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On the measurable dynamics of real rational functions

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Abstract. Let f be a real rational function with all critical points on the extended real axis and of even order. Then:
(1) f carries no invariant line field on the Julia set unless it is doubly covered by an integral torus endomorphism (a Lattés example); and
(2) $f|J(f)$ has only finitely many ergodic components.

1. *Introduction*

In this paper, we study the measurable dynamics of a real rational function f, which satisfies the following two conditions:
(C1) any critical point c of f is contained in $S^1 = \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$;
(C2) any critical point c of f is of even order.

Let $\mathcal{F}$ denote the set of all real rational functions satisfying these two conditions. Our main result is the following theorem.

THEOREM 1. *Any $f \in \mathcal{F}$ carries no invariant line field on the Julia set $J(f)$ unless it is a Lattés example.*

THEOREM 2. *Let $f \in \mathcal{F}$. $f|J(f)$ has only finitely many ergodic components.*

It is conjectured that these theorems hold for all rational functions, see [**16, 18, 22**]. The question in the first main theorem is closely related to the well-known Fatou conjecture: in the space $\mathbf{Rat}_d$ of all rational maps (and in the space $\mathbf{Poly}_d$ of all polynomials) of degree $d \geq 2$, hyperbolic ones form an open and dense subset. Our first main theorem implies the following corollary.

COROLLARY 3. *Let f be a real rational map of degree $d \geq 2$ for which all critical points are on the real axis. Suppose that f is structurally stable in the space $\mathbf{Rat}_d$. Then f is hyperbolic.*

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Proof of Corollary 3. Since f is structurally stable, the Teichmüller space of f has dimension $2d - 2$. In particular, all the critical points are non-degenerate which implies that condition (C2) holds. Thus $f \in \mathcal{F}$. It follows from Theorem 1 that f carries no invariant line field on the Julia set. By Theorem 9.4 of [**22**], f is hyperbolic. □

Some progress in the direction of this work had been made by several authors, mostly on the case that f is a quadratic polynomial. For a non-renormalizable polynomial, the absence of an invariant line field follows from the work of Yoccoz (see [**9**]), while McMullen [**20**] proved this for infinitely renormalizable quadratic polynomials which are 'robust', including all real ones. McMullen's method has been refined by Levin and van Strien [**12**] who generalized the result to all real unimodal polynomials. For polynomials with more critical points, the only result, to my knowledge, is in [**7, 8**], where the authors prove the absence of invariant line fields for a real polynomial with only real critical points and such that f is infinitely renormalizable and of bounded type. The ergodic decomposition was proved by Lyubich [**14**] for a smooth one-dimensional dynamics with non-flat critical points and in the category of holomorphic dynamics, by Prado [**25**] for a real unimodal polynomial. In the proof of all these results, some kind of 'bound' plays a crucial role.

In this paper, Theorems 1 and 2 will be proved using the same method and the real bounds developed in [**26**] play a crucial role in our proof.

The main difficulty is to study the measurable dynamics on the part $J_c = \{z \in J(f) : \omega(z) = \omega(c)\}$ for a recurrent critical point c with a minimal ω-limit set. It will be done in the same outline as in McMullen [**20**]. If a large real bound exists, then we shall transfer it to a 'complex bound' for a *quasi-polynomial-like mapping*, as in [**12**]. In the converse case, our argument will depend heavily on the essentially bounded geometry of the Cantor set $\omega(c)$. Comparing this work with that in [**20**], a new ingredient is the fact that the absence of a large real bound implies that c is 'uniformly persistently recurrent' in some sense, which is also proved in [**26**].

Let us briefly outline the structure of this paper. Section 2 contains some technical preliminaries, including some distortion lemmas. In §3, we recall the definition and give some sufficient condition for the absence of invariant line fields. In §4, the notions of real box mappings and quasi-polynomial-like mappings are recalled. Section 5 is devoted to the study of the measurable dynamics on the subset of the Julia set consisting of points accumulating on a reluctantly recurrent critical point. It turns out that we will only need to consider the part J_c for a persistently recurrent critical point. In §6, we study the dynamics of $f|J_c$ when a large bound exists for c. In §7, we shall study the converse case and then complete the proof of the main theorems.

2. *Preliminaries*

2.1. *Distortion lemmas.* We state some well-known distortion lemmas. Let $\Delta(r) = \{z \in \mathbb{C} : |z| < r\}$ and let $\Delta = \Delta(1)$.

LEMMA 2.1. *For any $\delta > 0$ and $N \in \mathbb{N}$, there is a constant $\epsilon = \epsilon(N, \delta)$ such that the following holds.*

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If $h : U \to V$ is a proper map of degree N between two hyperbolic topological disks U, V, then for any $z_0 \in U$, we have $h(\{z \in U : d_U(z, z_0) < \epsilon\}) \supset \{w \in V : d_V(h(z_0), w) < \delta\}$, where d_U and d_V are the hyperbolic metric of U and V respectively.

Moreover, $\epsilon \to 0$ as $\delta \to 0$ for a fixed N.

LEMMA 2.2. *For any $0 < r < 1$ and $N \in \mathbb{N}$, there is a constant $C = C(r, N) > 1$ with the following property.*

If $U \subset \mathbb{C}$ is a topological disk and $h : U \to \Delta$ is a proper map of degree $\leq N$, then for any component U_0 of $h^{-1}(\Delta(r))$, we have $\operatorname{diam}(U_0)^2/\operatorname{area}(U_0) \leq C$, where the diameter and area are measured in the Euclidean metric of U.

LEMMA 2.3. *For any $\delta > 0$ and $N \in \mathbb{N}$, there is a constant $\epsilon = \epsilon(N, \delta) > 0$ with the following property.*

Let U be a simply connected subset of $\hat{\mathbb{C}}$ and $h : U \to \Delta$ be a proper map of degree $\leq N$. Let $X \subset \Delta$ be a measurable set with $m(\Delta(1/2) \cap X)/m(\Delta(1/2)) \leq 1 - \delta$, then for any component U_0 of $h^{-1}(\Delta(1/2))$, we have $m(U_0 \cap h^{-1}(X))/m(U_0) \leq 1 - \epsilon$, where m denotes the Lebesgue measure on $\hat{\mathbb{C}}$.

All these lemmas follow easily from the following fact:

$$\{h : \Delta \to \Delta;\ h \text{ is holomorphic and proper of degree } \leq d,\ h(0) = 0\}$$

is a compact set in the topology of uniform convergence on compact sets. We leave the (easy) proofs of this fact and the lemmas to the readers.

2.2. *Real rational functions.* Let f be a real rational function satisfying the condition (C1).

We shall use S^1 to denote the extended real line $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and use $\operatorname{Par}(f)$, $C(f)$ to denote the set of parabolic periodic points and the set of critical points of f, respectively. All the metrics in this section are the spherical metric on $\hat{\mathbb{C}}$. Let $P(f)$ denote the post-critical set of f, that is,

$$P(f) = \bigcup_{c \in C(f)} \overline{\bigcup_{n=1}^{\infty} \{f^n(c)\}}.$$

The map $f|S^1 : S^1 \to S^1$ can be considered as a one-dimensional dynamical system, which has no wandering interval, due to [**19**].

PROPOSITION 2.1. *Let $y \in J(f) \cap S^1 - \operatorname{Par}(f)$. For any $\epsilon > 0$, there is a $\delta > 0$ such that for any interval $I \subset S^1$ and any $n \in \mathbb{N}$, if $f^n(I) \subset B(y, \delta)$, then $|I| \leq \epsilon$.*

Proof. Otherwise, there is a positive integer $\epsilon_0 > 0$, and for any $k \in \mathbb{N}$, there is an interval $I_k \subset S^1$ with $|I_k| \geq \epsilon_0$, and a positive integer n_k such that

$$f^{n_k}(I_k) \subset B(y, 1/k).$$

After passing to a subsequence, we may assume that there is a non-degenerate interval $I \subset I_k$ for all k. Then, $\liminf_n |f^n(I)| = 0$. Thus, there is an attracting or parabolic periodic point p of f, such that $\sup_{x \in I} d(f^n(x), f^n(p)) \to 0$ as $n \to \infty$.

Since $y \in \omega(x)$ for all $x \in I$, we have that y is contained in the orbit of p, and so y is an attracting or parabolic periodic point of f, which is impossible. □

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For any critical point c, there is a unique involution τ_c defined by $f(\tau_c(x)) = f(x)$ on a neighbourhood U_c in S^1. We shall fix for any critical point such a neighbourhood U_c such that they are pairwise disjoint. An interval $I \subset S^1$ is called *symmetric* if it contains a critical point, say c and if $I \subset U_c$ and $\tau_c(I) = I$.

For any open interval $I \subset S^1$, let $\hat{\mathbb{C}}_I$ denote the hyperbolic surface $\hat{\mathbb{C}} - (S^1 - I)$. Let $D(I)$ denote the disk such that $D(I) \cap S^1 = I$ and the boundary $\partial D(I)$ intersects S^1 orthogonally. The following is an observation of Sullivan, see [**29**].

LEMMA 2.4. *There is a universal constant $\theta > 0$ such that $D(I) = \{z \in \hat{\mathbb{C}}_I : d(z, I) < \theta\}$, where d denotes the hyperbolic distance in $\hat{\mathbb{C}}_I$.*

PROPOSITION 2.2. *Let $f \in \mathcal{F}$. Then either of the following holds:*

(i) *$J(f) = \hat{\mathbb{C}}$ and f is ergodic; or*

(ii) *for a.e. $z \in J(f)$, and any $y \in \omega(z) - \mathrm{Par}(f)$, there is a recurrent critical point c such that $\omega(z) \ni c$ and $\omega(c) \ni y$.*

Proof. Let $E \subset J(f)$ be a measurable set of positive measure such that $f(E) \subset E \pmod 0$ and $m(\hat{\mathbb{C}} - E) > 0$. Let $z \in E$ be an arbitrary Lebesgue density point of E. Let $y \in \omega(z)$ be an arbitrary point which is not a parabolic periodic point. We claim that there is some recurrent critical point c of f such that $c \in \omega(z)$ and $y \in \omega(c)$.

First of all, we choose a small positive number $\epsilon > 0$ so that

(1) for any $c \in C(f)$ which is not recurrent, we have $f^n(c) \notin B(c, \epsilon)$ for any $n \in \mathbb{N}$; and

(2) for any $c \in C(f) - \omega(z)$, we have $f^n(z) \notin B(c, \epsilon)$ for any $n \in \mathbb{N}$.

Let $\delta > 0$ be a small positive number. Let $D_0 = B(y, \delta/2) \subset D = B(y, \delta)$ be small disks centred at y. There exists a sequence of positive integers $n_1 < n_2 < \cdots$ such that $f^{n_k}(z) \to y$ as $k \to \infty$. Let A_k (A_k^0, respectively) denote the component of $f^{-n_k}(D)$ ($f^{-n_k}(D_0)$, respectively) containing z.

If $f^{n_k} : A_k \to D$ has a uniformly bounded degree, then since z is a Lebesgue density point of E, by Lemmas 2.2 and 2.3, $m(D_0 - E) = 0$. Since $E \subset J(f)$, we then have $D_0 \subset J(f)$. So $f^n(D_0) = \hat{\mathbb{C}}$ for some $n \in \mathbb{N}$ and hence $E = \hat{\mathbb{C}} \pmod 0$, which is a contradiction. So after passing to a subsequence we may assume that $f^{n_k}|A_k$ has a degree which tends to infinity as $k \to \infty$. For the same reason, we may assume that $f^{n_k}|A_k^0$ also has a large degree.

Thus, there are $0 \le p_k < q_k < n_k$ such that $f^{p_k}(A_k)$ and $f^{q_k}(A_k)$ contain the same critical point c_k and $f^{n_k - p_k}|f^{p_k}(A_k)$ has a uniformly bounded degree. By Lemma 2.1, the hyperbolic diameter of $f^{p_k}(A_k^0)$ in $f^{p_k}(A_k)$ is uniformly bounded from above. By passing to a further subsequence, we may assume that $c_k = c$ for all k.

If δ was chosen small, then it follows from Proposition 2.1 that $\max\{|f^n(A_k) \cap S^1| : p_k \le n \le n_k\} < \epsilon$. Since the forward orbit of c enters $f^{q_k}(A_k) \cap S^1$, which is contained in $B(c, \epsilon)$, it follows from (1) that c is a recurrent critical point. Since $f^{p_k}(A_k^0) \cap S^1 \neq \emptyset$, and $f^{p_k}(A_k) \subset \hat{\mathbb{C}}_{f^{p_k}(A_k) \cap S^1}$, and since the hyperbolic diameter of $f^{p_k}(A_k^0)$ in the hyperbolic surface $f^{p_k}(A_k)$ is bounded from above, we have that the spherical diameter of $f^{p_k}(A_k^0)$ is less than ϵ (provided that δ is sufficiently small), which implies that $c \in \omega(z)$ by (2).

