Math. Proc. Camb. Phil. Soc. (1989), **105**, 169
Printed in Great Britain

An 'excursion' approach to an annealing problem

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(*Received* 20 *January* 1988)

1. *Introduction*

The following problem is the simplest of a group of problems motivated by work of Durran and Truman [1] on planetesimals.

Suppose that F is a strictly increasing continuous function on a finite or infinite interval (a_1, a_2) containing 0, with $F(0) = 0$. (We are interested in cases in which $F(a_1+) = -\infty$ if a_1 is finite, and $F(a_2-) = +\infty$ if a_2 is finite.) Suppose that a process $\tilde{Y}$ with values in (a_1, a_2) is a (weak) solution of the stochastic differential equation

$$d\tilde{Y}_t = -F(\tilde{Y}_t)\,dt + \tilde{\epsilon}(t)\,d\tilde{B}_t, \tag{1·1}$$

where $\tilde{B}$ is a Brownian motion and where $\tilde{\epsilon}(\cdot)$ is a decreasing deterministic function on $[0, \infty)$ such that $\tilde{\epsilon}(0)$ is finite and $\tilde{\epsilon}(t) \downarrow 0$ as $t \uparrow \infty$. (The assumptions on F and $\tilde{\epsilon}$ remain in force throughout the paper; they are not repeated.) The problem reads: *under what conditions is it true that* $\tilde{Y}_t \to 0$ *almost surely* (*as* $t \to \infty$)?

Undoubtedly, the powerful techniques of large-deviation theory would crack this problem. However, the purpose of this paper is to show how excursion ideas (applied via the theory of elastic Brownian motion) give a simple explanation of the answer provided by the following Theorem 1. (Of course, excursion ideas are not foreign to large-deviation theory.) That the method extends to more interesting situations is shown in the immediately following paper by Chan.

We assume throughout the remainder of this paper that

$$\int_0^\infty \tilde{\epsilon}(s)^2\,ds = \infty,$$

the case when this integral is finite being left to the reader. Define

$$\gamma_t := \inf\left\{u : \int_0^u \tilde{\epsilon}(s)^2\,ds > t\right\}, \quad \epsilon(t) := \tilde{\epsilon}(\gamma_t). \tag{1·2}$$

(Here and elsewhere, the symbol ':=' signifies 'is defined to equal'. We shall use 'a.s.' as shorthand for 'almost surely', but we do not promise to give every possible 'a.s.' qualification.)

THEOREM 1. *The following statements are equivalent*:

(i) $\tilde{Y}_t \to 0$ (a.s.);

(ii) $I(c) := \int_0^\infty \exp\{-c\epsilon(u)^{-2}\}\,du = \int_0^\infty \tilde{\epsilon}(u)^2 \exp\{-c\tilde{\epsilon}(u)^{-2}\}\,du < \infty$ *for all* $c > 0$;

(iii) $\tilde{I}(c) := \int_0^\infty \exp\{-c\tilde{\epsilon}(u)^{-2}\}\,du < \infty$ *for all* $c > 0$;

(iv) $\epsilon(u)^2 \log u \to 0$ $(u \to \infty)$.

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It is easy to show that (ii) and (iii) are equivalent and that (iv) implies (iii). Martin Barlow explained to us an argument due to John Hawkes which shows that (iii) implies (iv); this proceeds as follows. If

$$A := \limsup \tilde{\epsilon}(u)^2 \log u > 0,$$

then we can choose a sequence (u_n) with $u_n > 2u_{n-1}$ such that $\tilde{\epsilon}(u_n)^2 \log u_n > \frac{1}{2}A$, for all n; but then, if $c > 0$ is such that $q := 2c/A < 1$, we have

$$\int_{u(n-1)}^{u(n)} \exp\{-c\tilde{\epsilon}(u)^{-2}\}\,du \geqslant (u_n - u_{n-1}) \exp\{-c\tilde{\epsilon}(u_n)^{-2}\} \geqslant \tfrac{1}{2}u_n^{1-q},$$

and $\tilde{\mathrm{I}}(c) = \infty$. (We used $u(k)$ as a neater notation for u_k in the limits for the integral.)

The main part of Theorem 1, namely that (i) and (ii) are equivalent, remains to be proved.

