question

Two circles have equations $x^2 + y^2 - 4x = 0$ and $x^2 + y^2 - 6x - 12y + 36 = 0$.

(a) Show that the y-axis is a tangent to both circles.

(b) Find the equation of the line through the centres of both circles.

(c) Determine the gradient of a line other than the y-axis which is a tangent to both circles.

This was the final question in an A-level mock exam. It was meant to be challenging, but is unquestionably within the AS specification, and meets Ofqual criteria for problem-solving in that it allows multiple approaches. In fact it inadvertently turned out to be extremely instructive and gives insights into a number of different aspects of A-level mathematics.

One theme occurs several times. If an ostensibly quadratic equation $ax^2 + bx + c = 0$ turns out on simplification to have $a = 0$, this may indicate an infinite solution which may be relevant and intelligible in context. Obviously the problem of infinity can be avoided by writing $u = 1/x$ to obtain $u = 0$ as one solution. In this particular question the variable is the gradient of a line, and of course $u = 0$ corresponds to the y-axis.

Four solutions

Parts (a) and (b) are routine and do not need comment.

(a) The centres and radii are (2, 0), 2; (3, 6), 3 respectively. As for each circle the radius equals the distance of the centre from the y-axis, that line is a tangent to both circles.

(b) $y = 6x - 12$ (Routine, but a hint for part (c).)

(c) *Method* 1 (AS): Use symmetry about the line of centres to say that another tangent has the same y-intercept as in part (b), as in Figure 1. Use a standard AS technique to find the equation of a line through $(0, -12)$ which is tangent to one circle.

The general line through $(0, -12)$ is $y = mx - 12$.

Substitute into the equation of one circle*:

$$(x - 2)^2 + (mx - 12)^2 = 4.$$

Equal roots implies that '$b^2 = 4ac$', so

$$16(1 + 6m)^2 = 4(m^2 + 1) \times 14$$

$$\Rightarrow \qquad 192m + 16 = 576.$$

(We can cancel m^2, as $m = \infty$ gives the y-axis.)

$$m = \frac{35}{12} = 2.9166\ldots .$$

(In the exam, decimal answers were allowed.)

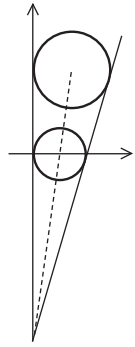


FIGURE 1

* Obviously we choose the circle with a simpler equation; using the other circle gives the same result of course.

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Method 2 (similar but Further Maths): The distance of (2, 0) from the *y*-axis is 2, so find m such that the distance of (2, 0) from $y = mx - 12$ is also 2.

Using $\dfrac{|ax + by + c|}{\sqrt{a^2 + b^2}}$, the distance of (2, 0) from $y = mx - 12$ is $\dfrac{|2m - 12|}{\sqrt{1 + m^2}}$, so $\dfrac{|2m - 12|}{\sqrt{1 + m^2}} = 2$.

This simplifies to $m^2 - 12m + 36 = m^2 + 1$, so $m = \dfrac{35}{12}$ or infinity, as before.

Method 3 (A-level): Let θ be the size of the angle between the line of centres and the *y*-axis. Find $\tan\theta$, find $\tan 2\theta$, and take its reciprocal.

The gradient of $y = 6x - 12$ is 6, so the angle θ with the *y*-axis is $\tan^{-1}\frac{1}{6}$. Using the double angle formula, $\tan 2\theta = \dfrac{2 \times \frac{1}{6}}{1 - \left(\frac{1}{6}\right)^2} = \dfrac{12}{35}$. Hence $m = \dfrac{35}{12}$.

Method 4 (AS but an algebraic mess): Use $y = mx + c$ and '$b^2 = 4ac$' twice. First circle:

$$x^2 + m^2x^2 + 2mcx + c^2 - 4x = 0$$

$$\Rightarrow \qquad (1 + m^2)x^2 + 2mcx - 4x + c^2 = 0.$$

Put '$b^2 = 4ac$':

$$4(mc - 2)^2 = 4c^2(1 + m^2). \tag{1}$$

Second circle:

$$x^2 + m^2x^2 + 2mcx + c^2 - 6x - 12mx - 12c + 36 = 0$$

$$\Rightarrow \qquad (1 + m^2)x^2 + 2x(mc - 6m - 3) + (c - 6)^2 = 0.$$

Put '$b^2 = 4ac$':

$$4(mc - 6m - 3)^2 = 4(m^2 + 1)(c - 6)^2. \tag{2}$$

When (1) is multiplied out, the m^2 terms cancel (corresponding to $m = \infty$, in other words the *y*-axis) and we get

$$m = \frac{4 - c^2}{4c}. \tag{3}$$

You could substitute this into (2), but it's very messy. Instead, divide (2) by (1):

$$\frac{(mc - 6m - 3)^2}{(mc - 2)^2} = \frac{(c - 6)^2}{c^2}$$

and this implies *either*

$$\frac{mc - 6m - 3}{mc - 2} = \frac{c - 6}{c} \qquad \textit{or} \qquad \frac{mc - 6m - 3}{mc - 2} = \frac{6 - c}{c}.$$