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Since δ can be taken arbitrarily small, the claim follows. The proposition easily follows from the claim. □

Let $C_r(f) \subset C(f)$ denote the set of non-periodic recurrent critical points. For any $c, d \in C_r(f)$, we say $c \sim d$ if $\omega(c) \ni d$ and $\omega(d) \ni c$. An order is defined on the set of equivalence classes as follows: $[c] \prec [d]$ if $\omega(c) \ni d$ but $\omega(d) \not\ni c$. Let

$$J_c = \{z \in J(f) : \omega(z) \ni c \text{ but } \omega(z) \not\ni d \text{ for any } [d] \prec [c]\}.$$

It follows from Proposition 2.2 that either $J(f) = \hat{\mathbb{C}}$ and f is ergodic or

$$J(f) \subset \bigcup_{c \in C_r(f)} J_c \ (\mathrm{mod}\, 0).$$

3. *Invariant line fields*

Recall that *a line field* μ is a measurable Beltrami differential on $\hat{\mathbb{C}}$ such that $|\mu| = 1$ on a set E of positive measure and μ vanishes elsewhere. The set E is the *support* of μ. A line field is *holomorphic* on an open set U if $\mu = \bar{\phi}/|\phi|$ a.e. for ϕ a holomorphic quadratic differential on U. A line field μ is called *(f-)invariant* if $f^*(\mu) = \mu$ a.e. We say that f *carries an invariant line field on the Julia set* if there is an f-invariant line field μ with a support contained in its Julia set (up to a set of measure zero).

A rational map f is called *a Lattés example* if it is doubly covered by an integral torus endomorphism. Such a rational map carries an invariant line field on its Julia set, which is $\hat{\mathbb{C}}$. It is a conjecture that Lattés examples are the only rational maps which carry an invariant line field on the Julia set, see [**18, 22**].

It is useful to remember the following, which is Lemma 3.16 in [**20**].

LEMMA 3.1. *Let μ be an f-invariant line field which is holomorphic on a non-empty open set contained in the Julia set. Then f is a Lattés example.*

PROPOSITION 3.1. *Let $f \in \mathcal{F}$. Assume that f carries an invariant line field μ with support $E \subset J(f)$. Then either*

(i) *for a.e. $z \in E$ and for any $y \in \omega(z)$ which is not a parabolic periodic point, there exists $c \in C_r(f)$ such that $c \in \omega(z)$ and $\omega(c) \ni y$; or*

(ii) *f is a Lattés example.*

Proof. Take a Lebesgue density point z of E and let $y \in \omega(z) - \mathrm{Par}(f)$. If there is no $c \in C_r(f) \cap \omega(z)$ such that $\omega(c) \ni y$, then by the same argument as that in the proof of Proposition 2.2, we will obtain a sequence of proper mappings $f^{n_k} : A_k \to B(y, \delta)$ for some $\delta > 0$ such that $x \in A_k$ and $f^{n_k}|A_k$ has a uniformly bounded degree, and $f^{n_k}(x) \to y$ as $k \to \infty$. It then easily follows that μ is holomorphic on some non-empty open set and hence f is doubly covered by an integral torus endomorphism. □

We shall give more criteria for the non-existence of invariant line fields. The idea is taken from [**20**] and [**21**]: *Uniformly nonlinearity implies the absence of invariant line fields.*

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An invariant line field μ is called *almost continuous* at a point x if for any $\epsilon > 0$,

$$\frac{m(\{z \in B(x,r) : |\mu(z) - \mu(x)| \geq \epsilon\})}{m(B(x,r))} \to 0$$

as $r \to 0$, where $\mu = \mu(z)\, d\bar{z}/dz$ on a conformal coordinate around x. Note that the definition does not depend on the choice of the coordinate.

Given a rational function f, consider the collection $\mathcal{H}(f)$ of all holomorphic maps $h : U \to V$, where U, V are open sets such that there exist $i, j \in \mathbb{N}$ such that $f^i \circ h = f^j$ on U. Obviously for any element $h : U \to V$ in $\mathcal{H}(f)$, $h^*(\mu|V) = \mu|U$, i.e.

$$\mu(z) = \mu(h(z))\overline{h'(z)}/h'(z) \text{ a.e. on } U.$$

PROPOSITION 3.2. *Let f be a rational function of degree ≥ 2 and x be a point in $J(f)$. If there is a positive constant $C > 1$ and a positive integer $N \geq 2$ and a sequence $h_n : U_n \to V_n$ of elements of $\mathcal{H}(f)$ with the following properties:*

(i) *U_n, V_n are topological disks and*

$$\operatorname{diam}(U_n) \to 0, \ \operatorname{diam}(V_n) \to 0$$

as $n \to \infty$;

(ii) *h_n is a proper map with degree between 2 and N;*

(iii) *for some $u \in U_n$ such that $h_n'(u) = 0$ and for $v = h_n(u)$ we have*

$$\max_{z \in \partial U_n} d(z,u) \leq C d(u, \partial U_n)$$

and

$$\max_{z \in \partial V_n} d(z,v) \leq C d(v, \partial V_n);$$

(iv)

$$d(U_n, x) \leq C \operatorname{diam}(U_n), \quad d(V_n, x) \leq C \operatorname{diam}(V_n),$$

where diam and, d denote the diameter and the distance in the spherical metric respectively.

Then for any f-invariant line field μ, $x \notin \operatorname{supp}(\mu)$ or μ is not almost continuous at x.

Proof. By changing the coordinates using a Möbius transformation, we can assume that $x \in \mathbb{C}$. The conditions in (i), (iii) and (iv) also hold for the Euclidean metric by changing the constant C. Here, all distances, diameters and areas are measured in the Euclidean metric on $\mathbb{C}$ unless otherwise stated.

Fix any $u_n \in U_n$ such that $h_n'(u_n) = 0$ and let $v_n = h_n(u_n)$. Denote by α_n and β_n the linear transformation of $\mathbb{C}$ such that $\alpha_n' > 0, \beta_n' > 0$ and

$$\alpha_n(u_n) = \beta_n(v_n) = 0, \quad \operatorname{diam}(\alpha_n U_n) = \operatorname{diam}(\beta_n V_n) = 1.$$

Denote $X_n = \alpha_n U_n$ and $Y_n = \beta_n V_n$. And denote by H_n the function $\beta_n \circ h_n \circ \alpha_n^{-1} : X_n \to Y_n$. By condition (iii), we have

$$\min_{z \in \partial X_n} |z| \geq \max_{z \in \partial X_n} |z|/C \geq 1/(2C),$$

$$\min_{z \in \partial Y_n} |z| \geq \max_{z \in \partial Y_n} |z|/C \geq 1/(2C).$$

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Thus $X_n, Y_n \supset B(0, 1/(2C))$. So after passing to a subsequence we may assume that $(X_n, 0) \to (X, 0)$ and $(Y_n, 0) \to (Y, 0)$ as $n \to \infty$ in the Caratheodory topology for some topological disk X, Y and there is a holomorphic function $H : (X, 0) \to (Y, 0)$ such that for any compact set K in X, $H_n|K \to H|K$ uniformly as $n \to \infty$.

CLAIM. *The function H is not a constant function.*

Indeed, since $X_n \subset B(0, 1)$, for some $\delta = \delta(C) > 0$, in the hyperbolic surface X_n, the hyperbolic ball $B_{X_n}(0, \delta)$ is contained in the Euclidean ball $B(0, 1/(4C))$. By Lemma 2.1, there is a constant $\epsilon > 0$ which depends only on C and N, such that $H_n(B(0, 1/(4C)))$ contains the hyperbolic ball $B_{Y_n}(0, \epsilon)$ in the hyperbolic surface Y_n. Since $Y_n \supset B(0, 1/(2C))$, $B_{Y_n}(0, \epsilon) \supset B(0, \epsilon')$ for some $\epsilon' > 0$. For any $n \in \mathbb{N}$, let $z_n \in B(0, 1/(4C))$ be a point such that $|H_n(z_n)| = \epsilon'$. After passing to a further subsequence, we may assume that $z_n \to z$ as $n \to \infty$. Thus $|H(z)| = \epsilon'$. Note $H(0) = 0$, and hence the claim follows. □

Consequently, H is a proper map of degree between 2 and N.

Assume now that $|\mu(x)| = 1$ and that μ is almost continuous at x. Without loss of generality we can assume that $\mu(x) = 1$. Let us prove that for any $\delta > 0$, we have

$$\frac{m(\{z \in U_n : |\mu(z) - 1| \geq \delta\})}{m(U_n)} \to 0, \tag{1}$$

as $n \to \infty$.

Let $r_n = \max_{z \in U_n} d(x, z)$ and $s_n = d(u_n, \partial U_n)$. Then $r_n \to 0$ as $n \to \infty$. Thus, by the definition of almost continuity, we have

$$\frac{m(\{z \in B(x, r_n) : |\mu(z) - 1| \geq \delta\})}{m(B(x, r_n))} \to 0.$$

Note that by (iii) and (iv), we have

$$m(B(x, r_n)) \asymp r_n^2 \asymp s_n^2 \asymp m(U_n).$$

Thus, statement (1) holds.

Similarly, we can prove that for any $\delta > 0$,

$$\frac{m(\{z \in V_n : |\mu(z) - 1| \geq \delta\})}{m(V_n)} \to 0, \tag{2}$$

as $n \to \infty$.

Let $\mu_n = (\alpha_n^{-1})^*(\mu|U_n)$ be a Beltrami differential on X_n and $\nu_n = (\beta_n^{-1})^*(\mu|V_n)$ be a Beltrami differential on Y_n. Obviously $H_n^* \nu_n = \mu_n$. It follows from (1) and (2) that for any $\delta > 0$

$$m(\{z \in X_n : |\mu_n(z) - 1| \geq \delta\}) \to 0, \tag{3}$$

and

$$m(\{z \in Y_n : |\nu_n(z) - 1| \geq \delta\}) \to 0, \tag{4}$$

as $n \to \infty$.

Let L be a small Euclidean ball whose closure is contained in X such that H is univalent on a neighbourhood of $\bar{L}$. Let $\eta > 0$ be a constant such that $|H'(z)| \geq 2\eta$ for any $z \in L$.

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Since $H_n|L \to H|L$ uniformly, we have that for n sufficiently large, $H_n|L$ is univalent and also $|H_n'(z)| \geq \eta$ on L.

By (3) and (4), for any $\delta > 0$, we have

$$m\{z \in L : |\mu_n(z) - 1| \geq \delta\} \to 0$$

and

$$m\{w \in H_n(L) : |\nu_n(z) - 1| \geq \delta\} \to 0$$

as $n \to \infty$. Note that

$$m\{z \in L : |\nu_n(H_n(z)) - 1| \geq \delta\} \leq \eta^{-2} m\{w \in H_n(L) : |\nu_n(w) - 1| \geq \delta\} \to 0$$

as $n \to \infty$.

For any $\delta \in (0, 1)$, let

$$A_n(\delta) = \left\{ z \in L : \left| \frac{\overline{H_n'(z)}}{H_n'(z)} - 1 \right| \geq \delta \right\}$$

as $n \to \infty$. For a.e. $z \in A_n(\delta)$, we have $\mu_n(z) = \nu_n(H_n(z))\overline{H_n'(z)}/H_n'(z)$. Thus, for such a z, either $|\nu_n(H_n(z)) - 1| > \delta/10$, or $|\nu_n(z) - 1| > \delta/10$. Thus, we have

$$m(A_n(\delta)) \to 0$$

as $n \to \infty$. Consequently, we have that, for any $\delta > 0$,

$$m\left\{ z \in L : \left| \frac{\overline{H'(z)}}{H'(z)} - 1 \right| > \delta \right\} = 0.$$

Since δ can be taken to be arbitrarily small, we know that for a.e. $z \in L$, $\overline{H'(z)} = H'(z)$. But then H' is a constant function, and hence it is equal to 0, i.e. H is a constant function, which is a contradiction. □

To construct the family $\{h_n\}$ we usually first construct a sequence of restrictions of forward iterates of f, then try to pull them back to the neighbourhood of x.

For any annulus A with a finite modulus, we define the *core curve* of A to be the unique simple geodesic in A. In other words, if $\phi : A \to \{z \in \mathbb{C} : 1 < |z| < R\}$ is a conformal map, then the core curve is $\phi^{-1}(\{z \in \mathbb{C} : |z| = \sqrt{R}\})$. For any topological disk U, and any compact set $K \subset U$, we use $\mathrm{mod}(K, U)$ to denote the supremum of the modulus of an annulus which is contained in U and surrounds K.

COROLLARY 3.3. *Let f be a rational function of degree $d \geq 2$ and x be a point in the Julia set $J(f)$. If there exists a positive constant δ, a positive integer $N \geq 2$, sequences $\{s_n\}, \{p_n\}, \{q_n\}$ of positive integers, sequences $\{A_n\}, \{B_n\}$ of topological disks with the following properties:*

(i) *$f^{s_n} : A_n \to B_n$ is a proper map whose degree is between 2 and N;*

(ii) *$\mathrm{diam}(B_n) \to 0$ as $n \to \infty$;*

(iii) *$f^{p_n}(x) \in A_n$, $f^{q_n}(x) \in B_n$ and*

$$\mathrm{mod}(\{f^{p_n}(u) : u \in A_n, (f^{p_n})'(u) = 0\} \cup \{f^{p_n+s_n}(x), f^{q_n}(x)\}, B_n) > \delta;$$

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(iv) *for any* $n \in \mathbb{N}$, *there exists* $g_n^A : A_n \to \mathbb{C}$ *and* $g_n^B : B_n \to \mathbb{C}$ *such that*

$$f^{p_n} \circ g_n^A = id_{A_n}, \quad g_n^A(f_n^p(x)) = x$$

and

$$f^{q_n} \circ g_n^B = id_{B_n}, \quad g_n^B(f^{q_n}(x)) = x;$$

(v) $s_n \to \infty$, $p_n \to \infty$, $q_n \to \infty$ *as* $n \to \infty$;

then for any f*-invariant line field* μ, $\mu(x) = 0$ *or* μ *is not almost continuous at* x.

