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OPTIMALITY RESULTS FOR COUPON COLLECTION

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Abstract

We consider the coupon collection problem, where each coupon is one of the types $1, \ldots, s$ with probabilities given by a vector $\boldsymbol{p}$. For specified numbers $r_1, \ldots, r_s$, we are interested in finding $\boldsymbol{p}$ that minimizes the expected time to obtain at least r_i type-i coupons for all $i = 1, \ldots, s$. For example, for $s = 2$, $r_1 = 1$, and $r_2 = r$, we show that $p_1 = (\log r - \log(\log r))/r$ is close to optimal.

Keywords: Multinomial trial, Poissonization, likelihood ratio ordered, increasing failure rate

2010 Mathematics Subject Classification: Primary 60K99

1. Introduction

Consider multinomial trials with possible outcomes $1, \ldots, s$, with trial outcome probabilities given by the vector $\boldsymbol{p} = (p_1, \ldots, p_s)$, $\sum_{i=1}^{s} p_i = 1$. For a specified vector of integers $\boldsymbol{p} = (r_1, \ldots, r_s)$, let $N = N_{\boldsymbol{r}}(\boldsymbol{p})$ denote the number of trials needed until there have been at least r_i type-i outcomes for all $i = 1, \ldots, s$. We are interested in the probability vector $\boldsymbol{p}$ that minimizes $\mathbb{E}[N_{\boldsymbol{r}}(\boldsymbol{p})]$.

In Section 2 we show that if $r_j > r_i$ then the minimizing vector would have $p_j > p_i$. In Section 3 we specialize to the case where $s = 2$ and derive an expression for $\mathbb{E}[N]$ as well as an asymptotic formula. In Section 4 we further specialize the $s = 2$ model by supposing that $r_1 = 1$, and show that the optimal value of p_1 is very close to $(\log(r) - \log\log(r))/r$.

Applications of majorization to multinomial distribution problems is the topic of the wonderful research monograph of Marshall and Olkin [3, Chapter 11, Section E, pp. 304–308]. The authors presented results of Alam [1], Olkin [4], Rinott [5], and Wong and Yu [6] and include personal communications from Persi Diaconis, Frank Proschan, and David Siegmund. The results adapted to coupons are summarized below. Perhaps because the results are expressed more generally, they appear to have been somewhat overlooked, and even now, thirty six years after publication, weaker versions appear from time to time.

The number of coupons collected until at least r of each coupon type is represented is stochastically minimized by equal coupon type probabilities. This follows from [3, Proposition E.11.D, p. 306], with $s = r$, and $t = \infty$, a result due to Olkin [4]. The number of coupons collected until one coupon type appears r times is stochastically maximized by equal coupon probabilities. The $r = 2$, $k = 365$ case corresponds to the famous birthday problem. This result

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also follows from [3, Proposition E.11.D], with $s = 0$ and $t = r - 1$. The number of coupons collected until r different coupon types are represented is stochastically minimized by uniform probabilities. This result follows from [3, Proposition E.11.b, p. 305] (with $k = r - 1$), a result of Wong and Yue [6].

2. A monotonicity result

We show that if r_i is nondecreasing in i then in attempting to minimize $\mathbb{E}[N(p_1, \ldots, p_s)]$, we can restrict attention to probability vectors $\boldsymbol{p} = (p_1, \ldots, p_s)$ for which p_i is nondecreasing in i.

Theorem 1. *If $r_i < r_j$ and $p_i > p_j$ then*

$$\mathbb{E}[N(p_1, \ldots, p_i, \ldots, p_j, \ldots, p_s)] \geq \mathbb{E}[N(p_1, \ldots, p_j, \ldots, p_i, \ldots, p_s)].$$

