Combinatorics, Probability and Computing (2026), 1–31
doi:10.1017/S0963548326100509

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ARTICLE

Spanning *H*-subdivisions and perfect *H*-subdivision tilings in dense digraphs

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(Received 15 January 2025 UTC; revised 3 April 2026 UTC; accepted 3 June 2026 UTC)

Abstract

Given a (di)graph H, we say that a (di)graph H' is an H-subdivision if H' is obtained from H by replacing one or more edges with internally vertex-disjoint paths. Pavez-Signé conjectured that for every $\varepsilon > 0$, there exists a constant $C_0 > 0$ such that for every graph H with h edges and no isolated vertices, if G is a graph on $n \geq C_0 h$ vertices and minimum degree $\delta(G) \geq (1+\varepsilon)\frac{n}{2}$, then G contains a spanning H-subdivision. This conjecture was later resolved by Lee [European J. Combin. **124** (2025), 104059]. In this paper, we strengthen Lee's result. Specifically, we prove that for any digraph D on $n \geq C_0 h$ vertices, if the minimum semi-degree of D is at least $\frac{n+h}{2} - 1$, then D contains a spanning H-subdivision. The lower bound on the minimum semi-degree is best possible. Furthermore, we show that there exist constants $C > 0$ and $\alpha, \beta \in (0,1)$ such that for any integer partition $n = n_1 + \cdots + n_m \geq Cm$ with $n_i \geq |V(H)| + 3h$ for each i, and $\sum_{n_i < \alpha n} n_i \leq \beta n$, if a digraph of order $n \geq Cm$ has minimum semi-degree at least $\frac{n+m+h}{2} - 1$, then it contains m vertex-disjoint H-subdivisions whose orders are $n_1, \ldots, n_m$, respectively. The bound $\frac{n+m+h}{2} - 1$ is also optimal. This work partly answers a conjecture of Lee [Combin. Probab. Comput. **34** (2025), 421–444] and generalises a recent result from the same paper.

Keywords: Digraph; H-subdivision; minimum semi-degree; absorption method; extremal condition

2020 MSC Codes: Primary: 05C07; Secondary: 05C20, 05C35, 05C70

1. Introduction

Given a (di)graph H, we say that a (di)graph H' is an *H-subdivision* if H' is obtained from H by replacing one or more edges with internally vertex-disjoint paths. Also, we call an H-subdivision *spanning* if it covers all vertices of the host (di)graph. Because subdivisions maintain the topological structure of the original (di)graph, various problems related to subdivisions were proposed and extensively studied. A particular focus has been on identifying a sufficient condition that ensures the host graph contains a subdivision of H, making this an important topic in extremal graph theory.

Recently, Pavez-Signé [24] showed that for every $\varepsilon > 0$, there exists a constant C_0 such that for all $C \geq C_0$ and $k \geq 2$ the following holds. Let $\log k \leq d \leq k$ and let G be a graph on $n = Cdk$ vertices and minimum degree at least $(1+\varepsilon)\frac{n}{2}$. Then G contains a spanning subdivision of every k-vertex d-regular graph. Here, a graph is *d-regular* if every vertex in it has the degree d. In the same paper, Pavez-Signé also proposed the following conjecture.

Conjecture 1.1 ([24, Conjecture 3.1]). *For every $\varepsilon > 0$, there exists a positive constant C_0 such that for all $C \geq C_0$ and any integer h the following holds. Let G be a graph on $n = Ch$ vertices and*

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minimum degree at least $(1+\varepsilon)\frac{n}{2}$. *Then G contains a spanning H-subdivision of every graph H with h edges and no isolated vertices.*

We define the *minimum semi-degree* of a digraph D as $\delta^0(D) = \min\{\delta^+(D), \delta^-(D)\}$ and the *minimum degree* $\delta(D) = \min_{x\in V}\{d(x) : d(x) = d^+(x) + d^-(x)\}$. Lee [19] solved Conjecture 1.1 in a stronger form by considering digraphs.

Theorem 1.2 ([19, Theorem 1.2]). *Let $\varepsilon > 0$ be a positive real number. Then there exists a constant $C_0 > 0$ such that for all $C \geq C_0$ the following holds. Let D be a digraph on $n \geq Ch$ vertices and $\delta^0(D) \geq (\frac{1}{2}+\varepsilon)n$. Then D contains a spanning subdivision of any digraph H with h arcs and no isolated vertices.*

By replacing each edge in the graph G by two opposite arcs and fixing an arbitrary orientation $\vec{H}$, one can directly deduce that Conjecture 1.1 follows from Theorem 1.2. In this paper, we improve the result above of Lee, by considering the H-subdivision problem in general digraphs under a sharp minimum semi-degree condition.

Theorem 1.3. *There exists a constant $C_0 > 0$ such that for every integer $C \geq C_0$ and any digraph H with h arcs and minimum degree $\delta(H) \geq 1$, the following holds. If D is a digraph on $n \geq Ch$ vertices with $\delta^0(D) \geq \frac{n+h}{2} - 1$, then D contains a spanning H-subdivision.*

Clearly, if H is a 2-cycle, then a spanning H-subdivision in D is exactly a Hamilton cycle in D. Since the tight lower bound of $\delta^0(D)$ for D to contain a Hamilton cycle is $n/2$, the semi-degree bound in Theorem 1.3 is tight.

Theorem 1.3 also allows us to make progress on the following problem posed by Babu and Diwan [1], in a stronger form.

Problem 1.4. [1] *For every fixed graph H with no isolated vertices, does there exist an integer $f(H)$ such that every graph of order $n \geq f(H)$ and minimum degree at least $(n+1)/2$ contains a spanning H-subdivision?*

The spanning subgraph structure is a special case of perfect subgraph tiling. Kühn and Osthus [17] gave an almost exact value of minimum degree to guarantee that a graph contains a perfect H-tiling for every graph H up to an additive constant depending only on H. There are several results related to the result of Kühn and Osthus, see references [2, 8, 10–12, 18, 22].