Proof. By condition (iii) there is an annulus $\Omega_n \subset B_n$ surrounding $\{f^{p_n}(u) : u \in A_n, (f^{p_n})'(u) = 0\} \cup \{f^{p_n+s_n}(x), f^{q_n}(x)\}$ such that $\mathrm{mod}(\Omega_n) = \delta$. Let γ_n be the core curve of Ω_n and B_n' be the topological disk bounded by this curve. There is a constant $C(\delta) > 1$ such that for any $b \in \{f^{s_n}(u) : (f^{s_n})'(u) = 0\} \cup \{f^{p_n+s_n}(x), f^{q_n}(x)\}$,

$$\max_{z \in \gamma_n} d(z, b) \le C(\delta) d(b, \gamma_n).$$

Let A_n' be the component of $f^{-s_n}(B_n')$ contained in A_n. Let $U_n = g_n^A A_n'$, $V_n = g_n^B B_n'$ and

$$h_n = g_n^B \circ f^{s_n} \circ g_n^A : U_n \to V_n.$$

Then $\{h_n : U_n \to V_n\}$ is a sequence satisfying the hypothesis of Proposition 3.2. □

4. *Quasi-polynomial-like mappings*

In this section, we recall two notions: one is 'a real box mapping' which we used in [**26**]; and the other is 'a quasi-polynomial-like mapping', used by Levin and van Strien in [**12**], which turns out to be the complex counterpart of the first notion.

Fix an $f \in \mathcal{F}$. *A (real) maximal chain* is a sequence of open intervals $\{G_i\}_{i=0}^n$ such that $f(G_i) \subset G_{i+1}$ and G_i is the maximal interval with this property, for any $0 \le i \le n-1$. The *order* is the number of G_is containing a critical point. If $\{G_i\}_{i=0}^n$ is a maximal chain we shall say that G_0 is *a pull-back* of G_n. If G_i does not contain a critical point for any $0 \le i \le n-1$, then the chain is called *monotone* and G_0 is called a *monotone pull-back* of G_n. If G_i does not contain a critical point for any $0 < i < n$ but G_0 contains exactly one, we shall say that the chain is *unimodal*, and G_0 is a *unimodal pull-back* of G_n.

Recall that an open interval T is called *nice* if $f^n(\partial T) \cap T = \emptyset$ for any $n \in \mathbb{N}$. Clearly, if G and G' are two pull-backs of T then either they are disjoint or one is contained in the other. We call an interval T *properly periodic* if there is a positive integer $s \ge 1$, such that the interiors of $T, f(T), \ldots, f^{s-1}(T)$ are pairwise disjoint and $f^s(T) \subset T$, $f^s(\partial T) \subset \partial T$. In this case s is called the *period* of T and $f^s : T \to T$ will be called *a renormalization* of f. Let us say that f is *renormalizable at* c if there is a properly periodic interval containing c and that f is *infinitely renormalizable at* c if there are a sequence of properly periodic intervals $\{T_n\}$ containing c with periods $s_n \to \infty$ as $n \to \infty$. If f is not infinitely renormalizable at c, we shall say that f is *only finitely renormalizable at* c.

Let $c \in C_r(f)$ and let $c_1, c_2, \ldots, c_b$ be the critical points in $[c]$. Let $I_i \ni c_i$ be small nice intervals, $i = 1, 2, \ldots, b$. Let J_i^j, $0 \le j < r_i + 1$ be pairwise disjoint intervals contained in I_i such that $J_i^0 \ni c_i$, where $r_i \in \mathbb{N} \cup \{0, \infty\}$. A mapping

$$B : \bigcup_{i=1}^{b} \bigcup_{j=0}^{r_i} J_i^j \to \bigcup_{i=1}^{b} I_i \tag{5}$$

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is called a *(real) box mapping (induced by f) corresponding to c* if for each $1 \le i \le b$ and each $0 \le j < r_i + 1$ there is a $k(i,j) \in \{1, 2, \ldots, b\}$ and $p(i,j) \in \mathbb{N}$ with the following properties:

(B1) $B|J_i^j = f^{p_{i,j}}|J_i^j$;

(B2) there is a maximal chain $\{G_k\}_{k=0}^{p_{i,j}}$ with $G_{p_{i,j}} = I_{k_{i,j}}$ and $G_0 = J_i^j$, moreover the chain is unimodal if $j = 0$ and monotone otherwise;

(B3) $J_i^j \cap \mathrm{orb}([c]) \neq \emptyset$;

(B4) $\mathrm{orb}([c]) \cap I_i \subset \bigcup_{j=0}^{r_i} J_i^j$;

(B5) for any $x \in \mathrm{orb}([c]) \cap J_i^j$, we have $f(x), f^2(x), \ldots, f^{p_{i,j}-1}(x) \notin \bigcup_{i=1}^{b} I_i$.

Remark 4.1. In the language of [**26**], (5) is a box mapping of type II which is the first return map to its image.

The following is a natural way to construct box mappings. For any small nice interval $I \ni c$ and any $c' \in [c]$, $c' \neq c$, let $I(c')$ be the component containing c' of the domain of the first return map (of f) to I. Denote these intervals $I(c')$ by $I_2, I_3, \ldots, I_b$ and put $I_1 = I$. For any $2 \le i \le b$ let $r_i = 0$ and let $J_i^0 = I_i$. For any $x \in \mathrm{orb}([c]) \cap I$, let $J(x)$ denote the component of the domain of the first return map to I containing x. Let $r_1 + 1$ be the number of these intervals $J(x)$ and let $J_1^0 \ni c$, $J_1^1, \ldots$ be these intervals. Finally define $B_I : \bigcup_{i=1}^{b} \bigcup_{j=0}^{r_i} J_i^j \to \bigcup_{i=1}^{b} I_i$ to be the first return map of f to $\bigcup_{i=1}^{b} I_i$. We shall call B_I the box mapping *associated to* I. Note that if $\omega(c)$ is a minimal set, then r_1 is finite.

Let V_i, be topological disks containing c_i, $i = 1, 2, \ldots, b$, such that $V_{i_1} \cap V_{i_2} \cap \mathrm{orb}([c]) = \emptyset$ for any $1 \le i_1 < i_2 \le b$. For each $1 \le i \le b$, let U_i^j be topological disks contained in V_i, $0 \le j < r_i + 1$, for some $r_i \in \mathbb{N} \cup \{0, \infty\}$ such that for any $0 \le j_1 < j_2 < r_i + 1$, we have $U_i^{j_1} \cap U_i^{j_2} \cap \mathrm{orb}([c]) = \emptyset$.

A collection $\left\{F_{i,j} : U_i^j \to \bigcup_{i=1}^{b} V_i \,\middle|\, 1 \le i \le b, 0 \le j < r_i + 1\right\}$ is called *a quasi-polynomial-like mapping (induced by f) corresponding to c* if the following holds. For any $1 \le i \le b$ and any $0 \le j < r_i + 1$, we have $1 \le k_{i,j} \le b$ and $p_{i,j} \in \mathbb{N}$, such that:

(H1) $F_{i,j} = f^{p_{i,j}}|U_i^j$;

(H2) $F_{i,j}(U_i^j) = V_{k_{i,j}}$ and the map $F_{i,j}$ is a branched covering with a unique critical point c_i for $j = 0$ and is conformal otherwise;

(H3) for any $1 \le i \le b$ and $0 \le j < r_i + 1$, $\mathrm{orb}([c]) \cap U_i^j \neq \emptyset$;

(H4) for any $1 \le i \le b$, we have $\mathrm{orb}([c]) \cap V_i \subset \mathrm{orb}([c]) \cap \left(\bigcup_{j=0}^{r_i} U_i^j\right)$;

(H5) for any $1 \le i \le b$ and $0 \le j < r_i + 1$ and any $x \in \mathrm{orb}([c]) \cap U_i^j$, if $F_{i,j} = f^p$, then $f(x), f^2(x), \ldots, f^{p-1}(x) \notin \bigcup_{i=1}^{b} V_i$.

Remark. Quasi-polynomial-like mappings are not mappings in the classical sense. They naturally appear as appropriate holomorphic extensions of real box mappings. We allow intersection of these domains U_i^j in order to obtain an advantage in constructing extensions. Note that we even allow the real traces of U_i^j to intersect each other, which was excluded in [**12**].

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A quasi-polynomial-like mapping $\mathbf{F} = \{F_{i,j}\}$ determines a map

$$F : \bigcup_{i,j} \left(U_i^j - \bigcup_{(i',j') \neq (i,j)} U_{i'}^{j'} \right) \to \bigcup_i V_i$$

in a natural way. Let $K(\mathbf{F})$ be the set of points z such that $F^n(z)$ is well defined for any $n \in \mathbb{N}$. We shall call $K(\mathbf{F})$ the *Julia set* of $\{F_{i,j}\}$, which is always a measurable set. By the definition, $\mathrm{orb}([c]) \cap \left(\bigcup_i V_i \right)$ belongs to the set $K(\mathbf{F})$.

For any $x \in K(\mathbf{F})$, and any $n \geq 0$, there is a unique (i_n, j_n) such that $F^n(x) \in U_{i_n}^{j_n}$. So for any topological disk $F^n(x) \in A \subset V_{i_n}$, we can consider *the pull-back of* A *under* $\mathbf{F}$ *along the orbit* $x, F(x), \ldots, F^n(x)$, which is defined to be the topological disk P containing x, such that $F_{i_{m-1},j_{m-1}} \circ F_{i_{m-2},j_{m-2}} \circ \cdots \circ F_{i_0,j_0}(P) \subset U_{i_m}^{j_m}$ for all $0 \leq m < n$ and $F_{i_{n-1},j_{n-1}} \circ F_{i_{n-2},j_{n-2}} \circ \cdots \circ F_{i_0,j_0}(P) = A$. Note that such a topological disk P always exists. We shall write $P = \mathrm{Comp}_x(\mathbf{F}, n, A)$ and sometimes we will use $F^n : P \to A$ to denote the map $F_{i_{n-1},j_{n-1}} \circ \cdots \circ F_{i_0,j_0} | P$. Similarly, we shall use $V(\mathbf{F}, x)$ to denote the topological disk V_i containing x, and $U(\mathbf{F}, x)$ the topological disk U_i^j containing x. We shall sometimes omit the quasi-polynomial-like mapping $\mathbf{F}$ in these notations, provided that it is clear from the context which $\mathbf{F}$ is referred to.

LEMMA 4.1. *If* $0 \leq m < n$ *are two integers and* $x \in K(\mathbf{F})$, *then we have*

$$\mathrm{Comp}_x(n, V(F^n(x))) \subset \mathrm{Comp}_x(m, V(F^m(x))).$$

Given a real box mapping as in (5), we can construct a quasi-polynomial-like mapping in the following way. For any $1 \leq i \leq b$, let $V_i = \hat{\mathbb{C}}_{I_i}$. For any $0 \leq j < r_i + 1$, let U_i^j be the component of $f^{-p_{i,j}}(V_i)$ containing J_i^j, where $p_{i,j}$ is as in (B1). Let $F_{i,j} = f^{p_{i,j}} | U_i^j$. Then $\{F_{i,j}\}$ is a quasi-polynomial-like mapping. This construction can always be done but the extended map may not possess any geometric property.

5. *The measurable dynamics of* $f|J_c$ *for a recurrent critical point* c *with a non-minimal* ω*-limit set*

The purpose of this section is to consider the dynamics of $f|J_c$ for a recurrent critical point c such that $\omega(c)$ is not minimal. (J_c is defined at the end of §2.) The result is as follows.

THEOREM 5.1. *Let* $f \in \mathcal{F}$, *and* $c \in C_r(f)$ *be such that* $\omega(c)$ *is not minimal. Then*

(i) $m(J_c) = 0$ *or* $J(f) = \hat{\mathbb{C}}$ *and* f *is ergodic; and*

(ii) f *carries no invariant line field on* J_c.

We shall use the real bound developed in [**26**] to construct a quasi-polynomial-like mapping corresponding to c. Theorem 5.1 will be shown by a combinatorial argument.

The following is Proposition 6.2 in [**26**].

