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Spatial numerical range of an operator

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0. *Introduction.* Let X be a normed space and let T be an operator on X. Let $S(X)$ denote its unit sphere, $\{x\in X:\|x\| = 1\}$, $B(X) = \{x\in X:\|x\| \leqslant 1\}$ its unit ball, X' its dual and $\mathscr{B}(X)$ the normed algebra of bounded linear operators on X. Let Π be the subset of the Cartesian product $X\times X'$ defined by

$$\Pi = \{(x,f): x\in S(X), f\in S(X'), f(x) = 1\}.$$

Let $D(X,x) = \{f\in X': \|f\| = f(x) = 1\}$. The spatial numerical range $V(T)$ of an operator $T\in\mathscr{B}(X)$ is defined by

$$V(T) = \{f(Tx): (x,f)\in\Pi\},$$

and the numerical radius of T by

$$v(T) = \sup\{|\lambda|: \lambda\in V(T)\}.$$

It is clear that

$$V(T)\subset\{z: z\in\mathbb{C}, |z|\leqslant\|T\|\}.$$

We now proceed to introduce the notion of algebra numerical range of an element $a\in A$, where A is a complex unital normed algebra. Let A' be the dual of A.

The algebra numerical range $V(A,a)$ of an element $a\in A$ is

$$V(A,a) = V(L_a)$$

where L_a is the operator of left multiplication by a. In other words,

$$\{V(A,a) = f(a): f\in D(A,1)\}.$$

1. In (1), Bollobás explored the following two questions in detail:

(1) For which polynomials $p(z)$ can we have

$$\|a^n\| = n!(e/n)^n v^n(a) \quad\text{and}\quad p(a) = 0?$$

(2) If $\|a^n\| = n!(e/n)^n v^n(a)$, what do we know about $V(A,a^m)$, where $1\leqslant m\leqslant n$?

In the sequel, we wish to supplement Bollobás's results by proving a few results on the spatial and algebra numerical range when the equality holds in the power inequality.

Theorem 1·1. *Let* $T\in\mathscr{B}(X)$. If $\|T^n\| = n!(e/n)^n v(T)^n$ *for some* $n\geqslant 1$, *then*

$$0\in\bigcap_{k\geqslant 1} V(T^k)^-.$$

Proof. We may suppose without loss of generality that $v(T) = 1$. Since

$$\|T^n\| = (e/n)^n n! v(T)^n,$$

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given any ϵ, $(0 < \epsilon < 1)$, there exists (x,f) in $S(X) \times S(X')$ such that

$$f\left(\frac{n^n T^n x}{n!}\right)^2 = \left\|\frac{n^n T^n x}{n!}\right\|^2 > e^{2n} - \epsilon.$$

Let

$$F(z) = f((\exp(znT))\,x) = \sum_{k=0}^{\infty} c_k z^k,$$

where $z = e^{it}$, $-\infty < t < \infty$, then by Parseval's Theorem,

$$\begin{aligned}\sum_{k=0}^{\infty} |c_k|^2 &= \frac{1}{2\pi}\int_0^{2\pi} F(e^{ikt})\,F(e^{-ikt})\,dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} |F(e^{ikt})|^2\,dt \\ &\leqslant \exp(2nv(T)). \qquad (1)\end{aligned}$$

This implies that

$$\sum_{k \neq n}^{\infty} |c_k|^2 < \epsilon.$$

In particular $|c_{n+k}| < \epsilon^{\frac{1}{2}}$ for all $k > 0$. Let $\|T^n\| = g(n)$ and let $\|g(n)^{-1} T^n x\| = K$. It is clear that $K \leqslant 1$ and that

$$f(K^{-1} g(n)^{-1} T^n x) = \|K^{-1} g(n)^{-1} T^n x\| = 1. \qquad (2)$$

We have

$$c_{n+k} = \frac{f(n^{n+k} T^{n+k} x)}{(n+k)!}.$$

In view of (2), $f(g(n)^{-1} K^{-1} T^{n+k} x) = \eta \in V(T^k)$, for $k = 1, 2, \ldots$, and

$$|\eta| < \epsilon^{\frac{1}{2}} \prod_{r=1}^{k} \left(1 + \frac{r}{n}\right).$$

As ϵ is arbitrary, the proof of theorem is complete.