In particular, given a digraph H, a *perfect H-subdivision tiling* in a digraph D is a collection of vertex-disjoint subdivisions of H covering all vertices of D. It is natural to determine what the minimum degree threshold is to ensure the existence of a perfect H-subdivision tiling in a digraph. We use standard hierarchy notation, that is, we write $0 < \alpha \ll \gamma$ to indicate that α is chosen to be sufficiently small relative to γ such that all calculations needed in our proof are valid. In this paper, we also give the following result.

Theorem 1.5. *Let H be a digraph with h arcs and $\delta(H) \geq 1$. For any constants α, β with $0 < \alpha, \beta \ll 1$, there exists a constant $C_0 = C_0(H, \alpha, \beta)$ such that the following holds for every integer $C \geq C_0$ satisfying $0 < 1/C \ll \min\{\alpha, \beta\}$. Consider any integer partition $n = n_1 + \cdots + n_m$ such that $n_i \geq 3h + |V(H)|$ for each i, and $\sum_{n_i<\alpha n} n_i \leq \beta n$. If D is a digraph on $n \geq Cm$ vertices with $\delta^0(D) \geq \frac{n+m+h}{2} - 1$, then D contains a perfect H-subdivision tiling, in which, the orders of these H-subdivisions are $n_1, n_2, \ldots, n_m$, respectively.*

The bound on the minimum semi-degree is tight. We construct a counterexample following the idea from [23]. Consider the case where H is 2-cycle and choose the prescribed order $n_1, \ldots, n_m$ all odd, with $n_i \geq 8$ for each i. Here, the bound $\frac{n+m+h}{2} - 1$ simplifies to $\frac{n+m}{2}$, and a perfect tiling of H-subdivisions is equivalent to a perfect tiling of cycles. Let D be a complete 3-partite digraph with vertex classes X, Y, Z, where $|X| = m-1$, $|Y| = (n-m+2)/2$ and $|Z| = (n-m)/2$. Then $|V(D)| = n$. The semi-degrees are as follows: for any $x \in X$, $d^+(x) = d^-(x) = n-m+1$;

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for any $y \in Y$, $d^+(y) = d^-(y) = \frac{n+m}{2} - 1$; and for any $z \in Z$, $d^+(z) = d^-(z) = \frac{n+m}{2}$. Given that $n \geq Cm$ and $1/C \ll 1$, we have $n - m + 1 > \frac{n+m}{2} - 1$. Consequently, the minimum semi-degree is $\delta^0(D) = \frac{n+m}{2} - 1$. We now show that D contains no perfect tiling of m disjoint odd cycles. The key observation is that in this digraph, every odd cycle must contain at least one vertex from X. Since $|X| = m - 1 < m$, it is impossible to pack m such cycles. This completes the counterexample, establishing that the bound is best possible.

In the same way, replacing edges of the graph G with two arcs in both directions and fixing an arbitrary orientation $\vec{H}$, we obtain the following corollary from the theorem above. This corollary extends a result of Lee [20] that recently gave an asymptotic answer to the question on perfect H-subdivision tilings for sufficiently large graphs. Furthermore, this corollary also provides a partial answer to the conjecture raised in his same paper [20, Conjecture 6.1], which predicts the exact minimum degree condition that guarantees the existence of a perfect H-subdivision tiling in a graph.

Corollary 1.6. *There exists a constant $C_0 > 0$ such that for every integer $C \geq C_0$ and two constants α, β with $0 \ll 1/C \ll \alpha, \beta \ll 1$, the following holds. Suppose H is a graph with h edges and $\delta(H) \geq 1$. For any integer partition $n = n_1 + \cdots + n_m$ such that $n_i \geq 3h + |V(H)|$ for each i, and $\sum_{n_i < \alpha n} n_i \leq \beta n$, if G is a graph on $n \geq Cm$ vertices with $\delta(G) \geq \frac{n+m+h}{2} - 1$, then G contains a perfect H-subdivision tiling of order $n_1, \ldots, n_m$, respectively.*

Theorem 1.5 can also derive many disjoint subdigraph structures. For example, if H is a cycle of length 2, then Theorem 1.5 implies that there exist constants $C_0 > 0$ and α, β with $0 \ll 1/C_0 \ll \alpha, \beta \ll 1$ such that the following holds. For any integer partition $n = n_1 + \cdots + n_m \geq C_0 m$ such that $n_i \geq 8$ for each i, and $\sum_{n_i < \alpha n} n_i \leq \beta n$, if D is an n-vertex digraph with $\delta^0(D) \geq \frac{n+m}{2}$, then it contains m disjoint cycles of length $n_1, \ldots, n_m$, respectively. For other cycle results in digraphs, we refer to the references [7, 14–16, 21, 23, 26, 27].

Also, it is worth noting that in the proofs of Theorems 1.3 and 1.5, we need to utilise the absorption method that was introduced by Rödl, Ruciński and Szemerédi [25] as well as the stability method. Also, this work is a follow-up work to [5], and the proof method in this paper closely follows the strategy in [5].

Organisation. The rest of the paper is organised as follows. In Section 2, we begin by presenting relevant definitions and notations, followed by an outline of the proofs of Theorems 1.3 and 1.5. Section 3 presents the proofs of Theorems 1.3 and 1.5 for non-extremal case. In Subsection 3.1, we prove the absorbing and path-covering lemmas. In Subsection 3.2, we complete the proofs of Theorems 1.3 and 1.5 for non-extremal case. In Section 4, we first identify three extremal cases that D can belong to when it satisfies the extremal condition, and then we will prove that Theorems 1.3 and 1.5 hold in each of these cases. Finally, Section 5 contains some concluding remarks to wrap up the paper.

