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FURTHER ANALOGUES OF ANDREWS' SPT-FUNCTION IDENTITY

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Abstract

We give new analogues of Andrews' celebrated spt-function identity using basic hypergeometric transformations due to Verma and Jain ['Transformations between basic hypergeometric series on different bases and identities of Rogers–Ramanujan type', *J. Math. Anal. Appl.* **76**(1) (1980), 230–269].

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1. Introduction

In 2008, Andrews introduced a weighted count on partitions given by counting in a partition the number of times the smallest part appears. This function, denoted by $\mathrm{spt}(n)$ and defined as the number of smallest parts among all the partitions of n, is known to possess an intriguing modular structure and to satisfy the Ramanujan-type congruences

$$\begin{aligned}\mathrm{spt}(5n+4) &\equiv 0 \pmod 5,\\ \mathrm{spt}(7n+5) &\equiv 0 \pmod 7,\\ \mathrm{spt}(13n+6) &\equiv 0 \pmod{13},\end{aligned}$$

among other beautiful combinatorial and arithmetic properties. The origin of these arithmetic properties is attributed to the connection between its generating function and harmonic Maass forms.

Let N, m, n be nonnegative integers. Throughout, we use the following standard notation:

$$\begin{aligned}(a)_N = (a;q)_N &:= \prod_{k=0}^{N-1}(1-aq^k),\\ (a)_\infty = (a;q)_\infty &:= \lim_{N\to\infty}(a)_N \quad \text{where } |q|<1.\end{aligned}$$

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The basic hypergeometric series ${}_{r+1}\phi_r$ is defined by

$$ {}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q, z\right) := \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \cdots (a_{r+1};q)_n}{(q;q)_n (b_1;q)_n (b_2;q)_n \cdots (b_r;q)_n} z^n. $$

When we do not specify the base as in $(a)_n$, the base is q.

By standard combinatorial arguments, the generating function of the smallest parts function (spt-function) due to Andrews is given by

$$ \sum_{n=1}^{\infty} \operatorname{spt}(n) q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2 (q^{n+1};q)_\infty}. $$

Starting from this, Andrews' identity for the spt-function, which relates the spt-function to the partition function and the second Atkin–Garvan rank moment, is obtained using Watson's q-analogue of Whipple's theorem (${}_8\phi_7$-transformation):

$$ \begin{aligned} &{}_8\phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-N} \\ \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{N+1}; \end{matrix} q, \frac{a^2 q^{N+1}}{bcde}\right] \\ &\quad = \frac{(aq,\ aq/de;q)_N}{(aq/d,\ aq/e;q)_N}\ {}_4\phi_3\left[\begin{matrix} aq/bc, & d, & e, & q^{-N} \\ aq/b, & aq/c, & deq^{-N}/a; \end{matrix} q, q\right]. \end{aligned} $$

The spt-function identity of Andrews [3, Theorem 4] is

$$ \sum_{n=1}^{\infty} \operatorname{spt}(n) q^n = \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{n q^n}{1-q^n} + \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n (1+q^n)}{(1-q^n)^2} q^{3n(n+1)/2}. \tag{1.1} $$

The first sum on the right-hand side is the generating function for $np(n)$, where $p(n)$ denotes the number of partitions of n and the second is the generating function for $\frac{1}{2}N_2(n)$, where

$$ N_2(n) = \sum_{m=-\infty}^{\infty} m^2 N(m,n) $$

is the second Atkin–Garvan rank moment. Here, $N(m,n)$ denotes the number of partitions of n with rank m. The rank of a partition is defined to be the largest part minus the number of parts. For instance, the rank of the partition $5+3+3+1$ is $5-4=1$. See [9, 11] for more on the rank and [8] for more on rank moments.