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The first case simplifies to $c = -12$; the terms in m all cancel, which corresponds, once more, to $m = \infty$. Substituting $c = -12$ into (3) gives $m = \dfrac{35}{12}$.

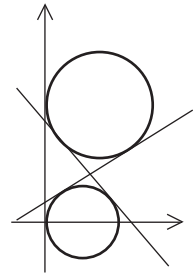


FIGURE 2

The second case simplifies to

$$2mc^2 - 12mc - 5c + 12 = 0.$$

Substituting (3) into this gives $c^3 - 6c^2 + 6c = 0$.
Once again $c = 0$ gives an infinite value for m (from (3)).
Otherwise $c^2 - 6c + 6 = 0$, so $c = 3 \pm \sqrt{3}$ and

$$m = \frac{-3 \mp 5\sqrt{3}}{12}.$$

These are the gradients of the *other* two common tangents to the two circles, (see Figure 2). These cannot be found by Methods 1 or 2, and can be found by Method 3 only with *ad hoc* modification. Obviously Method 4 is not a sensible approach under exam conditions, but it gives further insights into the problem and its solution.

Other possible solution strategies include drawing right-angled triangles and using compound angle formulae.

Further comments

Several candidates tried to use implicit differentiation, with the following outcome:

$$x^2 - 4x + y^2 = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x-2}{y}$$

$$x^2 + y^2 - 6x - 12y - 36 = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x-3}{y-6}.$$

Equating these gives $y = 6x - 12$

which is not the equation of the common tangent; it is the equation of the line through the centres.

So two queries arise:

(i) Why does this not give the right answer?

(ii) Why does it give the line through the centres?

The answer to (i) is that the common tangent touches the two circles in two different points, whereas the use of the same letter x and of the same letter y in the two equations means that (x, y) must represent the same point in both equations.

Let's look at it in more detail.

Consider the circle with equation $(x - a)^2 + (y - b)^2 = r^2$. From the equation we derive $\dfrac{dy}{dx} = -\dfrac{x-a}{y-b}$. What does this *mean*?

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If we substitute numerical values of x and y into the expression $-\dfrac{x-a}{y-b}$, we are finding the gradient of a line perpendicular to the segment joining a point $P(x, y)$ to the centre (a, b). P can be *any* point; it doesn't have to be on the circle, though of course that is how it is usually used. (In fact, as r has disappeared, we can think of P as being on a circle with centre (a, b) and a possibly different radius.)

If on the other hand we equate the expressions for $\dfrac{dy}{dx}$ for two different circles, for example $\left(\dfrac{dy}{dx} =\right) -\dfrac{x-a_1}{y-b_1} = -\dfrac{x-a_2}{y-b_2}$, we are saying something different.

Consider first the equivalent but slightly easier equation

$$\frac{y-b_1}{x-a_1} = \frac{y-b_2}{x-a_2}.$$

Here x and y are the coordinates of a point $P(x, y)$ for which the gradient from P to (a_1, b_1) is the same as the gradient from that same point P to (a_2, b_2). Therefore P lies on the line through (a_1, b_1) and (a_2, b_2).

As for the actual equation, when we write $-\dfrac{x-a_1}{y-b_1} = -\dfrac{x-a_2}{y-b_2}$ we are saying:

- the gradient of the perpendicular to the line joining $P(x, y)$ and (a_1, b_1) equals
- the gradient of the perpendicular to the line joining $P(x, y)$ and (a_2, b_2),

which is the same thing as in the easier equation.

The main conclusion, of course, is that partial differentiation is not a helpful way of thinking of the gradient of the tangent to a circle; pupils should always think "tangent is perpendicular to radius".

A second conclusion is that when two equations are combined, it is essential to make sure that the same letters stand for the same thing in both equations.

In terms of the actual question, of course: use a different method!

In case you were wondering: about 10% of my pupils got this question right.

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