2. Preparations for Theorems 1.3 and 1.5

2.1 Definitions and notations

For notations not defined in this paper, we refer the reader to [3]. Let $D = (V, A)$ be a digraph. The *out-neighbourhood* (resp., *in-neighbourhood*) of a vertex v in D is defined as $N^+(v) = \{u : vu \in A\}$ (resp., $N^-(v) = \{w : wv \in A\}$). The *out-degree* (resp., *in-degree*) of v in D, which is denoted by $d^+(v)$ (resp., $d^-(v)$), is the cardinality of $N^+(v)$ (resp., $N^-(v)$), that is, $d^+(v) = |N^+(v)|$ (resp., $d^-(v) = |N^-(v)|$). The *minimum out-degree* $\delta^+(D) = \min\{d^+(v) : v \in V\}$ and the *minimum in-degree* $\delta^-(D) = \min\{d^-(v) : v \in V\}$. For any $X \subseteq V$ and $\sigma \in \{-, +\}$, we define $N^\sigma(u, X) = N^\sigma(u) \cap X$ and $d^\sigma_X(u) = |N^\sigma(u, X)|$ for any vertex u in V. The cardinality of X is denoted by $|X|$, and we say X is an i-set if $|X| = i$. The subdigraph of D induced by X is denoted as $D[X]$. For another vertex set Y that is not necessarily disjoint from X, we use $e^+(X, Y)$ to represent the number of arcs from X to Y.

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Given digraphs D and H, let $\mathcal{P}(D)$ be the family of paths in D. An *H-subdivision* in D is a pair of mappings $f : V(H) \to V(D)$ and $g : A(H) \to \mathcal{P}(D)$ such that (*a*) $f(u) \neq f(v)$ for $u, v \in V(H)$ with $u \neq v$, and (*b*) for every arc $uv \in A(H)$, $g(uv)$ is a path from $f(u)$ to $f(v)$, and different arcs are mapped into internally vertex-disjoint paths in D.

We define the number of arcs of a path as its *length* and a *k-path* refers to a path of order k. We often represent the k-path P as $v_1 \cdots v_k$ when $V(P) = \{v_1, \ldots, v_k\}$ and call v_1 and v_k the *initial* and the *terminal* of P, respectively. Furthermore, for two disjoint vertex sets X and Y in V, if the initial and the terminal of P belongs to X and Y, respectively, then we say that P is an (X, Y)-path. In particular, we write an X-path instead of (X, X)-path if $X = Y$. Additionally, we say an (X, Y)-path P is *minimal* if there is no (X, Y)-path P' with $|V(P')| < |V(P)|$. All paths in digraphs refer to directed paths, and we use the term *disjoint* instead of vertex-disjoint for simplicity. Given a family of graphs $\mathcal{F}$, denote by $|\mathcal{F}|$ the number of graphs in $\mathcal{F}$ and we write $V(\mathcal{F}) = \bigcup_{T \in \mathcal{F}} V(T)$.

For a positive integer t, we simply write $\{1, \ldots, t\}$ as $[t]$. Throughout this paper, we will omit floor and ceiling signs when they are not essential. For the real numbers a and b, we use $a \pm b$ to represent an unspecified real number in the interval $[a - b, a + b]$.

To summarise this subsection, we provide the following extremal condition for a constant ε', where $0 < \varepsilon' \ll 1$. In particular, we say the digraph D is *stable with parameter ε'* (for simplicity, *ε'-stable*) if D does not satisfy the following extremal condition (**EC**).

Extremal condition with parameter ε'(EC(ε')): Let D be a digraph of order n. There exist two (not necessarily disjoint) vertex sets U_1 and U_2 in D with $|U_i| \geq (1/2 - \varepsilon')n$ for every $i \in [2]$ such that $e^+(U_1, U_2) \leq (\varepsilon' n)^2$.

Based on the extremal condition above, we now introduce a key combinatorial lemma concerning the internal structure of digraphs, which will serve as an essential technical tool in the analysis of host digraphs satisfying this condition.

For a digraph H, a pair (U_1, U_2) is called a *bipartition* of H, written as $H = (U_1, U_2)$, if $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 = V(H)$, and $e(H) = e(U_1) \cup e(U_2) \cup e(U_1, U_2)$. Here, $e(X)$ denotes the set of arcs in the induced subdigraph $H[X]$ with $X \in \{U_1, U_2\}$, and $e(U_1, U_2)$ represents the arcs in the bipartite digraph (U_1, U_2). Furthermore, we also define the quantity $m_H = \min_{H=(U_1,U_2)} \big||U_1| + e(U_2) - (|U_2| + e(U_1))\big|$.

Lemma 2.1. *Let H be a digraph with $\delta(H) \geq 1$, and let $h := e(H) \geq 2$. Then there exists a bipartition (U_1, U_2) of $V(H)$ such that $\big||U_1| + e(U_2) - (|U_2| + e(U_1))\big| \leq h/2 - 1$.*

Proof. For a bipartition (U_1, U_2) of $V(H)$, define $f(U_1, U_2) := |U_1| + e(U_2) - (|U_2| + e(U_1))$. Choose a bipartition (U_1, U_2) such that $|f(U_1, U_2)|$ is minimal. Suppose for a contradiction that $|f(U_1, U_2)| > h/2 - 1$. Since $|f(U_1, U_2)|$ is an integer, this implies $|f(U_1, U_2)| \geq \lfloor h/2 \rfloor \geq 1$. By swapping U_1 and U_2 if necessary, we may assume that $t := f(U_1, U_2) > 0$.

Claim 2.2. *Every vertex in U_1 has total degree at least* 2.

Proof. Let $u \in U_1$. Moving u from U_1 to U_2 yields a new bipartition (U_1', U_2'). The change in f is $f(U_1', U_2') = f(U_1, U_2) + d(u) - 2 = t + d(u) - 2$. By the minimality of $|f(U_1, U_2)|$, we have $|t + d(u) - 2| \geq t$. Hence, either $d(u) - 2 \geq 0$ or $d(u) - 2 \leq -2t$. If the latter holds, then $d(u) \leq 2 - 2t \leq 0$ (since $t \geq 1$), contradicting the assumption that $\delta(H) \geq 1$. Therefore, $d(u) \geq 2$ for all $u \in U_1$. □

Claim 2.3. *Every vertex in U_2 has total degree at most* 2.