COROLLARY 1·2. *Let $T \in \mathscr{B}(X)$. If $\|T^n\| = n!\,(e/n)^n\, v(T)^n$ for some $n \geqslant 1$, then*

$$0 \in \bigcap_{k=1}^{\infty} V(\mathscr{B}(X), T^k).$$

We now state a simple lemma due to Bollobás (1) and make use of it to deduce a few corollaries.

LEMMA 1·3. *Let A be a complex unital Banach algebra and let $a \in A$. Let*

$$\|a^n\| = n!\,(e/n)^n\, v(a)^n$$

and let $f \in A'$ be a tangent functional at a^n (i.e. $f \in S(A')$, $f(a^n) = n!\,(e/n)^n\, v(a)^n$). Then $f(a^k) = 0$, $k \neq n$.

COROLLARY 1·4. *Let A be a complex unital Banach algebra and let $a \in A$. Let*

$$p(z) = \sum_{l=1}^{k} p_l z^l.$$

If $\|a^n\| = n!\,(e/n)^n\, v(a)^n$ for some n, then $0 \in V(A, p(a))$.

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Proof. Suppose $\|a^n\| = n!\,(e/n)^n v^n(a)$ then by Lemma 1·3 there exists $f \in S(A')$ such that $f(a^n) = n!\,(e/n)^n v(a)^n$ and $f(a^k) = 0$, $k \neq n$.

We have

$$f(p(a)\,g(n)^{-1}a^n) \in V(A, p(a)),$$
$$\subset V(A, p(a)),$$

where $g(n) = n!\,(e/n)^n v(a)^n$.

Since $f(a^k) = 0$, $k \neq n$, the proof of the corollary is complete.

COROLLARY 1·5. *Let A be a complex unital Banach algebra and let $a \in A$. If*

$$\|a^n\| = n!\,(e/n)^n v(a)^n, \quad n = m, m+1, \ldots,$$

where $m \geqslant 1$, then there does not exist a polynomial $p(z)$ such that $p(a) = 0$.

We remark in passing that Corollary 1·2 is also an immediate consequence of Lemma 1·3.

We now make use of Lemma 1·3 to give an alternative proof of the following result due to Crabb (4).

THEOREM 1·6. *Let $0 \neq T \in \mathscr{B}(X)$. If $\|T^n\| = n!\,(e/n)^n v(T)^n$ for some positive integer n, and that $p(T) = 0$ for some non-zero polynomial p, then 0 is an eigenvalue of ascent greater than n.*

Proof. Let $p(T) = a_0 I + \ldots + a_m T^m = 0$, and let $F \in S(\mathscr{B}(X)')$ be such that

$$F(T^n) = n!\,(e/n)^n v(T)^n.$$

Then by Lemma 1·3, $F(T^k) = 0$, $k \neq n$. Applying F to $p(T)$, we have $a_n = 0$. Since $a_0 T^k + \ldots + a_m T^{m+k} = 0$, for $k = 1, 2, \ldots, n$, we have $a_0 = a_1 = \ldots = a_n = 0$. Then there exists $x \in X$ such that

$$T^n\left(\frac{p(T)}{T^{n+1}}x\right) \neq 0 \quad \text{while} \quad T^{n+1}\left(\frac{p(T)}{T^{n+1}}x\right) = 0.$$

This completes the proof of the theorem.