Example 1. To understand the 'boundary case' of the theorem, let us check that if $(a_1, a_2) = (-\infty, \infty)$ and

$$d\tilde{Y}_t = -\tilde{Y}_t\,dt + (\log t)^{-\frac{1}{2}}\,d\tilde{B}_t, \quad (t \geqslant 2), \tag{1·3}$$

and $\tilde{Y}_2 = 0$, then, almost surely,

$$\liminf \tilde{Y}_t = -1, \quad \limsup \tilde{Y}_t = +1. \tag{1.4}$$

To establish this, note that, directly from (1·3),

$$e^t \tilde{Y}_t = \int_2^t e^s (\log s)^{-\frac{1}{2}}\,d\tilde{B}_s = \beta(u_t), \quad (t \geqslant 2),$$

where β is a $\mathrm{BM}^0(\mathbb{R})$ (a Brownian motion on $\mathbb{R}$ started at 0), and

$$u_t := \int_2^t e^{2s}(\log s)^{-1}\,ds \sim \tfrac{1}{2}e^{2t}(\log t)^{-1}.$$

The results at (1·4) now follow from the law of the iterated logarithm for β. ∎

Return now to the general case of equation (1·1). Define

$$Y_t := \tilde{Y}_{\gamma(t)}, \quad W_t := \int_0^{\gamma(t)} \tilde{\epsilon}(s)\,d\tilde{B}_s,$$

where $\gamma(t)$ is as defined at (1·2). (We make the familiar notational switches such as that from γ_t to $\gamma(t)$ for typographical convenience.) Then W is a Brownian motion, and equation (1·1) takes the form

$$dY_t = -g(t, Y_t)\,dt + dW_t, \tag{1·5}$$

where

$$g(t, y) = \epsilon(t)^{-2} F(y). \tag{1·6}$$

From now on, our attention is focused on equation (1·5) rather than on equation (1·1), so that we think of our theorem in the form:

THEOREM 2. $Y_t \to 0$ (*a.s.*) *if and only if*

$$\mathrm{I}(c) := \int_0^\infty \exp\{-c\epsilon(u)^{-2}\} < \infty \tag{1·7}$$

for all $c > 0$.

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The method of proof is to compare the process Y with processes on which it is possible to do explicit calculations.

We mention one further illustrative example before beginning our proof of Theorem 1.

Example 2. Let $a(t)$ be a differentiable function on some interval of the form (t_0, ∞) such that

$$t^{-\frac{1}{2}}a(t) \text{ increases, while } t^{-1}a(t) \text{ decreases.} \tag{1·8}$$

Consider $\tilde{Y}(t) = a(t)^{-1}\tilde{B}(t)$, where $\tilde{B}$ is a $\mathrm{BM}^0(\mathbb{R})$ process. Then

$$d\tilde{Y}(t) = -a'(t)\,a(t)^{-1}\,\tilde{Y}(t)\,dt + a(t)^{-1}\,d\tilde{B}(t).$$

Our method shows that $\tilde{Y}(t) \to 0$ almost surely if and only if

$$\int_{t_0}^{\infty} a(u)^{-2} \exp\{-ca(u)\,a'(u)\}\,du < \infty$$

for all $c > 0$, while Kolmogorov's test (see §1·8 of Itô and McKean [**2**]) implies the criterion

$$\int_{t_0}^{\infty} u^{-\frac{3}{2}}a(u) \exp\{-cu^{-1}a(u)^2\}\,du < \infty$$

for all $c > 0$. We leave it to the reader to show that each of these criteria is equivalent to the statement

$$(t \log\log t)^{\frac{1}{2}}/a(t) \to 0.$$

Of course, we can prove the full version of Kolmogorov's test by applying our excursion argument to Ornstein–Uhlenbeck processes; this is very similar to Motoo's proof of the Kolmogorov result in §4·12 of Itô and McKean [**2**].

Brief discussion of annealing. Consider equation (1·1) when $F = \nabla U$, where U is a potential function perhaps possessing several local minima, but having a unique global minimum at the point c. Roughly speaking, the idea of the method of annealing is that if $\epsilon(t)$ behaves like a suitable multiple of $(\log t)^{-\frac{1}{2}}$, then the process $\tilde{Y}(t)$ converges *in distribution* to the point c. Our theorem (and Example 1) show that $\tilde{Y}(t)$ does not tend to c almost surely and that, if the potential barrier is not too high, then the $\tilde{Y}$ process will continually (but ever more rarely) tunnel out of the deepest well. References to some papers on annealing are included in the following paper by Chan.