PROPOSITION 5.2. *For any* $\rho > 0$ *and any* $\epsilon > 0$, *there is a real box mapping* $B : \bigcup_{i=1}^b \bigcup_{j=0}^{r_i} J_i^j \to \bigcup_{i=1}^b I_i$ *corresponding to* c *such that* $\max_{i=1}^b |I_i| \leq \epsilon$; *and for any* $1 \leq i \leq b$, *there exists a symmetric interval* T_i *containing the* ρ*-neighbourhood of* I_i, *with the following properties:*

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(i) *if* $B|J_i^j = f^p$ *and* $B(J_i^j) \subset I_k$, *then the maximal chain* $\{G_n\}_{n=1}^p$ *with* $G_p = T_k$ *and* $G_1 \supset f(J_i^j)$ *is a monotone chain; and*

(ii) *if* G_0 *is the component of* $f^{-1}(G_1)$ *containing* J_i^j, *then* $J_i^j \subset T_i$; *moreover,*

(iii) *if* $J_i^j \neq I_i$, *then* $G_0 \subset I_i$.

COROLLARY 5.3. *There is a quasi-polynomial-like mapping* $\mathbf{F} = \left\{F_i^j : U_i^j \to \bigcup_{i=1}^b V_i \,\middle|\, 1 \le i \le b, 0 \le j < r_i + 1\right\}$ *corresponding to* c, *such that* $\sum_{i=1}^b r_i = \infty$ *and for some* $C > 0$ *and* $N \in \mathbb{N}$, *we have:*

(i) *for each* $1 \le i \le b$, *there is a topological disk* $W_i \supset\supset V_i$, *such that* $\mathrm{mod}(W_i - \overline{V_i}) \ge C$;

(ii) *for any* $1 \le i \le b$ *and* $0 \le j < r_i + 1$, *if* $F_{i,j} = f^{p_{i,j}}$ *and* $F_{i,j}(U_i^j) = V_k$, *then there is a topological disk* $X_i^j \supset U_i^j$ *such that* $f^{p_{i,j}} : X_i^j \to W_k$ *is a proper map of degree* $\le N$. *Moreover, if* $U_i^j \neq V_i$, *then* $X_i^j \subset V_i$.

Proof. Let $B : \bigcup_{i=1}^b \bigcup_{j=0}^{r_i} J_i^j \to \bigcup_{i=1}^b I_i$ be a real box mapping satisfying the requirements in Proposition 5.2 for a large ρ and small ϵ. Since $\omega(c)$ is not minimal, $\sum_i r_i = \infty$ if ϵ is sufficiently small. We shall extend this box mapping to a quasi-polynomial-like mapping. To fix the notation, we assume that $J_i^0 = I_i$ if and only if $b' + 1 \le i \le b$ for some $1 \le i \le b'$. By possibly interchanging the indices, we can always assume this to be so. For each $1 \le i \le b$, let $S_i \supset I_i$ be the symmetric interval such that T_i is the $\rho/2$-neighbourhood of S_i and let $W_i = D(S_i)$. For any $1 \le i \le b'$, define $V_i = D(I_i)$.

For any $b' + 1 \le i \le b$, let $s = s_i$ be the minimal positive integer such that $B^{s_i}(I_i) \subset I_{i'}$ for some $1 \le i' \le b'$. Such a positive integer always exists and is no more than $b - b'$. If $B^{s_i}|I_i = f^{k_i}|I_i$, then let V_i be the component of $f^{-k_i}(D(I_{i'}))$ containing I_i. Let $\tilde{V}_i \supset V_i$ be a component of $f^{-k_i}(D(T_{i'}))$, then $f^{k_i} : \tilde{V}_i \to D_{T_{i'}}$ is a proper map of degree uniformly bounded from above. Since $V_{i'}$ has a small diameter in the hyperbolic Riemann surface $W_{i'}$, V_i has a small diameter in the hyperbolic Riemann surface $\tilde{V}_i$. Thus V_i is compactly contained in W_i and the annulus $W_i - \overline{V_i}$ has a large modulus.

For any $1 \le i \le b$ and $0 \le j < r_i + 1$, let $m = m(i, j)$ be such that $B(J_i^j) \subset I_m$ and let U_i^j (X_i^j, Y_i^j, respectively) be the component of $f^{-p_{i,j}}(V_m)$ ($f^{-p_{i,j}}(W_m)$, $f^{-p_{i,j}}(\hat{\mathbb{C}}_{T_m})$, respectively) containing J_i^j, where $p_{i,j} \in \mathbb{N}$ is such that $B|J_i^j = f^{p_{i,j}}|J_i^j$. Then $f^{p_{i,j}} : Y_i^j \to \hat{\mathbb{C}}_{T_m}$ is a proper map of a degree uniformly bounded from above. Since W_m has a small diameter in the hyperbolic Riemann surface $\hat{\mathbb{C}}_{T_m}$, by Lemma 2.1, X_i^j has a small diameter in the hyperbolic Riemann surface Y_i^j. Thus $X_i^j \subset V_i$ if $1 \le i \le b'$, since in that case, the real trace of Y_i^j is contained in I_i.

Define $F_{i,j}|U_i^j = f^{p_{i,j}}|U_i^j$. Then $\{F_{i,j}\}$ becomes a quasi-polynomial-like mapping as desired. □

PROPOSITION 5.4. *Assume that* $\omega(c)$ *is not minimal. Then for some* $N \in \mathbb{N}$, *and for any* $x \in J_c$, *there are some* $d \in [c]$ *and some neighbourhoods* $U \subset\subset U'$ *of* d *with the following property: for infinitely many positive integers* n, *there is a neighbourhood* P *of* x *such that* $f^n : P \to U'$ *is a proper map of degree* $\le N$, *and* $f^n(x) \in U$.

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Proof. Let $\mathbf{F} = \left\{F_{i,j} : U_i^j \to \bigcup_{i=1}^b V_i \,\middle|\, 1 \le i \le b, 0 \le j < r_i + 1\right\}$ be the quasi-polynomial-like mapping as in the previous proposition. The real trace of $\mathbf{F}$ is a real box mapping. Let $p_{i,j}, k_{i,j} \in \mathbb{N}$ be such that $F_{i,j} = f^{p_{i,j}}|U_i^j$, $F_{i,j}(U_i^j) \subset V_{k_{i,j}}$. We may assume that $r_1 = \infty$ and that for some $1 \le b' \le b$, $U_i^0 = V_i$ if and only if $b' + 1 \le i \le b$. Let F be the map determined by $\mathbf{F}$ as described at the top of page 967.

For any $1 \le i \le b$, let t_i be the minimal non-negative number such that

$$F^{t_i}(c_i) \subset \bigcup_{i=1}^{b'} V_i,$$

and let s_i be the non-negative integer such that $F^{t_i} = f^{s_i}$ on a neighbourhood of c_i. (In particular, $t_i = s_i = 0$ for $1 \le i \le b'$.)

For any $0 \le h < \infty$, and any $c' \in [c]$, let $a_h(c')$ be the minimal non-negative integer such that $f^{a_h(c')}(c') \in U_1^h$ and let $P_h(c')$ be the component of $f^{-a_h(c')}(U_1^h)$ containing c'. Then $f^{a_h(c')} : P_h(c') \to U_1^h$ is a proper map of a uniformly bounded degree. Write $q_h = p_{1,h} + s_{k_{1,h}}$. Then for some $1 \le i_h \le b'$, $f^{a_h(c')+q_h} : P_h(c') \to V_{i_h}$ is also a proper map of a uniformly bounded degree. For each $c' \in [c]$, if $f^{a_h(c')+q_h}(c') \in U_{i_h}^{j_h}$, then let $Q_h(c')$ be the component of $f^{-(a_h(c')+q_h)}(U_{i_h}^{j_h})$ which contains c'.

LEMMA 5.1. *For any $x \in J_c$ and any large h, if n_h is the minimal non-negative integer such that $f^{n_h}(x) \in \bigcup_{d\in[c]} Q_h(d)$, and if $c' = c'_h$ is the critical point such that $f^{n_h}(x) \in Q_h(c'_h)$, and if $A_h(x)$ is the component of $f^{-n_h}(P_h(c'))$ containing x, then the proper map $f^{n_h} : A_h(x) \to P_h(c')$ has a degree uniformly bounded from above.*

Proof. Let $E_k^h = f^k(A_h(x))$ for $0 \le k \le n_h$. We need to count the number of times a critical point appears in some E_k^h. Let $T_k^h = E_k^h \cap S^1$. Then $\{T_k^h\}_{k=0}^{n_h}$ is a (real) maximal chain. (We admit the situation that $T_k^h = \emptyset$ for some k.) Since all critical points of f are contained in S^1, it suffices to show that the maximal chain $\{T_k^h\}_{k=0}^{n_h}$ has a uniformly bounded order. Note that for h sufficiently large $\max_{k=0}^{n_h-1} |T_k^h|$ is small since $f|S^1$ has no wandering interval. So each E_k^h contains, at most, one critical point. For the same reason, any $d \in C(f) - C_r(f)$ appears in, at most, one of the domains E_k^h.

Assume that the maximal chain has order larger than $2\#C(f)$. Then there exists some $d \in C_r(f)$ such that $d \in T_{k_1}^h, T_{k_2}^h$ for some $0 \le k_1 < k_2 \le n_h$ and the maximal chain $\{T_k^h\}_{k=k_1}^{n_h}$ has order $\le 2\#C(f)$. Since $f^{n_h-k_1} : E_{k_1}^h \to P_h(c')$ has a degree bounded from above and $Q_h(c')$ has a small diameter in the hyperbolic Riemann surface $P_h(c')$, and since $|T_{k_1}^h|$ has a small diameter, $f^{k_1}(x)$ is close to d. If such a d appears for infinitely many hs, then it is contained in $\omega(x)$. Since $x \in J_c$, we must have $d \in [c]$. So there are positive integers $m_1 > m_2$ such that $f^{n_h-k_1} = F^{m_1}$, $f^{n_h-k_2} = F^{m_2}$ holds in a neighbourhood of d. It is then easy to see that $E_{k_2}^h \subset P_h(d)$ and $E_{k_1}^h \subset Q_h(d)$. Since $f^{k_1}(x) \in E_{k_1}^h$, this contradicts the minimality of n_h. Thus the maximal chain $\{T_k^h\}_{k=0}^{n_h}$ has order $\le 2\#C(f)$. The proof of the lemma is then completed. □

We continue the proof of Proposition 5.4. By now, we have, for each h large, a proper map $f^{n_h+a_h(c'_h)+q_h} : A_h(x) \to V_{i_h}$ of a uniformly bounded degree such that $f^{n_h+a_j(c'_h)+q_h}(x)$ is contained in $U_{i_h}^{j_h}$ for some $1 \le i_h \le b'$ and $0 \le j_h < r_{i_h} + 1$. By the previous proposition, for each h, there is a topological disk $X_{i_h}^{j_h} \subset V_{i_h}$ such that $f^{p_{i_h,j_h}} : X_{i_h}^{j_h} \to W_{k_{i_h,j_h}}$ is a proper map of a uniformly bounded degree. The proposition follows. □

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Proof of Theorem 5.1. Let $X \subset J_c$ be a measurable set of positive measure such that $f(X) \subset X \pmod 0$. Let $x \in X$ be a Lebesgue density point. Let $d \in [c]$, and $U \subset\subset U'$ be neighbourhoods of d as in the previous proposition. There exist $n_1 < n_2 < \cdots$ neighbourhoods A_j of x, such that $f^{n_j} : A_j \to U'$ has a uniformly bounded degree and $f^{n_j}(x) \in U$. Hence by Lemma 2.2 and 2.3, $m(U - X) = 0$. Since $f^n(U) = \hat{\mathbb{C}} \pmod 0$ for some $n \in \mathbb{N}$, we have that $X = \hat{\mathbb{C}} \pmod 0$. Therefore $J(f) = \hat{\mathbb{C}}$ and f is ergodic.

If f carries an invariant line field μ on J_c, let x be an almost continuous point of μ. One can construct a sequence of proper mappings $f^{n_j} : A_j \to U'$ as in the previous proof, which then implies that μ is holomorphic on a non-empty open set. Thus f is doubly covered by an integral torus endomorphism. In particular, f has no recurrent critical point. Contradiction! □

Reduction. From now on, we shall assume that for any $c \in C_r(f)$ with $\omega(c)$ non-minimal, the set J_c has measure zero. It follows from a similar argument as in the proof of Proposition 2.2 that for a.e. $z \in J(f)$, $\omega(z) \subset \mathrm{Par}(f) \cup \left(\bigcup_{c \in C_r(f), \omega(c) \text{ minimal}} \omega(c)\right)$. The last set is a disjoint union of finitely many minimal sets. Since the set $\{z \in J(f) : \omega(z) \subset \mathrm{Par}(f)\}$ is countable, we may assume that:

(1) $J(f) = \bigcup_{c \in C_r(f), \omega(c) \text{ minimal}} J_c \pmod 0$; and

(2) $J_c = \{z \in J(f) : \omega(z) = \omega(c)\} \pmod 0$ for any $c \in C_r(f)$ such that $\omega(c)$ is minimal.

6. *Large real bound implies the non-existence of an invariant line field*

We begin the study of the measurable dynamics of $f|J_c$ in the case that $\omega(c) \ni c$ is a minimal set. We shall assume the following throughout this section.

Assumption. There exists a sequence of symmetric nice intervals $I(n) \ni c$, $n = 1, 2, \ldots$, such that $|I(n)| \to 0$ and $d(\omega(c) \cap I(n), \omega(c) - I_n)/\mathrm{diam}(\omega(c) \cap I(n)) \to \infty$ as $n \to \infty$.