Proof. Let $v \in U_2$. Moving v from U_2 to U_1 yielding a bipartition (U_1'', U_2'') with $f(U_1'', U_2'') = t + 2 - d(v)$. Again, the minimality implies $|t + 2 - d(v)| \geq t$. Thus, either $2 - d(v) \geq 0$ or $2 - d(v) \leq -2t$. If the latter holds, then $d(v) \geq 2 + 2t$. Recalling that $t \geq \lfloor h/2 \rfloor$, we get $d(v) \geq 2t + 2 \geq 2\lfloor h/2 \rfloor + 2 > h$, which contradicts the obvious fact that $d(v) \leq e(H) = h$. Hence, $d(v) \leq 2$ for all $v \in U_2$. □

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Since each arc in U_1 contributes 2 to $\sum_{u\in U_1} d(u)$, each arc in U_2 contributes 2 to $\sum_{v\in U_2} d(v)$, and each crossing arc contributes 1 to each sum, we have

$$\sum_{u\in U_1} d(u) = 2e(U_1) + e(U_1, U_2), \quad \text{and} \quad \sum_{v\in U_2} d(v) = 2e(U_2) + e(U_1, U_2).$$

Combining these with the degree bounds from Claims 2.2 and 2.3, we obtain that

$$2|U_1| \le \sum_{u\in U_1} d(u) = 2e(U_1) + e(U_1, U_2), \quad \text{and} \tag{2.1}$$

$$\sum_{v\in U_2} d(v) = 2e(U_2) + e(U_1, U_2) \le 2|U_2|. \tag{2.2}$$

Recall that $h = e(H) = e(U_1) + e(U_2) + e(U_1, U_2)$ and $t = f(U_1, U_2) = |U_1| + e(U_2) - (|U_2| + e(U_1))$. We now derive bounds on the number of vertices $|V(H)| = |U_1| + |U_2|$. From equation (2.1), we have $e(U_1, U_2) \ge 2|U_1| - 2e(U_1)$. Substituting into the expression for h gives $h \ge e(U_1) + e(U_2) + 2|U_1| - 2e(U_1) = e(U_2) - e(U_1) + 2|U_1|$. Using the definition of t, we have $e(U_2) - e(U_1) = t - |U_1| + |U_2|$. Hence, $h \ge t - |U_1| + |U_2| + 2|U_1| = t + |U_1| + |U_2|$. So, $|V(H)| \le h - t$.

Similarly, for equation (2.2) we obtain $e(U_1, U_2) \le 2|U_2| - 2e(U_2)$. Substituting into h yields $h \le e(U_1) + e(U_2) + 2|U_2| - 2e(U_2) = e(U_1) - e(U_2) + 2|U_2|$. Since $e(U_1) - e(U_2) = |U_1| - |U_2| - t$, we get $h \le |U_1| - |U_2| - t + 2|U_2| = |U_1| + |U_2| - t$. So $|V(H)| \ge h + t$. Hence, we obtain $h + t \le |V(H)| \le h - t$, which implies $t \le 0$. However, by our assumption for contradiction, we have $t > 0$. Thus we have a contradiction. Therefore, the assumption is false, and we must have $t \le \frac{h}{2} - 1$. This completes the proof. □

Observation. The upper bound $h/2 - 1$ in Lemma 2.1 is tight. This can be seen by considering the case where H is a 2-cycle. Then $m_H = 0$ and $h/2 - 1 = 0$, which shows that the bound cannot be reduced further.

2.2 Proof structure for the main theorems

Let H be a digraph with h arcs and $\delta(H) \ge 1$, and let D be a digraph as described in Theorems 1.3 and 1.5. Since the proofs of Theorem 1.3 and Theorem 1.5 follow a similar line of reasoning, we will only present the detailed proof strategy for Theorem 1.3. The proof of Theorem 1.3 utilises the stability method, which is divided into two main cases:

The **extremal case**, when the digraph D is not ε'-stable;

The **non-extremal case**, when D is ε'-stable.

- **Non-extremal case**

For the non-extremal case, we divide the proof into the following two steps:

Step 1. Absorbing Lemma (Construction of an H-subdivision subdigraph). By the extremal condition with parameter ε and the probabilistic method, we construct an *absorbing* subdigraph H' that is a subdivision of H and possesses the remarkable property that for every vertex pair (u, v) in $D - H'$, every 'long' subdivided path inside H' contains at least one absorber for (u, v).

Step 2. Path-Covering Lemma (Lemma 3.6 in Subsection 3.1). This lemma allows us to cover all vertices of $D - H'$ with a bounded number of disjoint paths of arbitrary prescribed lengths.

Finally, by exploiting the absorbing property of H', we will absorb these disjoint paths of suitable lengths into H' to obtain the desired spanning H-subdivision. This completes the proof for the non-extremal case.

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The proof of Theorem 1.5 differs from that of Theorem 1.3 only in Step 1. Suppose that $n_1 \geq \cdots \geq n_t \geq \alpha n > n_{t+1} \geq \cdots \geq n_m$. Then we will construct t disjoint *absorbing* H-subdivisions $H'_1, \ldots, H'_t$ of orders are at most $n_1, \ldots, n_t$, respectively, and subsequently construct $(m-t)$ disjoint H-subdivisions, whose orders are $n_{t+1}, \ldots, n_m$, respectively.

- **Extremal case**

Step 3. For the extremal case, we show via the conventional structural method that D satisfies one of three extremal structures: **EC1**(ε), **EC2**(ε), or **EC3**($\varepsilon, \varepsilon_1$). Their descriptions are as follows.

In EC1(ε), with $O(\varepsilon n)$ exceptional vertices, the remaining vertices can be partitioned into two almost equal subsets W_1, W_2 such that each $D[W_i]$ is almost complete. In EC2(ε), analogously, after removing few exceptional vertices, the vertex set partitions into two almost equal subsets W_1, W_2 that form an almost complete bipartite digraph. In EC3($\varepsilon, \varepsilon_1$), again with few exceptional vertices, the remaining vertices of $V(D)$ can be partitioned into four disjoint vertex sets W_1, W_2, W_3, and W_4 such that $|W_1| \approx |W_3|$ and $|W_2| \approx |W_4|$; here, $D[W_1]$ and $D[W_3]$ are almost complete, and (W_2, W_4) is an almost complete bipartite digraph.

The proofs of Theorems 1.3 and 1.5 are completed in each extremal case through a detailed structural analysis.