2. *The building block*

Let b be a positive constant, and let B be a Brownian motion on $\mathbb{R}$. Skorokhod showed (see, for example, §V·6 of Rogers and Williams [**3**]) that there is a unique pair of processes (X, L) such that, firstly,

$$dX_t = dB_t - b\,dt + dL_t, \quad X_0 = 0, \tag{2·1}$$

and, secondly,

$$L \text{ is a continuous non-decreasing process which grows only when } x = 0. \tag{2·2}$$

Indeed, we have the explicit formula

$$L_t = \sup\{-(B_s - bs) : s \leqslant t\}.$$

Of course, L is local time at 0 for X.

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Now expand the sample space (for comments on this, see §4) to support an exponentially distributed random variable V of rate λ:

$$\mathbb{P}(V > t) = e^{-\lambda t} \quad (t \geqslant 0), \tag{2·3}$$

V being independent of B. Define

$$T := \inf\{t : L_t > V\}, \quad M := \sup\{X_t : t < T\}. \tag{2·4}$$

We say that $\{X_t : t < T\}$ is *elastic Brownian motion on* $[0, \infty)$, *with drift* $-b$ *and killing rate* λ, *driven by the Brownian motion* B, and write

$$\{X_t : t < T\} = \mathrm{EBMD}(-b, \lambda, B, V). \tag{2·5}$$

Note that $X(T-) = 0$. The process $\{X_t : t < T\}$ is a diffusion on $[0, \infty)$ with generator $\mathscr{G}$ satisfying

$$\mathscr{G}f = \tfrac{1}{2}f'' - bf' \quad \text{on} \quad \mathscr{D}(\mathscr{G}) = \{f : f'(0) = \lambda f(0)\}. \tag{2·6}$$

(Here, it is understood that X is a diffusion *relative to its own natural filtration*. Of course, the process X is *not* Markovian relative to a filtration $\{\mathscr{H}_t\}$ under which V is $\mathscr{H}_0$ measurable. The relevance of this remark will be apparent later.)

If, for the moment, we introduce the usual $\{\mathbb{P}^x : x \geqslant 0\}$ laws for the diffusion X with generator $\mathscr{G}$ as at (2·6), then, for $\delta > 0$, the function r on $[0, \delta)$ defined by

$$r(x) := \mathbb{P}^x(M > \delta), \quad x \in [0, \delta)$$

satisfies $\mathscr{G}r = 0$ on $[0, \delta)$ with $r(\delta) = 1$. Hence, for our $\mathbb{P} = \mathbb{P}^0$ measure,

$$\mathbb{P}(M > \delta) = 2b\{2b + \lambda(e^{2b\delta} - 1)\}^{-1} \quad (\delta > 0). \tag{2·7}$$

Next, for $\alpha > 0$, the bounded function q on $[0, \infty)$ defined by

$$q(x) := \mathbb{E}^x \int_0^T e^{-\alpha t}\, dt = \alpha^{-1}\{1 - \mathbb{E}^x e^{-\alpha T}\}$$

satisfies $(\alpha - \mathscr{G})q = 1$. Hence

$$\mathbb{E}\, e^{-\alpha T} = \lambda\{\lambda + (b^2 + 2\alpha)^{\frac{1}{2}} - b\}^{-1} \quad (\alpha > 0). \tag{2·8}$$

We shall be concerned only with the case when

$$\lambda b = 1. \tag{2·9}$$

In that case, it is immediate from (2·8) that

$$\mathbb{E}T = 1, \quad \mathrm{var}(T) = 1 + b^{-2}. \tag{2·10}$$

The essential intuitive point is that when (2·9) holds and b is large, then T has distribution close to the exponential distribution of rate 1.