Proof. To prove the result we will employ the Poissonization trick and assume that trials occur at times distributed according to a Poisson process with rate 1. Letting $T_m(\boldsymbol{p})$ denote the time until there have been r_m type-m outcomes, then it follows that $T_m(\boldsymbol{p}), m = 1, \ldots, T_s$ are *independent* gamma random variables with parameters $(r_m, p_m), m = 1, \ldots, s$. Letting $T(\boldsymbol{p})$ be the first time at which there have been at least r_m type-m outcomes for all $m = 1, \ldots, s$ then $T(\boldsymbol{p}) = \max_m T_m(\boldsymbol{p})$. Now suppose that $r_i < r_j$ and $p_i > p_j$, and let $\boldsymbol{p} = (p_1, \ldots, p_i, \ldots, p_j, \ldots, p_s)$ and $\boldsymbol{p}' = (p_1, \ldots, p_j, \ldots, p_i, \ldots, p_s)$. We claim that $T(\boldsymbol{p}) \geq_{\mathrm{st}} T(\boldsymbol{p})$. To show this, note that

$$T(\boldsymbol{p}) =_{\mathrm{st}} \max\Big(G(r_i, p_i),\ G(r_j, p_j),\ \max_{m \neq i,j} G(r_m, p_m)\Big),$$
$$T(\boldsymbol{p}') =_{\mathrm{st}} \max\Big(G(r_i, p_j),\ G(r_j, p_i),\ \max_{m \neq i,j} G(r_m, p_m)\Big),$$

where $G(a, b)$ represents a gamma random variable with parameter (a, b) and all the gamma random variables on the right-hand sides of the preceding stochastic equalities are independent. Consequently, to prove that $T(\boldsymbol{p}) \geq_{\mathrm{st}} T(\boldsymbol{p}')$, it suffices to show that

$$\max(G(r_i, p_i),\ G(r_j, p_j)) \geq_{\mathrm{st}} \max(G(r_i, p_j),\ G(r_j, p_i)),$$

which is equivalent to showing that

$$\begin{aligned} &\max(G(r_i, p_i),\ G(r_i, p_j) + G(r_j - r_i, p_j)) \\ &\quad \geq_{\mathrm{st}} \max(G(r_i, p_j),\ G(r_i, p_i) + G(r_j - r_i, p_i)), \end{aligned} \tag{1}$$

where the three gammas on each side are independent. Because $p_i > p_j$ implies that $G(r_j - r_i, p_j) \geq_{\mathrm{st}} G(r_j - r_i, p_i)$, (1) will hold if we can show that

$$\max(G(r_i, p_i),\ G(r_i, p_j) + G(r_j - r_i, p_i)) \geq_{\mathrm{st}} \max(G(r_i, p_j),\ G(r_i, p_i) + G(r_j - r_i, p_i)).$$

Thus, with X, Y, and Z being independent gamma random variables with respective parameters (r_i, p_i), (r_i, p_j), and $(r_j - r_i, p_i)$, it suffices to show that

$$\max(X,\ Y + Z) \geq_{\mathrm{st}} \max(X + Z,\ Y). \tag{2}$$

To show this, note that it easily follows from $p_i > p_j$ that X is a likelihood ratio smaller than Y, which implies that, for $y > x$,

$$\mathbb{P}(X = x,\ Y = y \mid X = x,\ Y = y \text{ or } X = y,\ Y = x) \geq \tfrac{1}{2}.$$

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Let $S = \{X, Y\}$ be the set of values of X and Y. Then, we claim that

$$\mathbb{P}(\max(X,\ Y+Z) \geq t \mid S = \{x, y\}) \geq \mathbb{P}(\max(X+Z,\ Y) \geq t \mid S = \{x, y\}). \tag{3}$$

To verify (3), suppose that $y > x$, and let $\alpha = \mathbb{P}(X = x,\ Y = y \mid S = \{x, y\}) \geq \frac{1}{2}$. Now (3) holds when $y > t$ since both sides are equal to 1 in that case. On the other hand, if $y < t$ then

$$\begin{aligned}\mathbb{P}(\max(X,\ Y+Z) \geq t \mid S = \{x, y\}) &= \alpha\mathbb{P}(Z \geq t-y) + (1-\alpha)\mathbb{P}(Z \geq t-x)\\ &\geq \alpha\mathbb{P}(Z \geq t-x) + (1-\alpha)\mathbb{P}(Z \geq t-y)\\ &= \mathbb{P}(\max(X+Z,\ Y) \geq t \mid S = \{x, y\}),\end{aligned}$$