2. Recently it has been shown by Crabb, Duncan and McGregor (6) that if A is a real Banach algebra and if $A \ni a = a^n$ for $n = 2$ or 3, then $\|a\| < ev(a)$. In particular if $a^n = a$, for $n = 2$ or 3, then $v(a) = 0$ implies $a = 0$. We show that this cannot hold for all n. In fact, if $n = 4k+1$ where $k > 2$, $a^n = a$ and $v(a) = 0$ then $\|a\| \leqslant 1$ and this is best possible. We supplement the result of Crabb, Duncan and McGregor by the following:

THEOREM 2·1. *Let A be a real unital Banach algebra and let $a \in A$. Let $p(x)$ be a polynomial with real coefficients and constant term zero. If $V(A, a) = \{0\}$ and $V(A, p(a)) = 0$, then $\|a\| \leqslant |W|$, where $W = \{\lambda : \lambda \in \mathbb{R},\ p(i\lambda) \in i\mathbb{R}\}$ and this is best possible.*

Proof. Let A_c be the complexification of A so that A is isometrically embedded in A_c. Then by (3*a*) Theorem 2·5 and Lemma 6·2,

$$a = ih \quad \text{and} \quad p(a) = ik,$$

for some $h, k \in H(A_c)$, where $H(A_c)$ denotes the set of Hermitian elements of A_c.

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Let $Sp(h) = S,$

then $p(iS) = Sp(A_c, p(ih)) \subset i\mathbb{R}$.

Clearly, $\|a\|_A = \|h\|_{A_c}$ and $S \subset W$, thus

$$\|a\| \leqslant |W|.$$

To show that the result is best possible, take $A = \mathbb{C}$, with modulus as the norm and pick $\lambda \in W$, then

$$V(\mathbb{C}, i\lambda) = \{0\}, \quad V(\mathbb{C}, p(i\lambda)) = V(\mathbb{C}, i\mu) = \{0\},$$

$\mu \in \mathbb{R}$ and this completes the proof of the theorem. In particular, if $p(x) = q(-x^2)$ then

$$|W| = \max\{r: r \in \mathbb{R}^+, q(+r^2) = 0\}.$$

3. Henceforth, we write $(X, \|\cdot\|)$ for the complex vector space X equipped with the norm $\|\cdot\|$, $N(X)$ for the set of equivalent norms on $(X, \|\cdot\|)$. The numerical index $n(X, \|\cdot\|)$ of a normed space $(X, \|\cdot\|)$ is defined by

$$n(X, \|\cdot\|) = \inf\{v(T): T \in \mathscr{B}(X), \|T\| = 1\}.$$

Since $v(T) \leqslant \|T\|$ $(T \in \mathscr{B}(X))$, it follows that $n(X, \|\cdot\|) \leqslant 1$ for any normed space $(X, \|\cdot\|)$. It was shown by Glickfeld (8) that if $(X, \|\cdot\|)$ is a complex normed space then $\|T\| \leqslant ev(T)$, so that $1/e \leqslant n(X, \|\cdot\|) \leqslant 1$. In fact Glickfeld (8) produced an operator on $\mathbb{C}^2$ for which $v(T) = 1/e$ and $\|T\| = 1$. Duncan, McGregor, Pryce and White (7) showed that if $(X, \|\cdot\|)$ is a complex two dimensional space, then there exists a family of norms $\{\|\cdot\|_t\}$, $(1/e \leqslant t \leqslant 1)$ such that $n(X, \|\cdot\|_t) = t$. In the sequel we show that

(i) if $(X, \|\cdot\|)$ is any complex normed space then there exists $\|\cdot\|' \in N(X)$ such that

$$n(X, \|\cdot\|') = 1/e,$$

and (ii) if $(X, \|\cdot\|)$ is any complex finite dimensional normed space then there exists a family of norms

$$\{\|\cdot\|_t\}, \quad (1/e \leqslant t \leqslant 1)$$

such that $n(X, \|\cdot\|_t) = t$.