In 2015, Andrews *et al.* [6] obtained new partition theoretic interpretations for $q\omega(q)$ and $\nu(-q)$, where $\omega(q)$ and $\nu(q)$ are the third-order mock theta functions

$$ \omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}} \quad \text{and} \quad \nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}, $$

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among a wide range of other results. Perhaps the most striking of these are the congruences

$$\begin{aligned}\mathrm{spt}_{\omega}(5n+3) &\equiv 0 \pmod 5,\\ \mathrm{spt}_{\omega}(10n+7) &\equiv 0 \pmod 5,\\ \mathrm{spt}_{\omega}(10n+9) &\equiv 0 \pmod 5,\end{aligned}$$

where $\mathrm{spt}_{\omega}(n)$ is the total number of smallest parts in all of the partitions of n in which each odd part is less than twice the smallest part. These three divisibility properties are a consequence of a q-series identity obtained using the following limiting case of a ${}_{10}\phi_9$-transformation due to Bailey (limiting case due to Andrews [2, (2.10)]):

$$\begin{aligned}&\lim_{N\to\infty} {}_{10}\phi_9\left(\begin{matrix}a,\ q^2\sqrt{a},\ -q^2\sqrt{a},\ p_1,\ p_1q,\ p_2,\ p_2q,\ f,\ q^{-2N},\ q^{-2N+1}\\ \sqrt{a},\ -\sqrt{a},\ \dfrac{aq^2}{p_1},\ \dfrac{aq}{p_1},\ \dfrac{aq^2}{p_2},\ \dfrac{aq}{p_2},\ \dfrac{aq^2}{f},\ aq^{2N+2},\ aq^{2N+1}\end{matrix};q^2,\ \frac{a^3q^{4N+3}}{p_1^2p_2^2f}\right)\\ &\quad=\frac{(aq;q)_\infty\left(\dfrac{aq}{p_1p_2};q\right)_\infty}{\left(\dfrac{aq}{p_1};q\right)_\infty\left(\dfrac{aq}{p_2};q\right)_\infty}\sum_{n=0}^{\infty}\frac{(p_1;q)_n(p_2;q)_n\left(\dfrac{aq}{f};q^2\right)_n}{(q;q)_n(aq;q^2)_n\left(\dfrac{aq}{f};q\right)_n}\left(\frac{aq}{p_1p_2}\right)^n.\end{aligned}\tag{1.2}$$

The q-series identity due to Andrews *et al.* [6, Lemma 6.1] is an analogue of the Andrews spt-function identity (1.1):

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2(q^{n+1};q)_n(q^{2n+2};q^2)_\infty}\\ &\quad=\frac{1}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}+\frac{1}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n(1+q^{2n})}{(1-q^{2n})^2}q^{n(3n+1)}.\end{aligned}\tag{1.3}$$

In a recent paper, Andrews and Bachraoui [4] established new partition theoretic interpretations, showing that $q\omega(q)$ and $\nu(-q)$ are generating functions for two-colour partitions in which the even or the odd parts may occur only in one colour, without appeal to such analogues. Additionally, even more recently, a newly discovered analogue of Andrews spt-function identity by Andrews and Kumar [7, Theorem 1.3] has led them to discover a new partition theoretic interpretation of the other third-order mock theta function

$$f(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2},$$

which in turn relates partitions with positive odd rank and two-colour partitions [7, Theorems 1.1 and 1.2]. This new analogue is obtained using the limiting case of another ${}_{10}\phi_9$-transformation due to Bailey (limiting case due to Andrews [2, (2.9)]),

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$$\lim_{N\to\infty} {}_{10}\phi_9\left(\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & r_1, & -r_1, & r_2, & -r_2, & q^{-N}, & -q^{-N} \\ \sqrt{a}, & -\sqrt{a}, & \frac{aq}{b}, & \frac{aq}{r_1}, & -\frac{aq}{r_1}, & \frac{aq}{r_2}, & -\frac{aq}{r_2}, & aq^{N+1}, & -aq^{N+1}; q, & -\frac{a^3q^{3+2N}}{br_1^2r_2^2} \end{matrix}\right)$$

$$= \frac{(a^2q^2;q^2)_\infty \left(\frac{a^2q^2}{r_1^2r_2^2};q^2\right)_\infty}{\left(\frac{a^2q^2}{r_1^2};q^2\right)_\infty \left(\frac{a^2q^2}{r_2^2};q^2\right)_\infty} \sum_{n=0}^{\infty} \frac{(r_1^2;q^2)_n (r_2^2;q^2)_n (-aq/b;q)_{2n}}{(q^2;q^2)_n (a^2q^2/b^2;q^2)_n (-aq;q)_{2n}} \left(\frac{a^2q^2}{r_1^2r_2^2}\right)^n,$$

and is given by

$$\sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{2n})^2(-q^{n+1};q)_n(q^{n+1};q)_\infty}$$

$$= \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} + \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n)}{(1-q^{2n})^2} q^{3n(n+1)/2}. \tag{1.4}$$

So far, these analogues have been produced using ${}_{10}\phi_9$-transformations. We derive more such analogues using higher parameter basic hypergeometric transformations.