where we used the fact that $\alpha \geq \frac{1}{2}$ and $\mathbb{P}(Z \geq t-y) \geq \mathbb{P}(Z \geq t-x)$, Hence, (3) is shown. Unconditioning now yields (2). Hence, we have shown that $T(\boldsymbol{p}) \geq_{\text{st}} T(\boldsymbol{p}')$, which implies that $\mathbb{E}[T(\boldsymbol{p})] \geq \mathbb{E}[T(\boldsymbol{p}')]$. The Theorem follows since $\mathbb{E}[T(\boldsymbol{p})] = \mathbb{E}[N(\boldsymbol{p})]$. □

Now suppose that for positive numbers $a_1, \ldots, a_s$ summing to 1, $r_i = a_i v$, $i = 1, \ldots, s$. With $\boldsymbol{p}(v) = (p_1(v), \ldots, p_s(v))$ being the probability vector that minimizes $\mathbb{E}[N_v(\boldsymbol{p})]$, it is quite intuitive that

$$\lim_{v\to\infty} \boldsymbol{p}(v) = (a_1, \ldots, a_s).$$

Indeed, if $\lim_{v\to\infty} \boldsymbol{p}(v)$ exists then it must be equal to $(a_1, \ldots, a_s)$. Suppose that

$$\lim_{v\to\infty} \boldsymbol{p}(v) = \boldsymbol{p}^* = (p_1^*, \ldots, p_s^*).$$

If $N^j(\boldsymbol{p}^*)$ denotes the number of trials needed to obtain $r_j = va_j$ type-j outcomes when the probabilities are given by $\boldsymbol{p}^*$, then it follows by the strong law of large numbers that

$$\lim_{v\to\infty} \frac{N^j(\boldsymbol{p}^*)}{va_j} = \frac{1}{p_j^*}, \qquad j = 1, \ldots, s.$$

Hence,

$$\lim_{v\to\infty} \frac{N_v(\boldsymbol{p}^*)}{v} = \lim_{v\to\infty} \max_j \frac{N^j(\boldsymbol{p}^*)}{v} = \max_j \frac{a_j}{p_j^*}.$$

However, from $\sum_j p_j^* = \sum_j a_j$, it follows that $\max_j (a_j/p_j^*) \geq 1$, with equality when $p_j = a_j$ for all j.

3. When $s = 2$

Suppose that $s = 2$, and let $p_1 = p$ and $p_2 = 1 - p = q$. We now derive an expression for $\mathbb{E}[N_{k,r}]$, the mean number of trials until there have been at least k type-1 and r type-2 outcomes. To begin, let T denote the number of trials until there have been r of type-2, and let $A = N_{k,r} - T$. Then,

$$\mathbb{E}[N_{k,r}] = \mathbb{E}[T] + \mathbb{E}[A] = \frac{r}{q} + \sum_{j=0}^{k-1} \frac{k-j}{p}\binom{j+r-1}{j} p^j q^r, \tag{4}$$

where $\mathbb{E}[A]$ was obtained by conditioning on T. Letting $B_{n,p}$ denote a binomial random variable with parameters (n, p), we have

$$\sum_{j=0}^{k-1} \binom{j+r-1}{j} p^j q^r = \mathbb{P}(B_{k+r-1,q} \geq r), \tag{5}$$

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since both sides represent the probability that r type-2 outcomes occur before k type-1 outcomes. Also,

$$\sum_{j=1}^{k-1} \frac{j}{p}\binom{j+r-1}{j} p^j q^r = \frac{r}{q}\sum_{j=1}^{k-1}\binom{j+r-1}{j-1} p^{j-1} q^{r+1} \tag{6}$$

$$= \frac{r}{q}\mathbb{P}(\text{at least } r+1 \text{ type-2s before } k-1 \text{ type-1s})$$

$$= \frac{r}{q}\mathbb{P}(B_{k+r-1,q} \geq r+1). \tag{7}$$

Proposition 1. *We have*

$$\mathbb{E}[N_{k,r}] = \frac{r}{q}\mathbb{P}(B_{k+r-1,q} \leq r) + \frac{k}{p}\mathbb{P}(B_{k+r-1,q} \geq r).$$

Proof. This directly follows from (4), (5), and (7). □

When $p = k/(k+r)$, we have the following asymptotic result.