3. Non-extremal case: Proofs of Theorems 1.3 and 1.5

Actually, we prove Theorems 1.3 and 1.5 in a stronger form. We show that if D satisfies the conditions of Theorem 1.3, then the following holds. For any sequence $l_1, \ldots, l_h$ with each $l_i \geq 3$ and $\sum_{i=1}^{h}(l_i - 1) + |V(H)| = n$, if either D is ε'-stable and the sum of the l_i less than αn is no more than βn, or D is not ε'-stable and $\max\{l_1, \ldots, l_h\} \leq (1/2 + 2\varepsilon^{1/2})n$, where $\alpha, \beta, \varepsilon'$ and ε are constants with $0 < \alpha, \beta \ll \varepsilon' \ll \varepsilon \ll 1$, then D contains a spanning H-subdivision, in which the paths have lengths $l_1, \ldots, l_h$, respectively. In addition, Theorem 1.5 holds if either D is ε'-stable and f is any mapping from $V(H)$ to $V(D)$, or D is not ε'-stable.

3.1 Absorbing and path-covering lemmas

Let $C_0 > 0$ be a constant and C be any integer with $C \geq C_0$, as shown in Theorems 1.3 and 1.5. Parameters ε', ρ and γ are chosen such that

$$0 < 1/C \ll \gamma \ll \rho \ll \varepsilon' \ll 1.$$

The following standard inequalities will be frequently used.

Lemma 3.1. [13] *Let X be a random variable with the expectation $\mathbb{E}X$, and a is any real number with $0 < a < 3/2$. Then the following statements hold.*

(1) *(Chernoff's inequality)* $\mathbb{P}(|X - \mathbb{E}X| > a\mathbb{E}X) < 2e^{-\frac{a^2}{3}\mathbb{E}X}$.
(2) *(Markov's inequality)* $\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}X}{a}$.

For a vertex pair (u, v) (possibly, $u = v$), we say that a 4-path $z_1z_2z_3z_4$ *absorbs* (u, v) if $z_2u, vz_3 \in A$. Also, a 4-path is called an *absorber* for (u, v) if it absorbs (u, v). This terminology reflects the fact that the 4-path $z_1z_2z_3z_4$ can be extended by absorbing a path with end-vertices u and v, resulting in a longer path with the same set of end-vertices. We say that a set F of disjoint subdigraphs absorbs a family U of vertex pairs if for every pair in U, there exists a 4-path in F that absorbs it, and the absorbing 4-paths for distinct vertices are distinct. Finally, for any vertex pair (u, v) (possibly $u = v$) in D, we denote by $\mathcal{A}_{uv}$ the family of all 4-paths absorbing (u, v). Then, we can conclude that for any vertex pair of D, there are at least γn^4 different 4-paths to absorb it.

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Fact 3.2. *Let D be an n-vertex digraph with $\delta^0(D) \geq (1/2-\rho)n$, and suppose that D is ε'-stable. The parameters ρ and ε' satisfy $0 < 1/C \ll \rho \ll \varepsilon' \ll 1$. Then for any vertex pair (u, v) of D, we have that $|\mathcal{A}_{uv}| \geq \rho n^4$.*

Proof. Let $U_1 = N^-(u)$ and $U_2 = N^+(v)$. By the lower bound of $\delta^0(D)$, we have that $|U_i| \geq (1/2-\rho)n$ for every $i \in [2]$. Since D is ε'-stable, we obtain that $e^+(U_1, U_2) > (\varepsilon' n)^2$. Then again by the lower bound of $\delta^0(D)$, we can deduce that for any given arc z_2z_3 with the vertex $z_2 \in U_1$ and $z_3 \in U_2$, we have that $|N^-(z_2) \setminus \{u, v, z_3\}| \geq (1/2-\rho)n - 3$ and $|N^+(z_3) \setminus \{u, v, z_2\}| \geq (1/2-\rho)n - 3$. This implies that the number of 4-paths $z_1z_2z_3z_4$ with $z_1 \in N^-(z_2) \setminus \{u, v, z_3\}$ and $z_4 \in N^+(z_3) \setminus \{u, v, z_1, z_2\}$ absorbing (u, v) is at least

$$(\varepsilon' n)^2 \cdot ((1/2-\rho)n - 3) \cdot ((1/2-\rho)n - 4) \geq (\varepsilon')^2 n^4/5 \geq \rho n^4.$$

This completes the proof of Fact 3.2. □

Before presenting the absorption lemma, it is necessary to introduce the following useful lemmas. Lemma 3.3 below states that for a digraph D with sufficiently large minimum semi-degree, we can get a small set $\mathcal{F}$ of disjoint absorbers, satisfying that for any vertex pair (u, v) in D, there are enough absorbers in $\mathcal{F}$ to absorb (u, v). Further, Lemma 3.4 below shows that we can partition this set $\mathcal{F}$ into l disjoint parts such that in each part, there is at least one absorber for any vertex pair (u, v), and that for each part, by using some vertices in $V(D - \mathcal{F})$ and the extremal condition with parameter ε (EC(ε)), we can get a path covering all absorbers of this part.

Lemma 3.3. *Let D be an n-vertex digraph with $\delta^0(D) \geq (1/2-\gamma)n$, and D is ε'-stable. Then there exists a family $\mathcal{F}$ of at most γn disjoint absorbers in D such that for every vertex pair (u, v), we have $|\mathcal{A}_{uv} \cap \mathcal{F}| \geq \gamma^2 n$.*

Proof. Let γ_1 be a real number such that $\gamma_1^2/4 \geq \gamma$ and $16\gamma_1^3 \leq \gamma$. We select a family $\mathcal{F}'$ of ordered 4-sets from $[V(D)]^4$ at random by including each of $n(n-1)(n-2)(n-3) \sim n^4$ of them independently with probability $\gamma_1^3 n^{-3}$ (some of the selected 4-sets may not be absorbing at all). By Chernoff's inequality, since $\mathbb{E}|\mathcal{F}'| = n^4 \cdot \gamma_1^3 n^{-3} = \gamma_1^3 n$, we have:

$$\mathbb{P}(|\mathcal{F}'| \geq 2\gamma_1^3 n) \leq \mathbb{P}(||\mathcal{F}'| - \mathbb{E}|\mathcal{F}'|| \geq \gamma_1^3 n) \leq 2e^{-\frac{1}{3}\mathbb{E}|\mathcal{F}'|}.$$