First, we consider the following ${}_{12}\phi_{11}$-transformation due to Verma and Jain [14, (1.5)], [13, (3.5)] and use it to arrive at another analogue of Andrews' spt-function identity. The identity is

$${}_{12}\phi_{11}\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, x, \omega x, \omega^2 x, y, \omega y, \omega^2 y, q^{-N}, \omega q^{-N}, \omega^2 q^{-N} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{x}, \frac{aq}{\omega x}, \frac{aq}{\omega^2 x}, \frac{aq}{y}, \frac{aq}{\omega y}, \frac{aq}{\omega^2 y}, aq^{N+1}, \omega^2 aq^{N+1}, \omega aq^{N+1} \end{matrix}; q; \frac{a^4q^{4+3N}}{x^3y^3}\right)$$

$$= \frac{(a^3q^3;q^3)_N (a^3q^3/x^3y^3;q^3)_N}{(a^3q^3/x^3;q^3)_N (a^3q^3/y^3;q^3)_N}$$

$$\cdot {}_6\phi_5\left(\begin{matrix} q^{-3N}, x^3, y^3, aq, aq^2, aq^3 \\ (aq)^{3/2}, -(aq)^{3/2}, a^{3/2}q^3, -a^{3/2}q^3, x^3y^3q^{-3N}/a^3 \end{matrix}; q^3; q^3\right). \tag{1.5}$$

This identity is one of several such identities between basic hypergeometric functions on different bases developed by the authors in [14] leading to a number of new identities of the Rogers–Ramanujan type.

REMARK 1.1. There is a minor error in this identity in Sills' paper [13, (3.5)] where a minus sign seems to appear before the last parameter a^4q^{4+3N}/x^3y^3 on the left-hand side.

THEOREM 1.2 (First analogue of Andrews' spt-function identity). *We have*

$$\sum_{n=1}^{\infty} \frac{q^{3n}}{(1-q^{3n})^2(q^{3(n+1)};q^3)_n(q^{3n+1};q)_\infty}$$

$$= \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{nq^{3n}}{1-q^{3n}} + \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n)}{(1-q^{3n})^2} q^{n(3n+5)/2}.$$

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Next, we consider the other ${}_{12}\phi_{11}$-transformation due to Verma and Jain [14, (1.6)], [13, (3.6)], to arrive at yet another analogue of Andrews' spt-function identity. This identity is

$$
{}_{12}\phi_{11}\left(\begin{matrix} a, q^3\sqrt{a}, -q^3\sqrt{a}, x, xq, xq^2, y, yq, yq^2, q^{-N}, q^{-N+1}, q^{-N+2} \\ \sqrt{a}, -\sqrt{a}, \frac{aq^3}{x}, \frac{aq^2}{x}, \frac{aq}{x}, \frac{aq^3}{y}, \frac{aq^2}{y}, \frac{aq}{y}, aq^{N+3}, aq^{N+2}, aq^{N+1} \end{matrix}; q^3; \frac{a^4q^{3+3N}}{x^3y^3}\right)
$$
$$
= \frac{(aq;q)_N(aq/xy;q)_N}{(aq/x;q)_N(aq/y;q)_N}\,{}_6\phi_5\left(\begin{matrix} \sqrt[3]{a}, \omega\sqrt[3]{a}, \omega^2\sqrt[3]{a}, x, y, q^{-n} \\ \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, \frac{xy}{a}q^{-n} \end{matrix}; q; q\right). \tag{1.6}
$$

THEOREM 1.3 (Second analogue of Andrews' spt-function identity). *We have*

$$
\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)(q^{3n};q^3)_\infty(-q;q)_{n-1}(q;q^2)_n}
$$
$$
= \frac{1}{(q^3;q^3)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + \frac{1}{(q^3;q^3)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n(1+q^{3n})}{(1-q^{3n})^2}q^{3n(3n+1)/2}.
$$

The paper contains two more sections. Section 2 is devoted to proofs of the two theorems stated above. In the final section, we discuss possible questions arising from our work.