Proposition 2. *With $v = k + r$ and $k = pv, r = qv$, we have*

$$\sqrt{v}[\mathbb{E}[N_{k,r}] - v] \to \frac{1}{\sqrt{2\pi pq}} \quad \text{as } v \to \infty.$$

Proof. From Proposition 1, we have

$$\mathbb{E}[N_{k,r}] = v\mathbb{P}(B_{v-1,q} \leq r) + v\mathbb{P}(B_{v-1,q} \geq r) = v + v\mathbb{P}(B_{v-1,q} = r).$$

Now, with Φ being the standard normal distribution function, we obtain

$$\begin{aligned}
\mathbb{P}(B_{v-1,q} = r) &= \frac{(v-1)!}{r!\,k!} p^r q^{k-1} \\
&= \binom{v}{r} p^r q^k \frac{k}{vq} \\
&= \binom{v}{r} p^r q^k \\
&= \mathbb{P}(B_{v,q} = r) \\
&\approx \Phi\left(\frac{1/2}{\sqrt{vpq}}\right) - \Phi\left(\frac{-1/2}{\sqrt{vpq}}\right) \\
&\approx \frac{1}{\sqrt{vpq}}\Phi'(0) \\
&= \frac{1}{\sqrt{2\pi\, vpq}},
\end{aligned}$$

and the result follows. □

For instance, if $v = 10^4$ and $p = 0.4$ then $(\mathbb{E}[N] - 10^4)10^2 = 0.814316$, whereas Proposition 2 yields the approximation 0.814338.

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4. When $s = 2$ with $r_1 = 1$

Suppose that $s = 2, r_1 = 1, r_2 = r$, and again let $p_1 = p, p_2 = 1 - p = q$. As in the preceding section

$$\mathbb{E}[N] = \frac{r}{1-p} + \frac{(1-p)^r}{p}.$$

Lemma 1. *There is a unique p_0 that minimizes $\mathbb{E}[N]$.*

Proof. Let

$$g(p) = \frac{\mathrm{d}}{\mathrm{d}p}\mathbb{E}[N] = \frac{r}{q^2} - \frac{q^{r-1}}{p^2}(1 + (r-1)p).$$

Since g increases from $-\infty$ to ∞ as p goes from 0 to 1, the result follows. □

Remark 1. If $r = 2$ then $\mathbb{P}(N = 3) = 3pq^2$, which is uniquely maximized when $p = \frac{1}{3}$. Since $g\frac{1}{3} \neq 0$, it follows that $\mathbb{E}[N]$ is not minimized when $p = \frac{1}{3}$, and, thus, that there is no value p that stochastically minimizes N.

Lemma 2. *With p_0 as defined in Lemma 1,*

$$p_0 < \frac{\log(r)}{r}, \qquad r \geq 3.$$

Proof. Let $p_{\mathrm{U}} = \log(r)/r$ and $q_{\mathrm{U}} = 1 - p_{\mathrm{U}}$, where the subscript 'U' denotes upper. Since g is increasing and $g(p_0) = 0$, the result will follow if we can show that $g(p_{\mathrm{U}}) > 0$, which is equivalent to

$$rp_{\mathrm{U}}^2 > q_{\mathrm{U}}^{r+1}(1 + (r-1)p_{\mathrm{U}}). \tag{8}$$

Now,

$$q_{\mathrm{U}}^r = \left(1 - \frac{\log(r)}{r}\right)^r \leq \left(\exp\left(-\frac{\log(r)}{r}\right)\right)^r = \exp(-\log(r)) = \frac{1}{r},$$

and, thus,

$$q_{\mathrm{U}}^{r+1}(1 + (r-1)p_{\mathrm{U}}) \leq \frac{1}{r}\left(1 - \frac{\log(r)}{r}\right)\left(1 - \frac{\log(r)}{r} + \log(r)\right) \leq \frac{1 + \log(r)}{r}.$$