Let $b = \#[c]$, and $c_1 = c, c_2, \ldots, c_b$ be the critical points contained in $[c]$.

PROPOSITION 6.1. *There is a constant ρ_0 such that for any $\rho \geq \rho_0$ the following holds.*

Let $I \ni c$ be a symmetric nice interval interval. Suppose that $|I|$ is very small and satisfies $d(\omega(c) \cap I, \omega(c) - I) \geq \rho\, \mathrm{diam}(\omega(c) \cap I)$, let $B_I : \left(\bigcup_{j=0}^{r} J^j\right) \cup \left(\bigcup_{i=2}^{b} I_i\right) \to \bigcup_{i=1}^{b} I_i$ be the real box mapping associated to I. Then there is a quasi-polynomial-like mapping $\mathbf{F} = \left\{F : U_i^j \to \bigcup_{i=1}^{b} V_i \;\middle|\; 1 \leq i \leq b, 0 \leq j < r_i + 1\right\}$ corresponding to c such that:

(i) *$r_1 = r$, $r_2 = r_3 = \cdots = r_b = 0$ and $U_i^0 = V_i$ for any $2 \leq i \leq b$;*

(ii) *$V_i \cap \omega(c) = I_i \cap \omega(c)$, $U_1^j \cap \omega(c) = J_1^j \cap \omega(c)$, for any $1 \leq i \leq b$ and any $0 \leq j \leq r$;*

(iii) *$F|(V_i \cap I_i) = B_I|(V_i \cap I_i)$, $F|(U^j \cap J^j) = B_I|(U^j \cap J^j)$, for any $2 \leq i \leq b$ and any $0 \leq j \leq r$;*

(iv) *there is topological disk D containing V_1 such that $D \cap \omega(c) = V_1 \cap \omega(c)$, and*

$$\mathrm{mod}(D - \overline{V_1}) \geq m(\rho) > 0,$$

where $m(\rho) \to \infty$ as $\rho \to \infty$.

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Proof. We first assume that $r \neq 0$.

Since $d(\omega(c) \cap I, \omega(c) - I)/\mathrm{diam}(\omega(c) \cap I)$ is very large, there is a round annulus X in $\hat{\mathbb{C}} - \omega(c)$ centred at c separating $\omega(c) \cap I$ from $\omega(c) - I$ with a large modulus. Take the core curve γ of X. It is a round circle centred at c. Let E_1 denote the topological disk bounded by γ. Let k_j $(0 \leq j \leq r)$, m_i $(2 \leq i \leq b)$ be the positive integers such that $B_I|I_i = f^{m_i}$ and $B_I|J^j = f^{k_j}$. Let $m_i'(2 \leq i \leq b)$ and $k_j'(0 \leq j \leq r)$ be the first return time of I_i, J^j to I, i.e. the minimal positive integers such that $f^{m_i'}(I_i) \subset I$ and $f^{k_j'}(J^j) \subset I$. Let E_i be the component of $f^{-m_i'}(E_1)$ containing c_i, $2 \leq i \leq b$, where c_i is the critical point of f which is contained in I_i. For $0 \leq j \leq r$, let D^j be the component of $f^{-k_j'}(E_1)$ containing $\omega(c) \cap J^j$. It is easy to see that if $B_I(I_i) \subset I_{i'}$, then $f^{m_i}(E_i) = E_{i'}$ and that if $B_I(J^j) \subset I_{i'}$, then $f^{k_j}(D^j) = E_{i'}$.

We claim that $D^j \subset E_1$ for any $0 \leq j \leq r$.

Indeed, since $X - \overline{E_1}$ is an annulus with a large modulus $(= \mathrm{mod}(X)/2)$ which is disjoint from $\omega(c)$, it is easy to show by Koebe's distortion theorem that any $\Omega \in \{E_i, D^j : 1 \leq i \leq b, 0 \leq j \leq r\}$ is an 'almost round' disk as seen from any $x \in \Omega \cap \omega(c)$, that is

$$\max_{z \in \partial\Omega} d(z, x) \leq (1 + \epsilon) d(x, \partial\Omega),$$

where $\epsilon > 0$ can be taken close to 0 if ρ is large. Since $r \geq 1$, we have $\mathrm{diam}(D^j) \leq 2(1+\epsilon)\,\mathrm{diam}(\omega(c) \cap I)$. Since $E_1 \cap S^1$ is an interval containing a large neighbourhood of $\omega(c) \cap I$, it follows that $D^j \subset E_1$.

$\left\{F_{1,j} : U^j \to \bigcup_i V_i, F_k : V_k \to \bigcup_i V_i \,\middle|\, 0 \leq j \leq r, 2 \leq k \leq b\right\}$ is a quasi-polynomial-like mapping satisfying the required conditions.

We now turn to the case that $r = 0$.

In this case there is a permutation $\sigma : \{1, 2, \ldots, b\} \to \{1, 2, \ldots, b\}$ such that $B_I(I_i) \subset I_{\sigma(i)}$, where B_I is the real box mapping associated to I, for otherwise the forward orbit of c will not enter some of the intervals I_i, which contradicts the hypothesis that $\omega(c)$ contains all critical points in $\bigcup_{i=1}^b I_i$. By possibly changing the subscript i, we may assume that $\sigma(i) = i + 1$ for any $1 \leq i \leq b - 1$ and $\sigma(b) = 1$. Let k_i be the positive integer such that $B_I|I_i = f^{k_i}$. Let $m = k_1 + k_2 + \cdots + k_b$.

Construct X, E_1 as in the case $r \neq 0$. Let D^0 be the pre-image of $f^{-m}(E_1)$ containing c.

$G = f^m : D^0 \to E_1$ is a proper map. For any $u \in D_0$ such that $G'(u) = 0$, $G^n(u)$ never escapes the domain D^0. Write $P = \overline{\{G^n(u) : G'(u) = 0, n \in \mathbb{N}\}}$, then $\mathrm{mod}(P, E_1) \geq \mathrm{mod}(\omega(c) \cap I, E_1) \geq \mathrm{mod}(X)/2$ is large. By Theorem 5.12 in [**20**], there is a domain $U' \subset D^0$ such that $f^m : U' \to V' = f^m(U')$ is a polynomial-like map with the same degree as G and $\mathrm{mod}(V' - \overline{U'})$ is large since f has no attracting cycle in D_0. Define $U^0 = U'$, $V_1 = V'$ and $V_s = f^{k_1 + k_2 + \cdots + k_{s-1}}(U^0)$. Then $\{f^{k_1} : U^0 \to V_1, f^{k_i} : V_i \to V_{i+1}, f^{k_b} : V_b \to V_1 \mid 2 \leq i \leq b - 1\}$ is a quasi-polynomial-like mapping as required. □

For $\delta > 0$ let $\mathcal{E}^\delta$ be the collection of quasi-polynomial-like mappings $\left\{F_{i,j} : U_i^j \to \bigcup_{i=1}^b V_i \,\middle|\, 1 \leq i \leq b, 0 \leq j \leq r_i\right\}$ corresponding to c induced by f such that $r_i = 0$, $U_i^0 = V_i$ for all $2 \leq i \leq b$ and such that there is a topological disk D containing V_1 such that $D \cap \omega(c) = V_1 \cap \omega(c)$ and $\mathrm{mod}(D - \overline{V_1}) \geq \delta$.

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For each quasi-polynomial-like mapping $\mathbf{F}$ in $\mathcal{E}^\delta$, and each $2 \le i_0 \le b$, we can associate to it another quasi-polynomial-like mapping $\tilde{\mathbf{F}}$ in the following way. Let $c_1 \in U^0$, $c_i \in V_i$ be the critical points of f. Denote $\tilde{V}_1 = V_{i_0}$. For each $1 \le i' \le b$, $i' \ne i_0$, let $k(i')$ be the minimal positive integer such that $F^{k(i')}(c_{i'}) \in \tilde{V}_1$ and let $\tilde{V}_{i'} = \mathrm{Comp}_{c_{i'}}(\mathbf{F}, k(i'), \tilde{V}_1)$. For each $x \in \omega(c) \cap \tilde{V}_1$, let $k(x)$ be the minimal positive integer such that $F^{k(x)}(x) \in \tilde{V}_i$ for some $1 \le i \le b$, and let $\tilde{U}(x) = \mathrm{Comp}_x(\mathbf{F}, k(x), \tilde{V}_i)$. Then define

$$\tilde{\mathbf{F}} = \left\{ F^{k(x)} : \tilde{U}(x)) \to \bigcup_{i=1}^{b} \tilde{V}_i,\ F^{k(i')} : \tilde{V}_{i'} \to \bigcup_{i=1}^{b} \tilde{V}_i \;\middle|\; x \in \omega(c) \cap \tilde{V}_1, 1 \le i' \le b, i' \ne i \right\}.$$

Note that there is a constant $\tilde{\delta} > 0$ depending only on δ (and f) such that $\tilde{F} \in \mathcal{E}^{\tilde{\delta}}$. Moreover, we can take $\tilde{\delta}$ arbitrarily large if δ is sufficiently large.

THEOREM 6.2. *If there is a sequence of symmetric nice intervals $I_n \ni c$ such that*

(1) $|I_n| \to 0$ *as* $n \to \infty$*; and*

(2) $d(\omega(c) \cap I_n, \omega(c) - I_n)/\mathrm{diam}(\omega(c) \cap I_n) \to \infty$ *as* $n \to \infty$*, then*

  (i) *if* $m(J_c) > 0$*, then* $f|J_c$ *is ergodic; and*

  (ii) J_c *carries no* f*-invariant line field.*

Proof. By Proposition 6.1, there is a sequence of quasi-polynomial-like mappings

$$F(n) = \left\{ F(n)_{1,j} : U^j(n) \to \bigcup_{i=1}^{b} V_i(n);\ F(n)_{k,0} : V_k(n) \to \bigcup_{i=1}^{b} V_i(n) \;\middle|\; 0 \le j \le r(n), 1 \le k \le b \right\}$$

corresponding to c (induced by f) which are contained in the class $\mathcal{E}^{\delta_n}$ for $\delta_n \to \infty$ as $n \to \infty$ such that $\mathrm{diam}(V_1(n)) \to 0$ as $n \to \infty$. Let $W(n) \supset V_1(n)$ be a topological disk such that $W(n) \cap \omega(c) = V_1(n) \cap \omega(c)$ and $\mathrm{mod}(W(n) - \overline{V_1(n)}) \ge \delta_n$.

(i) Let $E \subset J_c$ be a measurable set of positive measure with $f(E) \subset E \pmod 0$ and let x be a Lebesgue density point of E such that $\omega(x) = \omega(c)$ and $f^n(x) \notin C(f)$ for any $n \ge 0$.

For any $n \in \mathbb{N}$, let $k(n)$ be the minimal non-negative integer such that $f^{k(n)}(x) \in U^0(n) \cup \left(\bigcup_{i=2}^{n} V_i(n)\right)$ and let $1 \le i(n) \le b$ be such that $f^{k(n)} \in V_{i(n)}(n)$. Then $k(n) \to \infty$ as $n \to \infty$ since $\max_{i=1}^{b} \mathrm{diam}(V_i(n)) \to 0$. Let $A(n)$ be the component of $f^{-k(n)}(V_{i(n)}(n))$ containing x. Let $p(n)$ be the positive integer such that $f^{p(n)}(V_{i(n)}(n)) = V_1(n)$ if $i(n) \ne 1$ and let $p(n) = 0$ otherwise. Let $X(n)$ be the component of $f^{-p(n)}(W(n))$ containing $V_{i(n)}(n)$ and let $A'(n)$ be the component of $f^{-k(n)}(X(n))$ containing $A(n)$. Then $f^{p(n)} : X(n) \to W(n)$ is a proper map of a uniformly bounded degree, and $X(n) \cap \omega(c) = V_{i(n)}(n) \cap \omega(c)$. We claim that for n sufficiently large, $f^{k_n} : A'(n) \to X(n)$ is also a conformal map.

Assume that the claim fails. Let $k < k(n)$ be the maximal number such that $f^k(A'(n))$ contains a critical point d of f. First let us show that $d \in \omega(c)$. Since $f(d) \in f^{k+1}(A'(n))$, $f^{k+1}(A'(n)) \cap S^1 \ne \emptyset$. Since $f^{k(n)-k-1}|f^{k+1}(A'(n))$ is a conformal map, $f^{k(n)-k-1} : f^{k+1}(A'(n)) \cap S^1 \to X(n) \cap S^1$ is a diffeomorphism. In particular, $f^{k+1}(A(n)) \cap S^1 \ne \emptyset$.

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Since $\operatorname{diam}(f^{k+1}(A(n)))$ and $|f^{k+1}(A'(n)) \cap S^1|$ are small, we have that $f^{k+1}(x)$ is close to $f(d)$, and hence $f^k(x)$ is close to d. So we have $d \in \omega(x) = \omega(c)$ if n is large. Since $X(n) \cap \omega(c) = V_{i(n)}(n) \cap \omega(c)$, $d \in f^k(A(n))$. Since $f^{k(n)-k}(d) \in \bigcup_{i=1}^{b} V_i(n)$, there is some positive integer t such that $F(n)^t$ is well defined in a neighbourhood of d and coincides with $f^{k(n)-k}$. Then it is easy to see that $f^k(A(n)) \subset U^0(n) \cup \bigcup_{i=2}^{b} V_i(n)$. Since $f^k(x) \in f^k(A(n))$, this contradicts the minimality of $k(n)$.