THEOREM 3·1. *Let $X = (V, \|\cdot\|)$ be a complex vector space V equipped with the norm $\|\cdot\|$. Then there exists $\|\cdot\|' \in N(V)$ such that $n(V, \|\cdot\|') = 1/e$.*

Proof. Let $\|\cdot\|_e$ be a norm on $\mathbb{C}^2$ such that if $X_e = (\mathbb{C}^2, \|\cdot\|_e)$ and $T_e \in \mathscr{B}(X_e)$ is defined by

$$T_e(1, 0) = (0, 1), \quad T_e(0, 1) = (0, 0),$$

then $\|T_e\|_e = 1$ and $v(T_e) = 1/e$. Write $X = X_1 \oplus X_2$, where $X_1 = (\mathbb{C}^2, \|\cdot\|)$ and $X_2 = (W, \|\cdot\|)$. Define a new norm $\|\cdot\|'$ on V by

$$\|(x, y)\|' = \|x\|_e + \|y\|, \quad x \in \mathbb{C}^2, \quad y \in W.$$

As $\|\cdot\|$ and $\|\cdot\|_e \in N(\mathbb{C}^2)$, $\|\cdot\|' \in N(V)$. Put $Y = (V, \|\cdot\|')$ and define $T \in \mathscr{B}(Y)$, by

$$T((x, y)) = (T_e x, 0).$$

Then $\|T\|' = \|T_e\|_e = 1$ and $v(T) \geqslant v(T_e) = 1/e$. Let

$$(x, y) \in S(Y), \quad f \in S(Y'), \quad f((x, y)) = 1.$$

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Then $$\|x\|_e + \|y\| = 1, \quad f = (f_1, f_2) \in S(X_e') \times S(X_2)$$

and $$f_1(x_e) = \|x_e\|_e, \quad f_2(y) = \|y\|.$$

Consequently $$|f(T(x,y))| = |f_1(T_e x)| \leqslant 1/e.$$

Thus $$v(T) = 1/e.$$

Since $\|\cdot\|' \in N(V)$, proof of the theorem is complete.

We are unable to show that there exists $\|\cdot\|'' \in N(V)$ such that $n(V, \|\cdot\|'') = 1$. We are inclined to think that the answer to the above question is in the negative. Therefore it might be useful to introduce the concept of upper topological numerical index which could be taken as

$$\sup\{n(V,p): p \in N(V)\}.$$

THEOREM 3·2. *Let X be a finite dimensional complex normed space. Then there exists a family of norms $\{\|\cdot\|_t\}$, $(1/e \leqslant t \leqslant 1)$ such that $n(X, \|\cdot\|_t) = t$.*

Proof. There exists N_0 and N_1 such that $n(X, N_0) = 1$ and $n(X, N_1) = 1/e$ respectively. The existence of N_0 is well known and that of N_1 follows from Theorem 1·1.

Put $N_t = (1-t)N_0 + tN_1$ and

$$n(t) = n(X, N_t).$$

Choose α, β such that $0 < \alpha < 1 < \beta$ and

$$\alpha N_0 \leqslant N_1 \leqslant \beta N_0$$

and put $$K = \{T \in \mathscr{B}(X): \beta^{-1} \leqslant N_1(T) \leqslant \alpha^{-1}\}.$$

If $0 \neq T \in \mathscr{B}(X)$, put

$$k_t(T) = \frac{v(T, N_t)}{N_t(T)}.$$

It is easily checked that

$$K \supseteq \{T \in \mathscr{B}(X): N_t(T) = 1\},$$

so $$n(t) = \inf\{k_t(T): T \in K\}.$$

By (3*b*) Corollary 18·4, $k_t(T)$ is a continuous function on $K \times [0, 1]$. As K is compact $n(t)$ is a continuous function of t so for all $t_0 \in [1/e, 1]$, there exists $t \in [0, 1]$ such that

$$t_0 = n(t) = n(X, N_t)$$

and this completes the proof of the theorem.

We remark in passing that the existence of N_1 for finite dimensional normed spaces was proved independently by Bollobás (2), Crabb (5) and Glickfield (8).

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