2. Proof of the theorems

PROOF OF THEOREM 1.2. By [7, page 7], for $n \geq 1$,

$$
\lim_{a\to 1}\frac{(a;q)_n(q\sqrt{a};q)_n(-q\sqrt{a};q)_n}{(\sqrt{a};q)_n(-\sqrt{a};q)_n} = (q;q)_n(1+q^n).
$$

Letting $a \to 1$, $x = z$, $y = z^{-1}$ in (1.5), we have

$$
1 + \sum_{n=1}^{\infty}\frac{(1+q^n)(z)_n(\omega z)_n(\omega^2 z)_n(z^{-1})_n(\omega z^{-1})_n(\omega^2 z^{-1})_n(q^{-N})_n(\omega q^{-N})_n(\omega^2 q^{-N})_n}{(q/z)_n(q/\omega z)_n(q/\omega^2 z)_n(qz)_n(qz/\omega)_n(qz/\omega^2)_n(q^{N+1})_n(\omega^2 q^{N+1})_n(\omega q^{N+1})_n}q^{3Nn+4n}
$$
$$
= \frac{(q^3;q^3)_N^2}{(q^3/z^3;q^3)_N(q^3z^3;q^3)_N}\sum_{n=0}^{\infty}\frac{(q^{-3N};q^3)_n(z^3;q^3)_n(z^{-3};q^3)_n(q;q^3)_n(q^2;q^3)_n(q^3;q^3)_n}{(q^3;q^3)_n(q^{3/2};q^3)_n(-q^{3/2};q^3)_n(q^3;q^3)_n(-q^3;q^3)_n(q^{-3N};q^3)_n}q^{3n}.
$$

Now,

$$
\lim_{N\to\infty}\frac{(q^{-N})_n(\omega q^{-N})_n(\omega^2 q^{-N})_n}{(q^{N+1})_n(\omega^{-1}q^{N+1})_n(\omega^{-2}q^{N+1})_n}q^{3Nn} = (-1)^n q^{3n(n-1)/2}.
$$

Letting $N \to \infty$ in the identity above and simplifying yields

$$
1 + \sum_{n=1}^{\infty}\frac{(-1)^n(1+q^n)(z^3;q^3)_n(z^{-3};q^3)_n}{(z^{-3}q^3;q^3)_n(z^3q^3;q^3)_n}q^{n(3n+5)/2}
$$
$$
= \frac{(q^3;q^3)_\infty^2}{(z^{-3}q^3;q^3)_\infty(z^3q^3;q^3)_\infty}\sum_{n=0}^{\infty}\frac{(z^3;q^3)_n(z^{-3};q^3)_n(q;q)_{3n}}{(q^3;q^6)_n(q^6;q^6)_n(q^3;q^3)_n}q^{3n}.
$$

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Thus,

$$1+\sum_{n=1}^{\infty}\frac{(-1)^n(1+q^n)(1-z^3)(1-z^{-3})}{(1-z^3q^{3n})(1-z^{-3}q^{3n})}q^{n(3n+5)/2}$$
$$=\frac{(q^3;q^3)_\infty^2}{(z^{-3}q^3;q^3)_\infty(z^3q^3;q^3)_\infty}\sum_{n=0}^{\infty}\frac{(z^3;q^3)_n(z^{-3};q^3)_n(q;q)_{3n}}{(q^3;q^3)_{2n}(q^3;q^3)_n}q^{3n}.$$

Replacing z^3 by z and rearranging,

$$\sum_{n=0}^{\infty}\frac{(z;q^3)_n(z^{-1};q^3)_n(q;q)_{3n}}{(q^3;q^3)_{2n}(q^3;q^3)_n}q^{3n}$$
$$=\frac{1}{(q^3;q^3)_\infty^2}\left((zq^3;q^3)_\infty(z^{-1}q^3;q^3)_\infty+\sum_{n=1}^{\infty}\frac{(-1)^n(1+q^n)(z;q^3)_\infty(z^{-1};q^3)_\infty}{(1-zq^{3n})(1-z^{-1}q^{3n})}q^{n(3n+5)/2}\right). \tag{2.1}$$