3. *Comparison processes*

Suppose that $(b_k : k \in \mathbb{N})$ is a sequence of positive constants with $b_k \uparrow \infty$. Suppose further that $(V_k : k \in \mathbb{N})$ is a family of independent random variables, V_k being exponentially distributed with rate $\lambda_k := b_k^{-1}$, and that B is a Brownian motion independent of the sequence $(V_k : k \in \mathbb{N})$. Define recursively

$$S_0 := 0, \tag{3·1 (i)}$$

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$$\{X_t^{(k)}: t < T_k\} := \mathrm{EBMD}\,(-b_k, \lambda_k, B(S_{k-1}+\cdot), V_k), \quad (k \geqslant 1), \tag{3·1 (ii)}$$

$$S_k := S_{k-1} + T_k. \tag{3·1 (iii)}$$

Of course, $B(S_{k-1}+\cdot)$ signifies the Brownian motion

$$\{B(S_{k-1}+t): t \geqslant 0\};$$

see §4 for comments on this. Finally, define

$$Z_t := X^{(k)}(t - S_{k-1}), \quad (t \in [S_{k-1}, S_k)). \tag{3·1 (iv)}$$

We shall see shortly that $S_k \to \infty$ (a.s.), so that the 'comparison process' Z is (a.s.) defined for all t. We write

$$Z = \mathrm{COMP}\,(-b_k, V_k : k \in \mathbb{N}; B), \tag{3·2}$$

noting that the 'parameters of the function COMP' allow us to construct the paths of Z.

If we define the 'drift process' h via

$$h_t := b_k, \quad t \in [S_{k-1}, S_k), \tag{3·3}$$

then

$$dZ_t = dB_t - h_t\,dt + dL_t^Z, \tag{3·4}$$

where L^Z is a continuous-decreasing process which grows only when $Z = 0$.

Because $T_1, T_2, T_3, \ldots$ are independent, and

$$\mathbb{E}(T_k) = 1, \quad \mathrm{var}\,(T_k) = 1 + b_k^{-2} \to 1,$$

the usual argument based on Kronecker's Lemma shows that

$$S_k/k \to 1 \quad \text{(a.s.)}. \tag{3·5}$$

In particular, Z is (a.s.) defined for all t.

Let

$$M_k := \sup\{Z_t : t \in [S_{k-1}, S_k)\}.$$

Then, a.s. $Z_t \to 0$ if and only if $M_k \to 0$. Because of the Borel–Cantelli lemmas, result (2·7), and the facts that $b_k \to \infty$ and $\lambda_k = b_k^{-1}$, we see that

$$Z_t \to 0 \quad \text{(a.s.)} \quad \text{if and only if} \quad \Sigma \exp(-cb_k) < \infty \quad \text{for all} \quad c > 0. \tag{3·6}$$

4. *Extending the sample space, etc.*

We offer just a few words on this topic, and even these can well be skipped on a first reading of this paper.

Suppose that $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})^{(1)} = (\Omega^{(1)}, \mathscr{F}^{(1)}, \{\mathscr{F}_t^{(1)}\}, \mathbb{P}^{(1)})$ is a set-up carrying an $(\{\mathscr{F}_t^{(1)}\}, \mathbb{P}^{(1)})$ Brownian motion $B^{(1)}$. Suppose that $(\Omega, \mathscr{F}, \mathbb{P})^{(2)}$ is a triple supporting a sequence $(V_n^{(2)} : n \in \mathbb{N})$ of independent random variables, $V_n^{(2)}$ being exponentially distributed with rate n. Define

$$(\Omega, \mathscr{F}^0, \mathbb{P}) := (\Omega, \mathscr{F}, \mathbb{P})^{(1)} \times (\Omega, \mathscr{F}, \mathbb{P})^{(2)},$$

and, for $\omega = (\omega^{(1)}, \omega^{(2)}) \in \Omega$, define

$$B_t(\omega) := B_t^{(1)}(\omega^{(1)}), \quad V_n(\omega) := V_n^{(2)}(\omega^{(2)}).$$

We regard $(\Omega, \mathscr{F}, \mathbb{P})$ as an 'extension' of $(\Omega, \mathscr{F}, \mathbb{P})^{(1)}$ built to accommodate the sequence (V_n) independent of B, thinking of B as $B^{(1)}$ looked at in a new way.