Similarly, for every vertex pair (u, v), by Fact 3.2 applied with $\rho = \gamma_1$, we get that $\mathbb{E}|\mathcal{A}_{uv} \cap \mathcal{F}'| = \gamma_1 n^4 \cdot \gamma_1^3 n^{-3} = \gamma_1^4 n$. So by Chernoff's inequality again, we have that $\mathbb{P}(|\mathcal{A}_{uv} \cap \mathcal{F}'| \leq \gamma_1^4 n/3) \leq \mathbb{P}(||\mathcal{A}_{uv} \cap \mathcal{F}'| - \mathbb{E}|\mathcal{A}_{uv} \cap \mathcal{F}'|| \geq \gamma\gamma_1^3 n - \gamma_1^4 n/3) \leq 2e^{-\frac{1}{12}\mathbb{E}|\mathcal{A}_{uv} \cap \mathcal{F}'|}$. Thus, with probability $1 - o(1)$, as $n \to \infty$:

(1) $|\mathcal{F}'| < 2\gamma_1^3 n \leq \gamma n$, *and* $|\mathcal{A}_{uv} \cap \mathcal{F}'| > \gamma_1^4 n/3$ *for every vertex pair* (u, v).

Clearly, the expected number of intersecting pairs of 4-sets in $\mathcal{F}'$ is at most

$$n^4 \times \left(C_4^1 n^3 + C_4^2 n^2 + C_4^3 n\right) \times \left(\gamma_1^3 n^{-3}\right)^2 \leq n^4 \times 4 \times 4 \times n^3 \times \left(\gamma_1^3 n^{-3}\right)^2 = 16\gamma_1^6 n,$$

so by Markov's inequality: with $X :=$ the number of intersecting pairs of 4-sets in $\mathcal{F}'$ and $a = 17\gamma_1^6 n$, we can get that $\mathbb{P}(X \geq 17\gamma_1^6 n) \leq \frac{\mathbb{E}X}{a} = \frac{16\gamma_1^6 n}{17\gamma_1^6 n} = 16/17$. This implies that

(2) *with probability at least* $1/17$, *as* $n \to \infty$, *there are at most* $17\gamma_1^6 n$ *pairs of intersecting* 4*-sets in* $\mathcal{F}'$.

Hence, with positive probability, the family $\mathcal{F}'$ satisfies the properties both (1) and (2), which implies that there exists one such family, and, for simplicity, we define this family to be $\mathcal{F}''$. From $\mathcal{F}''$ we delete all 4-sets that intersect other 4-sets, as well as all 4-sets that are not absorbers, and denote by $\mathcal{F}$ the remaining subfamily. Clearly, by (1) and (2) again, we have that

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$|\mathcal{F}| \geq 2\gamma_1^3 n - 17\gamma_1^6 n \geq 3\gamma_1^3 n/2$. Also, the family $\mathcal{F}$ consists of disjoint absorbers and by (2) and (1), for every vertex pair (u, v),

$$|\mathcal{A}_{uv} \cap \mathcal{F}| > \frac{\gamma_1^4 n}{3} - 2 \cdot 17\gamma_1^6 n > \frac{\gamma_1^4 n}{4} \geq \gamma^2 n.$$

This completes the proof of the lemma. □

For two paths $P = a \cdots b$ and $Q = b \cdots d$ with $V(P) \cap V(Q) = \{b\}$, we denote the concatenated path as $P \circ Q$. This definition can be extended naturally to more than two paths. Further, the following conclusion holds.

Lemma 3.4. *Let D be an n-vertex digraph with $\delta^0(D) \geq (1/2 - \gamma)n$, and D is ε'-stable. Let $\mathcal{F}$ be the family with $|\mathcal{F}| = f$ obtained in Lemma 3.3. For any real β with $\gamma^2 < \beta < 1 - \gamma^2$ and any partition $f = f_1 + \cdots + f_l$ with $\beta f < f_i < (1 - \beta)f$ for each $i \in [l]$, where $l \geq 2$, there exists a partition $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_l$ with $|\mathcal{F}_i| = f_i$ such that*

(i) for any vertex pair (u, v) of $D - V(\mathcal{F})$ and any $i \in [l]$, $\mathcal{F}_i$ contains at least one absorber for (u, v).

(ii) for each $i \in [l]$, we further define $\mathcal{F}_i = \{F_{i,1}, \ldots, F_{i,f_i}\}$. Then there exists a path L_i in D of the form $L_i = F_{i,1} \circ P_{i,1} \circ F_{i,2} \cdots \circ F_{i,f_i-1} \circ P_{i,f_i-1} \circ F_{i,f_i}$, where for each $j \in [f_i - 1]$, the path $P_{i,j}$ has a length at most 3, and all paths L_i are disjoint.

Proof. Recall that the parameters γ_1, γ and ε' satisfy $16\gamma_1^3 \leq \gamma \ll \varepsilon'$. In the following, we first prove (i), and second give the proof of (ii).

We prove that there exists a partition $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_l$ satisfying our conditions with high probability. Let F_{uv}^j be a random variable that calculates the number of absorbers of (u, v) in $\mathcal{F}_j$ for $j \in [l]$. Since Lemma 3.3 gives $|\mathcal{A}_{uv} \cap \mathcal{F}| \geq \gamma^2 n$ and $|\mathcal{F}| < \gamma n$, it is not hard to see that

$$\mathbb{E}F_{uv}^j = \frac{|\mathcal{A}_{uv} \cap \mathcal{F}|}{|\mathcal{F}|} \cdot |\mathcal{F}_j| \geq \frac{\gamma^2 n}{\gamma n} \cdot |\mathcal{F}_j| = \gamma |\mathcal{F}_j|,$$

Then by Chernoff's bound with $a = \frac{1}{2}$, for any $j \in [l]$, due to $\beta f \leq |\mathcal{F}_j| \leq (1 - \beta)f$ and $f \geq \gamma^2 n$, we obtain that

$$\mathbb{P}\left(F_{uv}^j < \frac{\mathbb{E}F_{uv}^j}{2}\right) \leq \mathbb{P}\left(\left|F_{uv}^j - \mathbb{E}F_{uv}^j\right| > \frac{\mathbb{E}F_{uv}^j}{2}\right) < 2e^{-\frac{\mathbb{E}F_{uv}^j}{12}} \leq 2e^{-\frac{\gamma|\mathcal{F}_j|}{12}}. \tag{3.1}$$