Next, we use the identity [3, (2.1)]

$$-\frac{1}{2}\frac{d^2}{dz^2}((1-z)(1-z^{-1})f(z))\Big|_{z=1}=f(1)$$

to compute second derivatives. First, we consider $f(z)=(zq^3,q^3)_{n-1}(z^{-1}q^3,q^3)_{n-1}$, resulting in

$$-\frac{1}{2}\frac{d^2}{dz^2}((z;q^3)_n(z^{-1};q^3)_n)\Big|_{z=1}=(q^3;q^3)_{n-1}^2.$$

Again, taking $f(z)=(zq^3,q^3)_\infty(z^{-1}q^3,q^3)_\infty/(1-zq^{3n})(1-z^{-1}q^{3n})$ results in

$$-\frac{1}{2}\frac{d^2}{dz^2}\left(\frac{(z;q^3)_\infty(z^{-1};q^3)_\infty}{(1-zq^{3n})(1-z^{-1}q^{3n})}\right)\Big|_{z=1}=\frac{(q^3;q^3)_\infty^2}{(1-q^{3n})^2}.$$

Finally, replacing $q\to q^3$ in [3, (2.4)],

$$-\frac{1}{2}\frac{d^2}{dz^2}((zq^3;q^3)_\infty(z^{-1}q^3;q^3)_\infty)\Big|_{z=1}=(q^3;q^3)_\infty^2\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}}.$$

Differentiating both sides of (2.1) twice with respect to z and using these three results,

$$\sum_{n=1}^{\infty}\frac{(q^3;q^3)_{n-1}^2(q;q)_{3n}}{(q^3;q^3)_{2n}(q^3;q^3)_n}q^{3n}=\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}}+\sum_{n=1}^{\infty}\frac{(-1)^n(1+q^n)}{(1-q^{3n})^2}q^{n(3n+5)/2}.$$

The theorem follows by dividing throughout by $(q;q)_\infty$ and rewriting the left-hand side. □

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PROOF OF THEOREM 1.3. As in the proof of Theorem 1.2, for $n \geq 1$,

$$\lim_{a\to 1} \frac{(a;q^3)_n(q^3\sqrt{a};q^3)_n(-q^3\sqrt{a};q^3)_n}{(\sqrt{a};q^3)_n(-\sqrt{a};q^3)_n} = (q^3;q^3)_n(1+q^{3n}).$$

In addition, for $n \geq 1$,

$$\lim_{a\to 1} \frac{(\sqrt[3]{a};q)_n(\omega\sqrt[3]{a};q)_n(\omega^2\sqrt[3]{a};q)_n}{(\sqrt{a};q)_n(-\sqrt{a};q)_n} = \frac{(q^3;q^3)_{n-1}}{(q^2;q^2)_{n-1}}.$$

Then, letting $a \to 1$, $x = z$, $y = z^{-1}$ in (1.6), we have

$$1 + \sum_{n=1}^{\infty} q^{3Nn+3n} \cdot \frac{(1+q^{3n})(z;q^3)_n(zq;q^3)_n(zq^2;q^3)_n(z^{-1};q^3)_n(z^{-1}q;q^3)_n(z^{-1}q^2;q^3)_n(q^{-N};q^3)_n(q^{-N+1};q^3)_n(q^{-N+2};q^3)_n}{(q^3/z;q^3)_n(q^2/z;q^3)_n(\frac{q}{z};q^3)_n(zq^3;q^3)_n(zq^2;q^3)_n(zq;q^3)_n(q^{N+3};q^3)_n(q^{N+2};q^3)_n(q^{N+1};q^3)_n}$$
$$= \frac{(q;q)_N^2}{(q/z;q)_N(qz;q)_N} + \frac{(q;q)_N^2}{(q/z;q)_N(qz;q)_N} \sum_{n=1}^{\infty} \frac{(q^3;q^3)_{n-1}(z;q)_n(z^{-1};q)_n}{(q;q)_n(q^2;q^2)_{n-1}(\sqrt{q};q)_n(-\sqrt{q};q)_n} q^n.$$

Now,

$$\lim_{N\to\infty} \frac{(q^{-N};q^3)_n(q^{-N+1};q^3)_n(q^{-N+2};q^3)_n}{(q^{N+3};q^3)_n(q^{N+2};q^3)_n(q^{N+1};q^3)_n} q^{3Nn} = (-1)^n q^{3n(3n-1)/2}.$$