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In connection with filtrations, there is something of a conflict between what is natural for the Markovian and stochastic-integral arguments. Put $b_1 := \lambda_1^{-1}$ and consider

$$(X_t : t < T_1) := \mathrm{EBMD}\,(-b_1, \lambda_1, V_1, B). \tag{4·1}$$

As already remarked, this process (sent to a coffin state at time T_1) is Markovian relative to its own natural filtration $\{X_t^\circ\}$ where

$$X_t^\circ := \sigma(\{\omega : s < T_1(\omega), X_s \in \Gamma\} : s \leqslant t, \Gamma \in \mathscr{B}(\mathbb{R})).$$

For the stochastic-integral theory, the natural filtration to use is obtained by setting

$$\mathscr{F}_t^\circ := \mathscr{F}_t^{(1)} \times \mathscr{F}^{(2)}$$

and letting $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$ be the usual augmentation of $(\Omega, \mathscr{F}^\circ, \{\mathscr{F}_t^\circ\}, \mathbb{P})$. Note that B is an $\{\mathscr{F}_t\}$ Brownian motion and that, with the obvious interpretation, the times (S_k) used in constructing comparison processes become $\{\mathscr{F}_t\}$ stopping times. But, as previously remarked, the process $\{X_t : t < T_1\}$ at (4·1) is not Markovian relative to $\{\mathscr{F}_t\}$.

In this paper, we can use the very special form of equation (1·7) to obtain comparison results based on random times which are not stopping times. However, for certain extensions of our method, more careful consideration of filtrations, stopping times, etc. is needed.

5. *Proof of the 'only if' part of Theorem 2*

Let Y be a weak solution of the stochastic differential equation (1·5), where g is at (1·6). Tanaka's formula shows that

$$d|Y_t| = -\operatorname{sgn}(Y_t) F(Y_t) \epsilon(t)^{-2}\, dt + dB_t + dL_t^Y, \tag{5·1}$$

where B denotes the $\mathrm{BM}^0(\mathbb{R})$ process derived from W via $dB_t = \operatorname{sgn}(Y_t)\, dW_t$, and L^Y is local time at 0 for Y.

Choose $K > 1$, and set

$$b_k := \epsilon(Kk)^{-2}, \quad \lambda_k := b_k^{-1}, \quad (k \in \mathbb{N}). \tag{5·2}$$

Expand the sample space to carry a family $(V_k : k \in \mathbb{N})$ of independent random variables, the family being independent of B, and V_k being exponentially distributed with rate λ_k. Now introduce

$$Z := \mathrm{COMP}\,(-b_k, V_k : k \in \mathbb{N}; B). \tag{5·3}$$

Then, from (3·4),

$$dZ_t = -h_t\, dt + dB_t + dL_t^Z, \tag{5·4}$$

where, from (5·2) and (3·5), there exists (a.s.) a random time ρ_1 such that

$$h_t \geqslant \epsilon(t)^{-2}, \quad (t \geqslant \rho_1).$$

The reader can easily check that the fact that such times as ρ_1 are not stopping times causes no trouble.

Assume now that $Y_t \to 0$ (a.s.). Then, since F is continuous, there exists (a.s.) a random time $\rho_2 \geqslant \rho_1$ such that $|F(Y_t)| \leqslant 1$ for $t \geqslant \rho_2$, whence

$$\operatorname{sgn}(Y_t) F(Y_t) \epsilon(t)^{-2} \leqslant h_t \quad (t \geqslant \rho_2). \tag{5·5}$$

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Almost surely, we can find a random time $\rho \geqslant \rho_2$ such that $Z_\rho = 0$. For example, we can take

$$\rho := \inf\{S_k : S_k \geqslant \rho_2\}$$

where (S_k) is the sequence of times used in the construction of Z. Obviously

$$0 = Z_\rho \leqslant |Y_\rho|. \tag{5·6}$$

We assume for the moment that

$$Z_t \leqslant |Y_t| \quad \text{for} \quad t \geqslant \rho. \tag{5·7}$$

Then, since $Y_t \to 0$ (a.s.), it follows that $Z_t \to 0$ (a.s.), and hence from (3.6) that

$$\Sigma \exp(-cb_k) = \Sigma \exp\{-c\epsilon(Kk)^{-2}\} < \infty$$

for all $c > 0$. The 'only if' part of Theorem 2 follows. ∎

Proof of (5·7). The proof of (5·7) is a pathwise argument, and we can regard the sample point ω as fixed.