The result now follows since

$$rp_{\mathrm{U}}^2 = \frac{\log^2(r)}{r} > \frac{1 + \log(r)}{r},$$

where the preceding used that $\log^2(r) > 1 + \log(r)$ when $r > \exp((1 + \sqrt{5})/2)$. □

Lemma 3. *For $r \geq 8$,*

$$\frac{\log(r)}{r} < \frac{1}{1 + \sqrt{r}}.$$

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Proof. Let

$$h(x) = \log(x) - (\sqrt{x} - 1), \qquad x \geq 1,$$

and note that, for $x > 4$,

$$h'(x) = \frac{1}{2x}(2 - \sqrt{x}) < 0.$$

Hence, if $h(r') < 0$ for some $r' \geq 5$ then $h(r) < 0$ for all $r \geq r'$. Now, a direct computation yields that $h(13) < 0$ and so $\log(r) < \sqrt{r} - 1$ for $r \geq 13$. Since

$$\frac{r}{1 + \sqrt{r}} > \sqrt{r} - 1 \quad \text{when } r \geq 2,$$

it follows that

$$\frac{\log(r)}{r} < \frac{1}{1 + \sqrt{r}} \quad \text{when } r \geq 13.$$

As a computation yields that the inequality also holds for $r = 8, \ldots, 12$, the proof of the lemma is complete. □

Lemma 4. *Let $q_0^r < p_0$ for $r \geq 8$.*

Proof. Let $q_0 = 1 - p_0$. Since $g(p_0) = 0$ is equivalent to

$$\frac{r}{q_0} = \frac{q_0^r}{p_0}\left(r - 1 + \frac{1}{p_0}\right),$$

it follows that

$$\begin{aligned} q_0^r < p_0 \quad &\Longleftrightarrow \quad \frac{r}{q_0} < r - 1 + \frac{1}{p_0} \\ &\Longleftrightarrow \quad r\left(\frac{1}{q_0} - 1\right) < \frac{1}{p_0} - 1 \\ &\Longleftrightarrow \quad \frac{rp_0}{q_0} < \frac{q_0}{p_0} \\ &\Longleftrightarrow \quad \left(\frac{q_0}{p_0}\right)^2 > r \\ &\Longleftrightarrow \quad p_0 < \frac{1}{1 + \sqrt{r}}. \end{aligned}$$

The result now follows from Lemmas 2 and 3. □

Lemma 5. *If*

$$g(x) = \exp(-x) - \left(1 - \frac{x}{r}\right)^r, \qquad 0 \leq x \leq r,$$

then

$$\max_{0 \leq x \leq r} g(x) \leq \frac{1}{r} \exp\left(-\frac{r-1}{r}\right).$$

Proof. First, note that $g'(x) = (1 - x/r)^{r-1} - \exp(-x)$. Now, let $V_1, \ldots, V_{r-1}$ be independent uniform$(0, r)$ random variables, and set $M = \min(V_1, \ldots, V_{r-1})$. Then,

$$\mathbb{E}[M] = 1 \quad \text{and} \quad \bar{F}(x) \equiv \mathbb{P}(M > x) = \left(1 - \frac{x}{r}\right)^{r-1}.$$