Letting $N \to \infty$ in the above identity and simplifying,

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^n(1+q^{3n})(z;q)_{3n}(z^{-1};q)_{3n}}{(z^{-1}q;q)_{3n}(zq;q)_{3n}} q^{3n(3n+1)/2}$$
$$= \frac{(q;q)_\infty^2}{(z^{-1}q;q)_\infty(zq;q)_\infty} + \frac{(q;q)_\infty^2}{(z^{-1}q;q)_\infty(zq;q)_\infty} \sum_{n=1}^{\infty} \frac{(z;q)_n(z^{-1};q)_n(q^3;q^3)_{n-1}}{(q;q)_n(q^2;q^2)_{n-1}(q;q^2)_n} q^n.$$

Thus,

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^n(1+q^{3n})(1-z)(1-z^{-1})}{(1-zq^{3n})(1-z^{-1}q^{3n})} q^{3n(3n+1)/2}$$
$$= \frac{(q;q)_\infty^2}{(z^{-1}q;q)_\infty(zq;q)_\infty} + \frac{(q;q)_\infty^2}{(z^{-1}q;q)_\infty(zq;q)_\infty} \sum_{n=1}^{\infty} \frac{(z;q)_n(z^{-1};q)_n(q^3;q^3)_{n-1}}{(q;q)_n(q^2;q^2)_{n-1}(q;q^2)_n} q^n.$$

Rearranging,

$$1 + \sum_{n=1}^{\infty} \frac{(z;q)_n(z^{-1};q)_n(q^3;q^3)_{n-1}}{(q;q)_n(q^2;q^2)_{n-1}(q;q^2)_n} q^n$$
$$= \frac{1}{(q;q)_\infty^2} \left((zq;q)_\infty(z^{-1}q;q)_\infty + \sum_{n=1}^{\infty} \frac{(-1)^n(1+q^{3n})(z;q)_\infty(z^{-1};q)_\infty}{(1-zq^{3n})(1-z^{-1}q^{3n})} q^{3n(3n+1)/2} \right). \tag{2.2}$$

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Next, as in the proof of the previous theorem, we again use the identity

$$-\frac{1}{2}\frac{d^2}{dz^2}((1-z)(1-z^{-1})f(z))\Big|_{z=1} = f(1)$$

to differentiate twice with respect to z. Considering $f(z) = (zq,q)_{n-1}(z^{-1}q,q)_{n-1}$ results in a double derivative also found in Andrews' original work [3, Proof of Theorem 3],

$$-\frac{1}{2}\frac{d^2}{dz^2}((z;q)_n(z^{-1};q)_n)\Big|_{z=1} = (q;q)_{n-1}^2.$$

Again, considering $f(z) = (zq,q)_\infty(z^{-1}q,q)_\infty/(1-zq^{3n})(1-z^{-1}q^{3n})$ results in

$$-\frac{1}{2}\frac{d^2}{dz^2}\left(\frac{(z;q^3)_\infty(z^{-1};q^3)_\infty}{(1-zq^{3n})(1-z^{-1}q^{3n})}\right)\Big|_{z=1} = \frac{(q;q)_\infty^2}{(1-q^{3n})^2}.$$

Finally, [3, (2.4)] in Andrews' original work states

$$-\frac{1}{2}\frac{d^2}{dz^2}((zq;q)_\infty(z^{-1}q;q)_\infty)\Big|_{z=1} = (q;q)_\infty^2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.$$

Differentiating both sides of (2.2) twice with respect to z and using these three results,

$$\sum_{n=1}^{\infty}\frac{(q;q)_{n-1}^2(q^3;q^3)_{n-1}}{(q;q)_n(q^2;q^2)_{n-1}(q;q^2)_n}q^n = \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + \sum_{n=1}^{\infty}\frac{(-1)^n(1+q^{3n})}{(1-q^{3n})^2}q^{3n(3n+1)/2}.$$

The theorem follows by dividing throughout by $(q^3;q^3)_\infty$ and rewriting the left-hand side. □

3. Concluding observations

Several possible avenues for further investigation arise from our work.