Thus $f^{k(n)+p(n)} : A'(n) \to W(n)$ is a proper map of a uniformly bounded degree. Since x is a Lebesgue density point and $f^{k(n)+p(n)}(x) \in V(n)$, it follows that $m(E \cap V(n))/m(V(n)) \to 1$ as $n \to \infty$.

If $m(J_c) > 0$ and $f|J_c$ is not ergodic, then there is a measurable set $E \subset J_c$ such that $f^{-1}(E) = E \pmod 0$ and $m(E) > 0$, $m(J_c - E) > 0$. Then it follows that $m(E \cap V(n))/m(V_n) \to 1$ and $m((J_c - E) \cap V(n))/m(V(n)) \to 1$ as $n \to \infty$, which is obviously absurd.

(ii) Assume that f carries an invariant line field μ with support $E \subset J_c$. We need to show a contradiction.

Let $x \in E$ be an almost continuous point of μ such that $\mu(x) \neq 0$ and $f^n(x) \notin C(f)$ for any $n \geq 0$. By the same argument as in the previous proof, for each n large, there are $k(n) \in \mathbb{N}$, $1 \leq i(n) \leq b$ such that $f^{k(n)}(x) \in V_{i(n)}(n)$ and there is a univalent branch $h(n)$ of $f^{-k(n)}$, defined on $V_{i(n)}(n)$, mapping $f^{k(n)}(x)$ to x.

Case 1. There are infinitely many n such that $i(n) = 1$.

By passing to a subsequence, we may assume that $i(n) = 1$ for all $n \in \mathbb{N}$. Let $s(n)$ be the first return time of $U^0(n)$ to $V_1(n)$. Then $f^{s_n} : U^0(n) \to V_1(n)$ is a proper map whose degree is no less than 2 and bounded uniformly from above.

Since $f^{k(n)}(x) \in U^0(n) \subset V_1(n)$, it follows from Corollary 3.3 that this is impossible.

Case 2. For $n \gg 1$, $i(n) > 1$.

Let

$$\tilde{F}(n) : \left(\bigcup_{j=0}^{\tilde{r}(n)} \tilde{U}^j(n)\right) \cup \left(\bigcup_{i=2}^{b} \tilde{V}_i(n)\right) \to \bigcup_{i=1}^{b} \tilde{V}_i(n)$$

be the holomorphic box mapping associated to $F(n)$ and $i(n)$ as in the remark before Corollary 7.3. Then $\tilde{F}(n)$ belongs to the class $\mathcal{E}^{\tilde{\delta}_n}$ with $\tilde{\delta}_n \to \infty$. Let $\tilde{k}(n)$ be the minimal non-negative integer such that

$$f^{\tilde{k}(n)}(x) \in \tilde{U}^0(n) \cup \left(\bigcup_{i=2}^{b} \tilde{V}_i(n)\right)$$

and let $1 \leq \tilde{i}(n) \leq b$ be such that $f^{\tilde{k}(n)}(x) \in \tilde{V}_{\tilde{i}(n)}(n)$.

If there are infinitely many n such that $\tilde{i}(n) = 1$, then we return to case 1 and the proof is completed. So let us assume that for $n \gg 1$, $\tilde{i}(n) \neq 1$. So there is a univalent branch of $\tilde{h}(n) : \tilde{V}_{\tilde{i}(n)}(n) \to \hat{\mathbb{C}}$ of $f^{-\tilde{k}(n)}$ such that $\tilde{h}(n)(f^{\tilde{k}(n)}(x)) = x$. Let $\tilde{s}(n)$ be the first return time of $\tilde{V}_{\tilde{i}(n)}(n)$ to $\tilde{V}_1(n)$. Then $f^{\tilde{s}(n)} : \tilde{V}_{\tilde{i}(n)}(n) \to \tilde{V}_1(n)$ is a proper map whose degree is no less than 2 and bounded from above uniformly. A contradiction again follows from Corollary 3.3. □

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7. *Bounded geometry implies the non-existence of an invariant line field on the Julia set* We fix an $f \in \mathcal{F}$ and $c \in C_r(f)$ with $\omega(c)$ minimal. Assume that there is a constant $M > 0$ such that for any symmetric nice interval I containing c, we have

$$\frac{d(\omega(c) \cap I, \omega(c) - I)}{\operatorname{diam}(\omega(c) \cap I)} \leq M. \tag{6}$$

By taking M larger, we may assume that the inequality (6) holds for any small nice interval intersecting $\omega(c)$.

To conclude the proof of Theorem 1 and 2, it suffices to prove the following theorem.

THEOREM 7.1. *In the previous situation, we have:*
(1) $m(J_c) = 0$ *or* $f|J_c$ *is ergodic;*
(2) f *carries no invariant line field on* J_c.

Since f has a recurrent critical point, it has infinitely many periodic points on the extended real axis S^1. Using conjugacy by real Möbius transformation, we may assume that ∞ is a periodic point of f.

Take an arbitrarily small symmetric interval $T \ni c$ and let $B_T : \left(\bigcup_{j=0}^{r} S^j\right) \cup \left(\bigcup_{i=2}^{b} T_i\right) \to \bigcup_{i=1}^{b} T_i$ be the real box mapping associated to T. As described in §4, this real box mapping can be extended to a quasi-polynomial-like mapping $\mathbf{F} = \left\{F_{1,j} : U_i^j \to \bigcup_{k=1}^{b} V_k \,\middle|\, 1 \leq i \leq b, 0 \leq j \leq r_i\right\}$ with $V_i = \hat{\mathbb{C}}_{T_i}$ and $r_1 = r, r_2 = r_3 = \cdots = r_b = 0$. We shall fix $\mathbf{F}$ from now on. Let F be the map determined by $\mathbf{F}$ as described at the top of page 967.

From now on, all metrics are the Euclidean metric in $\mathbb{C}$ unless otherwise stated. All intervals are assumed to be contained in $\bigcup_{i=1}^{b} T_i$. For any two disjoint open intervals A_1, A_2, we shall use (A_1, A_2) to denote the maximal bounded open interval which is disjoint from $A_1 \cup A_2$, and similarly use $(A_1, A_2]$ to denote the minimal open interval which contains $(A_1, A_2) \cup A_2$.

For any $\delta > 0$, let $\mathcal{I}_\delta$ be the collection of intervals I satisfying the following conditions: I is a symmetric nice interval I containing a critical point in $\omega(c)$ such that the δ-neighbourhood of I is disjoint from $\omega(c) - I$; moreover, if f is only finitely renormalizable at c, I is a δ-nice interval and if f is infinitely renormalizable at c, I is a properly periodic interval. By Proposition 3.1 and Corollary 4.7 in [**26**], for some $\delta_0 > 0$, $\mathcal{I}_{\delta_0}$ contains an arbitrarily small interval containing c. (By saying that I is a δ-nice interval, we mean that for any $x \in \omega(c) \cap I$, if J is the component of the domain of the first return map to I, then I contains the δ-neighbourhood of J.)

Let I be a small interval in $\mathcal{I}_{\delta_0}$ and let $J_i, i = 1, \ldots, n$ be the components of the domain of the first return map to I which (are contained in $\bigcup_k T_k$ and) intersect $\omega(c) - I$ and let $J_0 = I$.

LEMMA 7.1. *There is a constant* $\delta_1 > 0$ *such that for any* $0 \leq i \leq n$,

$$\frac{d(J_i, \omega(c) - J_i)}{|J_i|} \geq \delta_1.$$

Proof. For any $i \neq 0$, let $r = r_i$ be the minimal positive integer such that $f^r(J_i) \subset I$. Let I' denote the δ_0-neighbourhood of I and let $D = D(I')$. Let U be the component of

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$f^{-r}(D)$ which contains J_i. Then $f^k : U \to D$ is a proper map with uniformly bounded degree. Obviously, $U \cap \mathbb{R}$ contains a definite neighbourhood of J_i, and is disjoint from $\omega(c) - J_i$. □

For any $0 \le i \le n$, let $1 \le k_i \le b$ be such that $J_i \subset T_{k_i}$.

LEMMA 7.2. *There exists $0 \le i, j \le n$ with $i \ne j$, $k_i = k_j$, such that there is no point in $\omega(c)$ between J_i and J_j, and such that*

$$d(J_i, J_j) \le C_1 \min(|J_i|, |J_j|), \quad C_1^{-1}|J_j| < |J_i| < C_1|J_j|,$$

where C_1 is a constant depending only on δ_1 and f.

Proof. Take $0 \le i \le n$ be such that

$$|J_i| = \min_{m=0}^{n} |J_m|.$$

First assume that both components of $T_{k_i} - J_i$ contain an interval of the form J_j. Let $j_1 \ne j_2$ be such that $J_{j_1}, J_{j_2} \subset T_{k_i}$ and such that (J_i, J_{j_1}) and (J_i, J_{j_2}) are disjoint from $\bigcup_m J_m$. It cannot happen that both $(J_i, J_{j_1}]$ and $(J_i, J_{j_2}]$ are much larger than $|J_i|$, for otherwise, a large neighbourhood of J_i is disjoint from $\omega(c) - J_i$, which contradicts assumption (6). Thus the lemma holds. Now let us consider the case that only a component, say L, of $T_{k_i} - J_i$ intersects $\bigcup_m J_m$. Let j be such that $J_j \subset L$ is closest to J_i. When I is small, J_i is also small, and hence the other component of $T_{k_i} - J_i$ is much larger than J_i. Similarly as before, we must have $(J_i, J_j]$ is not too large compared to J_i. So the lemma holds in this case as well. □

We shall say that a topological disk P is *admissible* if ∂P is a smooth curve disjoint from $\omega(c)$, and if one of the following holds:

(A1) $P \cap \omega(c) = \emptyset$; or

(A2) there exists a nice interval $A \subset P$ such that $P \cap \omega(c) = A \cap \omega(c)$.

Let $C > 1$ be a constant and let P be a admissible topological disk. We say that P is *C-bounded* if the following hold:

(D1) $l(\partial P)^2 \le C$ area (P);

(D2) for $x \in \omega(c) \cap P_i$, and each $z \in P_i$, we have

$$d(x, z) \le C d(x, \partial P_i);$$

(D3) $\mathrm{mod}(\omega(c) \cap P, P) \ge C^{-1}$;

(D4) there is a topological disk $Q \supset \overline{P}$, such that $(Q - P) \cap \omega(c) = \emptyset$, and

$$\mathrm{mod}(Q - \overline{P}) \ge C^{-1}.$$

We shall often consider a couple (P_1, P_2) of admissible topological disks. We shall say that the couple (P_1, P_2) is C-bounded if the following hold:

(T1) $P_1 \cap P_2 \cap \omega(c) = \emptyset$;

(T2) both of P_1 and P_2 are C-bounded;

(T3) $C^{-1} \operatorname{diam}(P_2) \le \operatorname{diam}(P_1) \le C \operatorname{diam}(P_2)$; and

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(T4) there is a piecewise smooth curve γ which joins P_1 and P_2, such that

$$l(\gamma) \leq C \operatorname{diam}(P_1),$$

and for each $p \in \gamma$, we have

$$B(p, C^{-1} \operatorname{diam}(P_1)) \cap \omega(c) = \emptyset.$$

For instance, for any $0 \leq i \leq n$, the Poincaré disk $D((1 + \delta_1/2)J_i)$ is a C-bounded admissible disk for some $C > 1$. For i, j as in Lemma 7.2 $(D((1 + \delta_1/2)J_i), D((1 + \delta_1/2)J_j))$ is C-bounded for some $C > 1$.

LEMMA 7.3. *Let P be a C-bounded admissible topological disk, and let p be a positive integer. Let P' be a pull-back of P under $\mathbf{F}$ and assume that the corresponding map $F^n : P' \to P$ has degree at most p. Then P' is a C'-bounded admissible topological disk, where $C' = C'(C, p)$ is a constant.*

Proof. First of all, it is easy to check that P' is admissible.

Let Q be a topological disk with $Q \cap \omega(c) = P \cap \omega(c)$ such that $\operatorname{mod}(Q - P) \geq 1/C$. Let $Q' \supset P'$ be the corresponding pull-back of Q. Then $F^n : Q' \to Q$ obviously has the same degree as $F^n | P'$. The lemma follows easily. □

Now let i, j be as in Lemma 7.2. Let p_i be minimal non-negative integers such that $F^{p_i}(J_i) \subset I$, and let p_j be similarly defined. Let

$$E_1 = \operatorname{Comp}_{J_i}(p_i, D((1 + \delta_1/2)I)),$$
$$E_2 = \operatorname{Comp}_{J_j}(p_j, D((1 + \delta_1/2)I)).$$

We can easily show that (E_1, E_2) is a C_2-bounded couple of admissible topological disks. Indeed, we can take the interval (J_i, J_j) to be the curve joining E_1 and E_2. Moreover, $E_1 \cup E_2$ is contained in a C_3-bounded topological disk D with $D \cap \omega(c) \subset E_1 \cup E_2$.