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Since M is an increasing failure rate random variable and because $\exp(-x)$ is the tail distribution function of an exponential with the same mean as M, it follows from [2] that $\bar{F}(x)$ crosses $\exp(-x)$ exactly once, and the cross is from above it to below it. Consequently, there is a unique $x^* > 0$ for which $g'(x^*) = 0$, and at which $g(x)$ is maximized. Now,

$$\begin{aligned} g(x^*) &= \exp(-x^*) - \left(1 - \frac{x^*}{r}\right)^r \\ &= \left(1 - \frac{x^*}{r}\right)^{r-1} - \left(1 - \frac{x^*}{r}\right)^r \\ &= \frac{x^*}{r}\left(1 - \frac{x^*}{r}\right)^{r-1} \\ &\le \frac{1}{r}\left(1 - \frac{1}{r}\right)^{r-1} \\ &\le \frac{1}{r}\exp\left(-\frac{r-1}{r}\right), \end{aligned}$$

where in the next to last inequality we use the fact that $x(1-x)^{r-1}$ is, in the interval $0 \le x \le 1$, maximized when $x = 1/r$. □

Theorem 2. *If*

$$p_{\mathrm{L}} = \frac{\log(r) - \log\log(r)}{r},$$

then $p_{\mathrm{L}} < p_0$ *for* $r \ge 4$, *where the subscript* 'L' *denotes lower.*

Proof. Since q^r/p is decreasing in p, and, by Lemma 4, $q_0^r/p_0 < 1$ when $r \ge 8$, the result will follow for $r \ge 8$ if we can show that $q_{\mathrm{L}}^r/p_{\mathrm{L}} > 1$, where $q_{\mathrm{L}} = 1 - p_{\mathrm{L}}$. Now,

$$\begin{aligned} q_{\mathrm{L}}^r &= \left(1 - \frac{\log(r) - \log\log(r)}{r}\right)^r \\ &\ge \exp(-(\log(r) - \log\log(r))) - \frac{1}{r}\left(\frac{r-1}{r}\right)^{r-1} \\ &= \frac{\log(r)}{r} - \frac{1}{r}\left(\frac{r-1}{r}\right)^{r-1}, \end{aligned}$$

where the inequality follows from Lemma 5. Thus,

$$\frac{q_{\mathrm{L}}^r}{p_{\mathrm{L}}} = \frac{\log(r) - ((r-1)/r)^{r-1}}{\log(r) - \log\log(r)}$$

and so the result will follow if we can show that

$$\log\log(r) > \left(\frac{r-1}{r}\right)^{r-1}.$$

However, $\log\log(r) > 1 > ((r-1)/r)^{r-1}$ for $r \ge 16$, and numerical computation shows that $q_{\mathrm{L}}^r/p_{\mathrm{L}} > 1$ for $r = 4, \ldots, 15$, which completes the proof. □

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TABLE 1.

r	p_{L}	p_0	$\mathbb{E}_{p_0}[N]-r-1$	$\mathbb{E}_{p_{\mathrm{L}}}[N]-\mathbb{E}_{p_0}[N]$
10	1.469×10^{-1}	1.756×10^{-1}	1.9558	0.1556
20	9.490×10^{-2}	1.128×10^{-1}	2.3522	0.1786
100	3.080×10^{-2}	3.500×10^{-2}	3.4373	0.1640
500	8.800×10^{-3}	9.600×10^{-3}	4.6841	0.1316
2000	2.790×10^{-3}	2.980×10^{-3}	5.8359	0.1057
10^8	1.551×10^{-7}	1.573×10^{-7}	15.6672	0.0279

Remark 2. (i) The intuitive choice $p=p^*\equiv 1/(r+1)$ leads to

$$\begin{aligned}\mathbb{E}[N] &= r+1+\left(\frac{r}{r+1}\right)^r(r+1)\\ &= r+1+\left(1-\frac{1}{r+1}\right)^{r+1}\frac{(r+1)^2}{r}\\ &\approx r+1+\frac{r+2}{\mathrm{e}}.\end{aligned}$$

This can be significantly improved by using $p=p_0$. For example, when $r=100$,

$$\mathbb{E}_{p_0}[N]-101=3.4373,\qquad \mathbb{E}_{p^*}[N]-101=37.524.$$

(ii) The excess $\mathbb{E}_{p_0}[N]-(r+1)$ grows slowly, for instance, being equal to 15.6672 when $r=10^8$.

(iii) As seen by Table 1, we have $\mathbb{E}_{p_{\mathrm{L}}}[N]\approx\mathbb{E}_{p_0}[N]$.

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