Notice that there are less than n^2 vertex pairs in $D - V(\mathcal{F})$. Since $|\mathcal{F}| \geq |\mathcal{A}_{uv} \cap \mathcal{F}| \geq \gamma^2 n$, we have that $|\mathcal{F}_j| \geq \beta f \geq \beta\gamma^2 n$. Then by equation (3.1), for $j \in [l]$, we get that

$$\sum_{\{u,v\} \subseteq V(D-\mathcal{F})} \mathbb{P}\left(F_{uv}^j < \frac{\mathbb{E}F_{uv}^j}{2}\right) < 2n^2 e^{-\frac{\gamma|\mathcal{F}_j|}{12}} \leq 2n^2 e^{-\frac{\beta\gamma^3 n}{12}} \to 0, \text{ as } n \to \infty.$$

Therefore with high probability, we obtain that $F_{uv}^j \geq \frac{\mathbb{E}F_{uv}^j}{2} \geq \frac{\gamma|\mathcal{F}_j|}{2} > 0$ for any $j \in [l]$. Recapping the definition of F_{uv}^j, this implies that we may always find l subsets $\mathcal{F}_1, \ldots, \mathcal{F}_l$ of absorbers in $\mathcal{F}$ such that for any vertex pair (u, v) of $D - V(\mathcal{F})$ and any $i \in [l]$, the family $\mathcal{F}_i$ contains at least one absorber of (u, v). This implies that (i) holds.

(ii) Let $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_l$ be a partition of $\mathcal{F}$ satisfying (i). For any index $i \in [l]$, we have that $|\mathcal{F}_i| = f_i$. In the following, we will prove (ii) by induction on k that for each $k \in [f_i]$, there exists a path S_k in D of the form $S_1 = F_{i,1}$ and for $k \geq 2$,

$$S_k = F_{i,1} \circ P_{i,1} \cdots \circ F_{i,k-1} \circ P_{i,k-1} \circ F_{i,k},$$

where each of the paths $P_{i,1}, \ldots, P_{i,k-1}$ has the length at most 3. Note that $L_i = S_{f_i}$.

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It is obvious for the case $k = 1$. Assume that the equation above is true for some $k - 1 \in [f_i - 1]$. Moreover, we suppose that the terminal of $F_{i,k-1}$ is b, and the initial of $F_{i,k}$ is a. Denote by D_{k-1} the subdigraph induced by the vertex set $V_{k-1} = V(D) \setminus V((S_{k-1} - b) \cup (\mathcal{F} - a))$ in D. By the proof of Lemma 3.3, we have that $|\mathcal{F}| \leq 2\gamma_1^3 n$, and then $|V(S_{k-1} \cup \mathcal{F})| < |\mathcal{F}|(4+4) = 8f \leq 8 \cdot 2\gamma_1^3 n < \gamma n$. This implies that $|N^+_{D_{k-1}}(b)|, |N^-_{D_{k-1}}(a)| \geq \delta^0(D) - |V(S_{k-1} \cup \mathcal{F})| \geq (1/2 - 2\gamma)n > (1/2 - \varepsilon')n$. Then by the extremal condition with $(U_1, U_2)_{EC(\varepsilon')} = (N^+_{D_{k-1}}(b), N^-_{D_{k-1}}(a))$, there is a path $P_{i,k-1}$ from b to a of length at most 3 in D_{k-1} such that it connects paths S_{k-1} and $F_{i,k}$. Note that $V(P_{i,k-1}) \setminus \{a, b\}$ is disjoint from $V(\mathcal{F} \cup S_{k-1})$, and so the desired path

$$S_k = S_{k-1} \circ P_{i,k-1} \circ F_{i,k}.$$

Hence, the proof of (*ii*) is completed. □

By using Lemmas 3.3 and 3.4, we will now present the following absorption lemma.

Lemma 3.5 (Absorbing Lemma). *Suppose that $C > 0$ is an integer and α, β are two constants with $0 < \alpha, \beta \ll 1$. Let $n = n_1 + \cdots + n_m$ with $n \geq Chm$, $n_i \geq |V(H)| + 3h$ for each $i \in [m]$, and $\sum_{n_i < \alpha n} n_i \leq \beta n$. For any digraph H with the vertex set $V(H) = \{v_1, \ldots, v_s\}$ and h arcs, and $\delta(H) \geq 1$, if D is a digraph with the order n and $\delta^0(D) \geq n/2$, and D is ε'-stable, then the following statements hold.*

(*i*) *For any sequence $\mathcal{N} = \{l_1, \cdots, l_h\}$ satisfying that each $l_i \geq 3$ and the sum of the l_i less than αn is not more than βn, the digraph D contains an H-subdivision, called H', with $|V(H')| \leq \gamma n$ and in which the length of internally disjoint mapped paths is at most $l_1, \ldots, l_h$, respectively.*

(*ii*) *The digraph D contains m disjoint H-subdivisions, called $H'_1, \ldots, H'_m$, such that $|\bigcup_{i=1}^m V(H'_i)| \leq 5\gamma n$, and for each $i \in [m]$, we have that $|V(H'_i)| \leq n_i$.*

(*iii*) *Finally, we define the digraph Q as follows:*

$$Q = \begin{cases} H', & \text{for } (i), \\ H'_1 \cup \cdots \cup H'_m, & \text{for } (ii). \end{cases}$$

In the constructions of (i) and (ii), Q contains t distinguished paths $L_1, \ldots, L_t$, such that each L_i contains an absorber for every vertex pair (u, v) of $D - Q$. Consequently, for any t pairwise disjoint paths $P_1, \ldots, P_t$ in $D - Q$, where P_i has initial c_i and terminal d_i, the path L_i can absorb P_i for each $i \in [t]$, and the same branch-vertices are maintained.