Let k be the minimal non-negative integer such that $F^k(d) \in E_1 \cup E_2$ for some critical point $d \in \omega(c)$. To fix the notation, we assume that $F^k(d) \in E_1$.

Let $D' = \operatorname{Comp}_d(k, D)$, and $\Omega_1 = \operatorname{Comp}_d(k, E_1)$. Then Ω_1 is a C_3-bounded topological disk by Lemma 7.3. Write $(a_k, b_k) = (J_i, J_j)$ such that $a_k \in E_1$. Let $a \in \Omega_1$ be a point such that $F^k(a) = a_k$. Let $\gamma' \subset D'$ be an arbitrary lift with initial point a of the curve $\gamma = (a_k, b_k)$ under F^k, and let b be the endpoint of γ'. Let Ω_2 be the topological disk in D' such that $b \in \Omega_2$ and such that $F^k(\Omega_2) = E_2$.

Let K be the real pull-back of J_i along the orbit $\{d, F(d), \ldots, F^k(d)\}$. This is a symmetric nice interval contained in Ω_1. Let $0 \leq p'_j \leq p_j$ be minimal such that $L = F^{p'_j}(J_j)$ contains a critical point. Note that L is also a symmetric nice interval and is of the form $J_{j'}$ for some $0 \leq j' \leq n$. Let $\Omega'_2 = F^{p'_j + k}(\Omega_2) = F^{p'_j}(E_2)$. Note that for some $\delta_2 > 0$, $K, L \in \mathcal{I}_{\delta_2}$.

Note that $F^k | D'$ has a bounded distortion outside a small neighbourhood of $\omega(c) \cap \Omega_1$.

LEMMA 7.4. *The couple (Ω_1, Ω_2) of admissible topological disks is C_4-bounded for some $C_4 > 1$.*

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Let K and L be the symmetric nice intervals constructed as before. In the case that f is only finitely renormalizable at c, since both of K and L are in $\mathcal{I}_{\delta_2}$, by Theorem 1.4 in [**26**] and by assumption (6), each of K and L has, at most, $N = N(\delta)$ *children*. (If $\{G_i\}_{i=0}^{p}$ is a unimodal chain such that both G_p and G_0 are symmetric nice intervals containing a critical point in $\omega(c)$, then we say that G_0 is a child of G_p.) This holds in the case that f is infinitely renormalizable at c as well: in fact, K, L, I are then all periodic intervals and hence each of them has only one child.

Let

$$B_K : \left(\bigcup_{j=0}^{r} K_1^j\right) \cup \left(\bigcup_{i=2}^{b} K_i\right) \to \bigcup_{i=1}^{b} K_i$$

denote the real box mapping associated to K. And let

$$R = R_K : \left(\bigcup_{j=0}^{r} K_1^j\right) \cup \left(\bigcup_{i=2}^{b} K_i\right) \to K_1 = K$$

denote the first return map to K. Write $R|K_1^j = f^{s_1^j}$ and denote by D_j the components of $f^{-s_1^j}(\Omega_1)$ containing K_1^j. For $2 \le i \le b$, write $R|K_i = f^{s_i}|K_i$ and let $\tilde{\Omega}_i$ be the component of $f^{-s_i}(\Omega_1)$ containing K_i. Let $s_i' \le s_i$ be the positive integer such that $B_K|K_i = f^{s_i'}$.

PROPOSITION 7.2. *Let $x \in J_c \cap K(\mathbf{F})$ be a point such that $F^m(x) \notin \omega(c)$ for any $m \ge 0$. Assume that I is sufficiently small. Then there is a constant $C_5 > 1$ depending only on f, a domain $\Omega \in \{\tilde{\Omega}_i, 2 \le i \le b; \Omega_1; \Omega_2\}$ and a non-negative integer k such that there is a univalent branch h of f^{-k} defined on Ω and*

$$d(x, h(\Omega)) \le C_5 \operatorname{diam}(h(\Omega)).$$

Since the proof of this proposition is relatively long, we first show how it implies Theorem 7.1.

Proof of Theorem 7.1. (1) Assume that $X_i \subset J_c$ are two measurable sets of positive measure such that $f(X_i) \subset X_i \pmod 0$, $i = 1, 2$. Assume that $m(X_1 \cap X_2) = 0$, and we shall deduce a contradiction.

Let $I \in \mathcal{I}_{\delta_0}$ be a small interval as before. Let $y \in X_1$ be a Lebesgue density point of X_1 such that $f^m(y) \notin \omega(c)$ for all $m \ge 0$. We also assume that $f^m(y)$ are all Lebesgue density points of X_1. Then it is not difficult to show that $x = f^m(y) \in K(\mathbf{F})$ for some positive integer m. Let Ω and h be as desired by Proposition 7.2 for I and x. Note that there is a positive integer p, such that $f^p : h(\Omega) \to D = D((1 + \delta_1/2)I)$ is a proper map of uniformly bounded degree. When I is sufficiently small, we have that $m(h(\Omega) \cap X_1)/m(h(\Omega))$ is close to 1 and hence so is $m(D \cap X_1)/m(D)$. Similarly, we can prove that $m(D \cap X_2)/m(D)$ is close to 1, which is obviously absurd.

(2) If f carries an invariant line field μ on J_c, then there is a point $x \in K(\mathbf{F})$ such that $f^m(x) \notin \omega(c)$ for all m, and such that $\mu(x) \ne 0$, and μ is almost continuous at x. Given any small interval $I \in \mathcal{I}_{\delta_0}$ as before, again let Ω be the domain as desired by Proposition 7.2 for I and x.

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Note that there is a non-negative integer p, and a symmetric interval $A \in \mathcal{I}_{\delta'}$ for some $\delta' > 0$ which is uniformly bounded from zero, such that $f^p|\Omega$ is a conformal map onto a C-bounded admissible topological disk Q with $A \subset Q$ and $Q \cap \omega(c) \subset A$. Indeed, if $\Omega \neq \Omega_2$, then we just take $p = 0$. In $\Omega = \Omega_2$, then by the previous construction, the statement is still true with $A = L$.

Thus, there is a univalent branch g of f^{k+p}, defined on Q, such that $d(x, Q)/\mathrm{diam}(Q)$ is uniformly bounded. Obviously, such a map g extends to a univalent map defined on $D = \mathbb{C}_{A'}$, where A' is the maximal open interval with $A' \cap \omega(c) = A \cap \omega(c)$.

Let e be the critical point in A, and let $A_0 \ni e$ be the component of the domain of the first return map to A. Then by Proposition 6.3 in [**26**], we have that $|A_0|/|A|$ is uniformly bounded from zero. Let s be the return time of A_0 to A. Let $Y = D((1 + \delta'/2)A)$ and let X be the component of $f^{-s}(Y)$ which contains A_0. Then $f^s : X \to Y$ is a proper map which has a critical point and is of uniformly bounded degree. Note that $\mathrm{mod}(D - X \cup Y)$ is uniformly bounded from zero, and hence g has a uniformly bounded distortion on $X \cup Y$.

Let $U = g(X)$ and $V = g(Y)$, and let $h = g \circ f^s \circ g^{-1} : U \to V$. Then $h \in \mathcal{H}(f)$ is a proper map with a critical point and of bounded degree. Since $\max_{z \in Y} d(e, z)/d(e, \partial Y)$ is uniformly bounded, so is $\max_{z \in U} d(g(e), z)/d(g(e), \partial U)$. Similarly, we have a uniform upper bound on $\max_{z \in V} d(z, g(f^s(e)))/d(g(f^s(e)), \partial V)$. Also, since $d(U, x), d(V, x)$ are not so large compared to $\mathrm{diam}(g(Q))$, and since $\mathrm{diam}(Q) \asymp \mathrm{diam}(Y) \asymp \mathrm{diam}(X)$, we have a uniform bound on $d(U, x)/\mathrm{diam}(U)$ and on $d(x, V)/\mathrm{diam}(V)$.

Choose a sequence of intervals $I_m \in \mathcal{I}_{\delta_0}$ such that $|I_m| \to 0$ as $m \to \infty$. Define D_m, X_m, Y_m, s_m, g_m, h_m as before, and apply Proposition 3.2, we obtain a contradiction. □

Proof of Proposition 7.2. The proof will be divided in two cases. Case 1 will be proved similarly to the case with decay geometry (a large real bound), while case 2 has to be proved differently.

LEMMA 7.5. *For any $x \in \bigcup_{j=0}^{r} D_j - \Omega_1$, there is a piecewise smooth curve γ joining x to $\partial\Omega_1$, such that*

(1) $l(\gamma) \leq \delta_3^{-1} \mathrm{diam}(\Omega_1)$;

(2) *for any $z \in \gamma$, we have*

$$B(z, \delta_3 \mathrm{diam}(\Omega_1)) \cap \omega(c) = \emptyset,$$

where $\delta_3 > 0$ is a constant.

Proof. Let K' be the maximal open symmetric interval such that $K' \cap \omega(c) = K \cap \omega(c)$. By (6), we know that the diameter of K' is comparable to the diameter of Ω_1. Let $A = \hat{\mathbb{C}}_{K'}$. For any $0 \leq j \leq r$, let A_j be the component of $f^{-s_1^j}(A)$ which contains K_1^j. Then $f^{s_1^j} : A_j \to A$ has the same degree as $f^{s_1^j} : D_j \to \Omega_1$ and hence is uniformly bounded from above. Since the hyperbolic diameter of Ω_1 in A is uniformly bounded, so is the hyperbolic diameter of D_j in A_j, by Lemma 2.1.

We claim that $A_j \subset A$ for any $0 \leq j \leq r$. Let $\tau : K \to K$ be the involution such that $f \circ \tau = f$. Consider the chain $\{G'_m\}_{m=0}^{s_1^j}$ and also the chain $\{G_m\}_{m=0}^{s_1^j}$ with $G'_{s_1^j} = K'$, $G_{s_1^j} = K$ and $G'_0 \supset G_0 \supset K_1^j$. Since $K' - K$ is disjoint from $\omega(c)$, so is

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$G_1' - G_1$. Thus both $G_0' - G_0$ and $\tau(G_0') - \tau(G_0)$ are disjoint from $\omega(c)$. So $G_0' \subset K'$. Since $A_j \cap \mathbb{R} = G_0'$, we have $A_j \subset \hat{\mathbb{C}}_{G_0'} \subset A$.

Thus, the hyperbolic diameter of D_j in A is uniformly bounded from above. The lemma follows easily. □

Continuation of proof of Proposition 7.2.

Case 1. For any non-negative integer k, we have

$$F^k(x) \notin \bigcup_{j=0}^{r} D_j - \Omega_1.$$

Let $\tilde{\Omega}_1 = \Omega_1$. Let

$$k_i = \min\{k \in \mathbb{N} \cup \{0\} : F^k(x) \in \tilde{\Omega}_i\}$$

for $1 \le i \le b$. Let $1 \le i_0 \le b$ be such that $k_{i_0} \le k_i$ for any $1 \le i \le b$. We claim that $F^{k_{i_0}} : \mathrm{Comp}_x(k_{i_0}, \tilde{\Omega}_{i_0}) \to \tilde{\Omega}_{i_0}$ is a conformal map, and hence we can take $\Omega = \tilde{\Omega}_{i_0}$ to complete the proof of the proposition.

We prove the claim by contradiction. Assume that such a map is not conformal. Let $A_m = \mathrm{Comp}_{F^m(x)}(k_{i_0} - m, \tilde{\Omega}_{i_0})$ for $0 \le m \le k_{i_0}$. Then there will be a maximal $m_0 < k_{i_0}$ such that A_{m_0} contains a critical point $e \in \omega(c)$. Then $F^{k_{i_0}-m_0}(e) \in \tilde{\Omega}_{i_0}$. So there is a positive integer p such that $F^{k_{i_0}-m_0}(c) = B_K^p(c)$. Let $m_0 < m_1 < \cdots < m_p = k_{i_0}$ be the integers such that $B_K^j(c) = F^{m_j - m_0}(c)$ for any $0 \le j \le p$. Let $0 \le i_1 \le b$ be such that $B_I^{p-1}(c) = F^{m_{p-1}-m_0}(c) \subset \tilde{\Omega}_{i_1}$. If $i_1 \ne 1$, then $F^{m_{p-1}}(x) \in \tilde{\Omega}_{i_1}$, this contradicts the minimality of k_{i_0}. So $i_1 = 1$ and hence $F^{m_{p-1}}(x) \in \bigcup_{j=0}^{r} D_j$. Since there is no point in the forward orbit of x which is contained in $\bigcup_{j=0}^{r} D_j - \Omega_1$, we have $F^{m_{p-1}}(x) \in \tilde{\Omega}_1 = \Omega_1$, which contradicts the minimality of k_{i_0} again.

Case 2. There is a non-negative integer k such that

$$F^k(x) \in \bigcup_{j=0}^{r} D_j - \Omega_1.$$

In this case we shall show that there is a univalent branch h of $f^{-k'}$ defined on $\Omega = \Omega_1$ or Ω_2 such that $d(x, h(\Omega)) \le C_5 \,\mathrm{diam}(h(\Omega))$, where k' is the non-negative integer such that $F^k = f^{k'}$ in a neighbourhood of x.