(1) In [6] and [7], the spt-function identities led to the connections with third-order mock theta functions discussed earlier in Section 1. In both cases, this connection is established using a nonweighted version of the sum on the left-hand side of the identities in (1.3) and (1.4), and working on this closely related series after initial use of the deep four-parameter identity due to [1, Theorem 1]:

$$\sum_{n=0}^{\infty}\frac{(B;q)_n\,(-Abq;q)_n\,q^n}{(-aq;q)_n\,(-bq;q)_n} = \frac{-a^{-1}(B;q)_\infty\,(-Abq;q)_\infty}{(-bq;q)_\infty\,(-aq;q)_\infty}\sum_{m=0}^{\infty}\frac{(A^{-1};q)_m}{(-B/a;q)_{m+1}}\left(\frac{Abq}{a}\right)^m + (1+b)\sum_{m=0}^{\infty}\frac{(-a^{-1};q)_{m+1}\,(-ABq/a;q)_m\,(-b)^m}{(-B/a;q)_{m+1}\,(Abq/a;q)_{m+1}}. \quad (3.1)$$

For instance, it is shown in [6, Theorem 3.1] that

$$q\omega(q) = \sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)(q^{n+1};q)_n(q^{2n+2};q)_\infty}.$$

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It is possible that such mock theta connections or relations with other partition theoretic objects also exist for our spt-function identities. One attempt at this by removing the quadratic factor in the denominator of the identity in Theorem 1.2 leads to the simplification

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^{3n}}{(1-q^{3n})(q^{3(n+1)};q^3)_n(q^{3n+1};q)_\infty} &= \sum_{n=0}^{\infty}\frac{q^{3n+3}}{(q^{3n+3};q^3)_{n+2}(q^{3n+4};q)_\infty}\\ &= \frac{q^3}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(q;q)_{3n+3}}{(q^{3n+3};q^3)_{n+2}}q^{3n}\\ &= \frac{q^3}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(q^3;q^3)_n(q;q^3)_{n+1}(q^2;q^3)_{n+1}}{(q^3;q^6)_{n+1}(-q^3;q^3)_{n+1}}q^{3n}.\end{aligned}$$

Andrews' identity (3.1) does not seem useful here due to the presence of more parameters in these series. We believe this can be dealt with by considering a higher parameter identity analogous to Andrews' identity. Some instances of such identities are [5, Theorem 1.1] and [10, Theorem 1.3]. Our investigation here is preliminary and a different approach to simplifying and transforming the series is worth considering.

(2) There are more such analogues to be discovered. Some identities that could possibly yield such analogues are [14, (1.7) and (1.8)], a ${}_{14}\phi_{13}$-identity [14, (2.29)] and a similar higher parameter identity [14, (2.30)], all due to Verma and Jain, and several similar identities found by McLaughlin and Zimmer [12]. However, appropriate choices of the substitutions are necessary. For instance, making the substitutions $a = q$, $x = z$, $y = z^{-1}$ in [14, (1.7)], letting $N \to \infty$ and differentiating twice with respect to z as in the earlier proofs leads us to the series

$$\sum_{n=0}^{\infty}\frac{(q;q^3)_n(q;q)_{n-1}}{(1-q^n)(q;q)_{2n}}q^n$$

on the right-hand side which is new. However, we have observed that making the substitutions $a \to 1$, $c = q$, $d = q^2$, $x = z$, $y = z^{-1}$ in [14, (2.29)] leads us again to the identity in Theorem 1.2. Further multi-series generalisations of the ${}_{10}\phi_9$-identities and the ${}_{14}\phi_{13}$-identities due to Verma and Jain [14, (4.1), (4.3), (4.4), (4.5)] could lead to generalised spt-function identities.

(3) There is resemblance and symmetry between our newfound spt-identities in Theorems 1.2 and 1.3, and those by the previous authors. For instance, upon bringing the lone product $(q;q)_\infty$ in our identity in Theorem 1.3 to the left-hand side, the first sum on the right-hand side of our identity and that of Andrews, Dixit and Yee in (1.3) are the same, namely the generating function for $np(n)$. The second sum in our identity can be obtained by substituting $q \to q^{3/2}$ in the second sum in their identity. This calls for an examination of the underlying theory behind this resemblance more closely.

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