Assume for the purposes of contradiction that for some $\eta > 0$,

$$\tau := \inf\{t > \rho : Z_t \geqslant |Y_t| + 2\eta\} < \infty, \tag{5·8}$$

and set $$\sigma := \sup\{t < \tau : Z_t \leqslant |Y_t| + \eta\}.$$

It follows from (5·6) that $\sigma > \rho$.

Between σ and τ we have $Z_\cdot > \eta$, so that

$$L^Z_\tau - L^Z_\sigma = 0.$$

Hence, on comparing (5·4) with (5·1) in the light of (5·5), we see that

$$Z_\tau - Z_\sigma \leqslant |Y_\tau| - |Y_\sigma|, \tag{5·9}$$

whereas our assumption (5·8) and the definitions of τ and σ imply that the left-hand side of (5·9) exceeds the right-hand side by η. This contradiction establishes (5·7). ∎

6. *Proof of the 'if' part of Theorem 2*

Again suppose that Y satisfies (1·5). Suppose now that $\mathrm{I}(c) < \infty$ for all $c > 0$, where $\mathrm{I}(c)$ is the integral at (1·7). We must prove that for every $\delta > 0$,

$$\limsup Y_t \leqslant \delta \quad \text{(a.s.)}. \tag{6·1}$$

(For the same argument will apply to $-Y$, and we shall have proved that $Y_t \to 0$ almost surely.)

Fix δ in $(0, \frac{1}{2}a_2)$, $K < 1$, and define

$$Z := \mathrm{COMP}(-b_k, V_k : k \in \mathbb{N}; W), \tag{6·2}$$

where $$b_k := F(\delta)\epsilon(K(k-1))^{-2} \tag{6·3}$$

and $(V_k : k \in \mathbb{N})$ is a sequence of independent random variables independent of W, V_k having the exponential distribution of rate $\lambda_k := b_k^{-1}$. Because $\mathrm{I}(c) < \infty$ for all $c > 0$, result (3·6) shows that

$$Z_t \to 0 \quad \text{(a.s.)}. \tag{6·4}$$

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We now have the following situation:

$$dZ_t = -h_t\,dt + dW_t + dL_t^z, \tag{6·5}$$

$$dY_t = -g(t, Y_t)\,dt + dW_t, \tag{6·6}$$

where it is almost surely true that

$$Y_\cdot \leqslant \delta \quad \text{at a sequence of times tending to } \infty, \tag{6·7(i)}$$

and for some random time ρ_2,

$$h_t \leqslant g(t, Y_t) \quad \text{whenever} \quad t \geqslant \rho_2 \quad \text{and} \quad Y_t \geqslant \delta. \tag{6·7(ii)}$$

Property (6·7(i)) is obvious from the definition of Y; and property (6·7(ii)) follows from (6·3) and (3·5).

Because of (6·7(i)), we can choose a random time $\rho > \rho_2$ at which

$$Y_\rho \leqslant Z_\rho + \delta. \tag{6·8}$$

We claim that

$$Y_t \leqslant Z_t + \delta \quad \text{for all} \quad t \geqslant \rho. \tag{6·9}$$

Once (6·9) is proved, result (6·1) follows from (6·4) and (6·9), and Theorem 2 is completely proved.

Proof of (6·9). The argument mirrors that used to prove (5·7). Since it is again a pathwise argument, we may regard ω as fixed.

Assume that for some $\eta > 0$,

$$\tau := \inf\{t > \rho : Y_t \geqslant Z_t + \delta + 2\eta\} < \infty,$$

and set

$$\sigma := \sup\{t < \tau : Y_t \leqslant Z_t + \delta + \eta\}.$$

By (6·8), we have $\sigma \geqslant \rho$. Between times σ and τ we have $Y_\cdot > \delta + \eta$ so that

$$g(t, Y_t) \geqslant h_t, \quad (\sigma \leqslant t \leqslant \tau). \tag{6·10}$$

On comparing (6·5) with (6·6) in the light of (6·10), we see that our assumption that

$$(Y_\tau - Y_\sigma) - (Z_\tau - Z_\sigma) = \eta > 0$$

is absurd. Result (6·9) is proved, and our paper is complete. ∎

The authors thank Richard Durran and Aubrey Truman, and also Martin Barlow (and John Hawkes), James Norris, and Chris Rogers.

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