Let P_1, P_2 be admissible topological disks, we say that the triplet (m, P_1, P_2) is *bounded by a constant* $C > 1$ if (P_1, P_2) is a C-bounded couple and if there is a piecewise smooth curve γ, joining $F^m(x)$ to P_1, such that

$$l(\gamma) \le C \,\mathrm{diam}(P_1)$$

and such that for any $z \in \gamma$, we have

$$B(z, C^{-1} \,\mathrm{diam}(P_1)) \cap \omega(c) = \emptyset.$$

Remark 7.1. Let (m, P_1, P_2) be a C-bounded triplet as before. Then (m, P_2, P_1) is C'-bounded for some $C' > 1$ depending only on C. Indeed, there is a curve ρ joining ∂P_1 and ∂P_2, with $l(\rho)/\mathrm{diam}(P_1)$ uniformly bounded, such that for each point $z \in \rho$, $B(z, C^{-1} \,\mathrm{diam}(P_1))$ is disjoint from $\omega(c)$. Let $\gamma' = \gamma * \partial P_1 * \rho$. This is a piecewise smooth curve joining $F^m(x)$ to ∂P_2, with similar properties as γ.

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Convention. For a Jordan curve we always give it an anti-clockwise orientation.

LEMMA 7.6. *Let m be a non-negative integer and $P \subset V(F^m(x))$ be a C-bounded admissible topological disk. Let $\rho \subset V(F^m(x)) - \omega(c)$ be a path with initial point $F^m(x)$ and endpoint $w \in \partial P$ with $l(\rho) \leq C \operatorname{diam}(P)$. Assume that for any $y \in \rho$, we have $B(y, C^{-1}\operatorname{diam}(P)) \cap \omega(c) = \emptyset$. Let ρ' be the lift of ρ under F^m with initial point x and z the endpoint of ρ'. Assume that the lift of ∂P, considered as a loop based at w, with initial point z is a closed Jordan curve. Then there is a constant C' depending only on C, and a topological disk $Q \subset U_x(m)$ such that $F^m : Q \to P$ is a well-defined conformal map and*

$$d(x, Q) \leq C' \operatorname{diam}(Q).$$

Remark. There is a well-defined proper map $F^m : U_x(m) \to V(F^m(x))$ and the lifts should be considered under the map.

Proof. Let γ denote the lift of ∂P with initial point z and let Q denote the topological disk bounded by γ, then $F^m : Q \to P$ is a conformal mapping. Let h denote the inverse of this conformal mapping. The function h has an analytic continuation along the path ρ^{-1}.

By the assumption, there is a uniformly bounded integer p, and round disks $B_i = B(y_i, r_i)$, $1 \leq i \leq p$ which are disjoint from $\omega(c)$, such that

$$\gamma \cup \partial P \subset \bigcup_{i=1}^{p} B\left(y_i, \frac{r_i}{2}\right).$$

From Koebe's principle, it follows that h has a uniformly bounded distortion on $\gamma \cup \partial P$. Since h is conformal on P, it also has uniformly bounded distortion on P. Thus the lemma holds. □

COROLLARY 7.3. *Let P_i, $i = 1, 2$ be admissible topological disks and let m be a non-negative integer. Assume that for $i = 1, 2$, there is a nice interval $A_i \subset P_i$ such that $P_i \cap \omega(c) = A_i \cap \omega(c)$. Assume also that the triplet (m, P_1, P_2) is bounded by a constant $C > 0$. If $\operatorname{diam}(P_1)$ is sufficiently small, then there is a constant C' depending only on C and an integer $i \in \{1, 2\}$ such that either of the following holds.*

(1) *There exists a topological disk $Q \subset U_x(m)$ such that $F^m : Q \to P_i$ is a conformal map and $d(x, h(Q)) \leq C' \operatorname{diam}(Q)$.*

(2) *There is a positive integer $m' < m$, two admissible topological disks $P_{i,1}$, $P_{i,2}$, such that the triplet $(m', P_{i,1}, P_{i,2})$ is C'-bounded. Moreover, there are nice intervals $A_{i,j} \subset P_{i,j}$, $j = 1, 2$ with the following properties:*
   (2.i) *$A_{i,j} \supset P_{i,j} \cap \omega(c) \neq \emptyset$, $j = 1, 2$;*
   (2.ii) *$f(A_{i,1}) = f(A_{i,2})$, $f(P_{i,1}) = f(P_{i,2})$, $F^{m-m'}(P_{i,1}) = P_i$;*
   (2.iii) *$F^{m-m'}|A_{i,1}$ is monotone and $F^{m-m'}(A_{i,1}) = A_i$.*
   Moreover, $\operatorname{diam}(P_{i,1})$ is also very small.

Proof. The proof is a modification of McMullen's argument in his approach to the absence of a line field, see [**20, 21**]. Let ξ_i be a piecewise smooth curve from $F^m(x)$ to ∂P_i,

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$i = 1, 2$, such that for some $\eta > 0$, some $C' > 1$, and each $y \in \xi_i$, we have

$$B(y, \eta \operatorname{diam}(P_1)) \cap \omega(c) = \emptyset,$$

and such that

$$l(\xi_i) \leq C' \operatorname{diam}(P_1).$$

By the definition of boundedness, such constants η, C' exist.

Let ξ_i' denote the lift of ξ_i with initial point x under F^m, $i = 1, 2$. Let z_i denote the endpoint of ξ_i' and ζ_i denote the lift of ∂P_i under F^m with initial point z_i.

If either of ζ_1 and ζ_2 is a Jordan curve then we are in case 1 by Lemma 7.6. Assume we are not in this case. Let m_i be the maximal integer such that $F^{m_i}(\zeta_i)$ is not a closed curve. Without loss of generality, assume that $m_1 \geq m_2$. Let Δ_i denote the domain bounded by $F^{m_i+1}(\zeta_i)$, then both Δ_1 and Δ_2 intersect $\omega(c)$.

CLAIM. *The set* $F^{m_1+1}(\zeta_1 \cup \zeta_2 \cup \xi_1' \cup \xi_2')$ *has a small diameter provided that* $\operatorname{diam}(P_1)$ *is small.*

Indeed, by construction, the map

$$F^{m-m_1-1} : F^{m_1+1}(\zeta_1 \cup \zeta_2 \cup \xi_1' \cup \xi_2') \to \partial P_1 \cup \partial P_2 \cup \xi_1 \cup \xi_2$$

is a conformal map. Since (m, P_1, P_2) is bounded, the map has a uniformly bounded distortion. Thus, to prove this claim, it suffices to show the real trace of Δ_1 is small when $\operatorname{diam}(P_1)$ is also small. But $\Delta_1 \cap \mathbb{R}$ is an interval intersecting $\omega(c)$, and it is mapped diffeomorphically to a small interval $P_1 \cap \mathbb{R}$, and so it must be small due to the absence of a wandering interval.

So we may assume that $m_1 > m_2$. By the same reasoning, the set $F^{m_1}(\zeta_1 \cup \zeta_2 \cup \xi_1' \cup \xi_2')$ is close to a critical point, say e, which is contained in the set $\omega(x) = \omega(c)$.

Let U be a definite neighbourhood of e such that $f|U : U \to f(U)$ is a branched covering with a unique critical point e and let $\phi : U \to U$ be the prime transformation of the branched covering such that the lift of $\partial f(U)$, considered as a loop based at $f(z)(z \in \partial U)$, with initial point z under f is ended by $\phi(z)$.

Let

$$\rho = \xi_2 * \xi_1^{-1} * \partial P_1 * \xi_1.$$

Then ρ is a piecewise smooth curve from $F^m(x)$ to ∂P_2. By choosing $\eta > 0$ smaller, and C' larger, we may assume that

$$l(\rho) \leq C' \operatorname{diam}(P_2)$$

and also that for any $y \in \rho$, we have

$$B(y, \eta \operatorname{diam}(P_2)) \cap \omega(c) = \emptyset.$$

Let ρ' be the lift of ρ with initial point x under F^m and z' the endpoint of ρ'.

Observation. The endpoint $F^{m_1}(z')$ of $F^{m_1}(\rho')$ is $\phi(F^{m_1}(z_2))$.

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In fact, since the pull-back of ξ_i with initial point $F^{m_1}(x)$ under F^{m-m_1} is ended by $F^{m_1}(z_i)$, the pull-back of ξ_i with initial point $\phi(F^{m_1}(x))$ is ended by $\phi(F^{m_1}(z_i))$, $i = 1, 2$. So the pull-back of ξ_1^{-1} with initial point $\phi(F^{m_1}(z_1))$ is $\phi(F^{m_1}(x))$. The pull-back of ∂P_1 with initial point $F^{m_1}(z)$ is ended by $\phi(F^{m_1}(z))$ again by the definition of ϕ. The observation follows.

Note also that the lift of ∂P_2 under F^{m-m_1} with initial point $\phi(F^{m_1}(z_2))$ is the Jordan curve $\phi(F^{m_1}(\zeta_2))$, which bounds the topological disk $\phi(F^{m_1-m_2-1}(\Delta_2))$. Let $P_{2,1} = F^{m_1-m_2-1}(\Delta_2)$ and $P_{2,2} = \phi(P_{2,1})$. Then $f(P_{2,1}) = f(P_{2,2})$. If $P_{2,2} \cap \omega(c) = \emptyset$, then the lift of ∂P_2 under F^k with initial point z' is obviously a Jordan curve and hence by Lemma 7.6, we are in case 1. So let us assume $P_{2,2} \cap \omega(c) \neq \emptyset$. Obviously both $P_{2,1}$ and $P_{2,2}$ are admissible topological disks. Let $m' = m_1$. Then it is easy to check that the triple $(m', P_{2,1}, P_{2,1})$ is uniformly bounded using Koebe's Distortion Theorem. For $j = 1, 2$, let $A_{2,j} \subset P_{2,j}$ be the interval such that $F^{m-m_1}(A_{2,j}) = A_2$. It is easy to check properties (2.i)–(2.iii) for $i = 2$. □

We can complete the proof of Proposition 7.2 now.

Continuation of proof of Proposition 7.2. If $\Omega_2 \cap \omega(c) = \emptyset$, then it follows from Lemma 7.6 that we can take $\Omega = \Omega_2$ to conclude the proof. So assume that $\Omega_2 \cap \omega(c) \neq \emptyset$. Let $A_i \subset \Omega_i$ be the nice interval intersecting $\omega(c)$ such that $A_i \supset \Omega_i \cap \omega(c)$. The triplet (k, Ω_1, Ω_2) is uniformly bounded. Recall that $A_1 = K$ and A_2 is a monotone pull-back of L.

Applying Corollary 7.3 to the triplet (k, Ω_1, Ω_2), we have two possibilities. If we are in case 1 in that corollary, then the proof is completed. Assume that we are in case 2. Then we have $i_0 \in \{1, 2\}$ and another triplet $(k_1, \Omega_{i_0 1}, \Omega_{i_0 2})$ which is also uniformly bounded. Let $A_{i_0 j}$ be the nice interval as in that corollary, $j = 1, 2$. Remember that both $A_{i_0 1}$ and $A_{i_0 2}$ are monotone pull-backs of A_{i_0} and intersect $\omega(c)$.

Apply Corollary 7.3 to the triple $(k_1, \Omega_{i_0 1}, \Omega_{i_0 2})$, and so on. Either we complete the proof within $N + 1$ steps, or we will have $i_0, i_1, \ldots, i_N \in \{1, 2\}$ and nice intervals

$$A_j, A_{i_0 j}, A_{i_0 i_1 j}, \ldots, A_{i_0 i_1 \cdots i_N j}, \quad (j = 1, 2),$$

intersecting $\omega(c)$ such that for any $0 \leq s \leq N$, $A_{i_0 i_1 \cdots i_s 1}$ and $A_{i_0 i_1 \ldots i_s 2}$ are monotone pull-backs of $A_{i_0 i_1 \cdots i_s}$ which are symmetric with respect to a critical point in $\omega(c)$. For any $i \in \{1, 2\}$, let i' denote the element of $\{1, 2\} - \{i\}$. Let

$$\mathbb{S} = \{A_{i_0 i'_1}, \ldots, A_{i_0 i_1 \cdots i'_N}, A_{i_0 i_1 \cdots i_N 1}, A_{i_0 i_1 \cdots i_N 2}\}.$$

Then $\mathbb{S}$ has $N + 1$ elements, which are all monotone pull-backs of A_{i_0}. For each $S \in \mathbb{S}$, let $k(S)$ be the minimal positive integer such that $f^{k(S)}(c(S)) \in S$ for some $c(S) \in \omega(c)$ and let $A(S)$ be the pull-back of S along the orbit $\{f^j(c(S))\}_{j=0}^{k(S)}$. Then $A(S)$ is a child of $A_1 = K$ (if $i_0 = 1$) or L (if $i_0 = 2$). It is easy to see that for $S, S' \in \mathbb{S}$ with $S \neq S'$, $A(S)$ and $A(S')$ are different. So we know that either K or L has at least $N + 1$ children, which is a contradiction. □

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