Proof. Suppose that γ is a real number satisfying that

$$0 < 1/C \ll \alpha, \beta, \gamma_1 < \gamma \ll \varepsilon'.$$

We shall repeatedly use the following simple observation. Since D is ε'-stable, for any two vertex sets $X, Y \subseteq V(D)$ with $|X|, |Y| \geq (1/2 - \varepsilon')n$, we have $e^+_D(X, Y) > (\varepsilon' n)^2$. In particular, if D' is an induced subdigraph of D obtained by deleting at most ηn vertices, where $\eta \ll \varepsilon'$, and $X, Y \subseteq V(D')$ still satisfy $|X|, |Y| \geq (1/2 - \varepsilon')n$, then since D' is induced, $e^+_{D'}(X, Y) = e^+_D(X, Y) > (\varepsilon' n)^2$. Thus there is an arc from X to Y in D'. In the applications below, the number of deleted vertices is always at most $(\beta + O(\gamma) + O(1/C))n \ll \varepsilon' n$, so the relevant neighbourhoods, which have size $(1/2 - o(\varepsilon'))n$ before the deletion, still have size at least $(1/2 - \varepsilon')n$.

(i) Without loss of generality, we assume that $l_1 \geq l_2 \geq \cdots \geq l_h$. Take t from $[h]$ such that t is the largest subscript satisfying $l_t \geq \alpha n$. Note that t is an absolute constant independent of n since $t \leq \frac{n}{\alpha n} = \frac{1}{\alpha}$. We define $W = f(V(H)) := \{u_1, \ldots, u_s\}$ and let $D' = D - W$. Clearly, $\delta^0(D') \geq n/2 - s \geq n/2 - 2h \geq (\frac{1}{2} - \frac{1}{Cm})n$ since $n \geq Chm$.

On the one hand, in the desired H-subdivision H', if there is a path of length $l_{t+1} < \alpha n$, say from u_1 to u_2, then we can greedily get this path by doing the following. We choose a vertex $w_3 \in N^+_{D'}(u_1)$ and $w_{i+1} \in N^+_{D'}(w_i) \setminus \{w_3, \ldots, w_i\}$ for all $i \in \{3, \ldots, l_{t+1} - 3\}$.

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Further, because $l_{t+1}+\cdots+l_h \le \beta n$ and $\delta^0(D') \ge (\frac{1}{2}-\frac{1}{Cm})n$, we have that $|N^+_{D'}(w_{l_{t+1}-2}) \setminus \{u_1, w_3, \ldots, w_{l_{t+1}-2}, u_2\}|, |N^-_{D'}(u_2) \setminus \{u_1, w_3, \ldots, w_{l_{t+1}-2}, u_2\}| \ge (1/2-\varepsilon')n$. Then this implies that there is an arc from the former to the latter, since D does not satisfy EC(ε) and $D' \subseteq D$. Using a similar process as shown below, we can obtain all paths $P_{t+1}, \ldots, P_h$ of length $l_{t+1}, \ldots, l_h$, respectively, of the desired H-subdivision H'.

In the following, let $D'' = D' - \bigcup_{j=t+1}^{h} P_j$. Clearly, $\delta^0(D'') \ge (1/2-\beta)n$. Then by Lemma 3.3, there is a family $\mathcal{F}$ of at most γn disjoint, absorbing 4-paths in D'' such that for every vertex pair (u, v), we have that $|\mathcal{A}_{uv} \cap \mathcal{F}| \ge \gamma^2 n$. Then by using Lemma 3.4, we can partition $\mathcal{F}$ into t distinct parts, say $\mathcal{F}_1, \ldots, \mathcal{F}_t$, such that $|V(\mathcal{F}_i)| \le l_i - 6$ for each $i \in [t]$, and in each part, there is an absorber for any vertex pair (u, v) of $V(D'' - \mathcal{F})$, and Lemma 3.4 also depicts that we can connect all absorbing paths in $\mathcal{F}_i$ into a path L_i of length no more than $l_i - 6$. Further, these paths $L_1, \ldots, L_t$ are disjoint and $|\bigcup_{i=1}^{t} V(L_i)| \le \gamma n$. Without loss of generality, write $L_i = a_i \cdots b_i$ for any $i \in [t]$, and suppose that in the desired H-subdivision H', the path from u_i to u'_i has the length at most l_i. Define $D^0 = D'' - \bigcup_{j=1}^{t} L_j$, and let $R_i = N^+_{D^0}(u_i)$ and $S_i = N^-_{D^0}(a_i)$ for each $i \in [t]$. Then we have that $|R_i|, |S_i| \ge \delta^0(D'') - \gamma n \ge (1/2-\varepsilon')n$, which implies that there exists an arc $x_i x'_i$ from R_i to S_i, since $D^0 \subset D$ does not meet EC(ε). Similarly, we can prove that there is an arc $y_i y'_i$ from $N^+_{D^0}(b_i)$ to $N^-_{D^0}(u'_i)$ for each $i \in [t]$. Then we get a path $u_i x_i x'_i L_i y_i y'_i u'_i$. Hence we obtain all paths of length at most $l_1, \ldots, l_t$, respectively.

To sum up, we get an H-subdivision H' in D, in which the length of internally disjoint mapped paths is at most $l_1, \ldots, l_h$, respectively. Clearly, $|V(H')| \le \beta n + 4h + |V(\mathcal{F})| \le \gamma n$. This completes the proof of (i).

(ii) In D, we randomly take the disjoint mapping vertex sets of $V(H)$ to be $f_1(V(H)) = W_1 := \{v^1_1, v^1_2, \ldots, v^1_s\}, \ldots, f_m(V(H)) = W_m := \{v^m_1, v^m_2, \ldots, v^m_s\}$, respectively. Let $D' = D - \cup_{i=1}^{m} W_i$, and clearly, since $n \ge Chm$,

$$\delta^0(D') \ge \frac{n}{2} - 2mh \ge \frac{n}{2} - \frac{2n}{C} \ge (1/2-\beta)n.$$

Without loss of generality, we assume that $n_1 \ge \cdots \ge n_t \ge \alpha n > n_{t+1} \ge \cdots \ge n_m$. In the following, we will first get t disjoint H-subdivisions $H'_1, \ldots, H'_t$ of order at most $n_1, \ldots, n_t$, respectively, such that $|\bigcup_{i=1}^{t} V(H'_i)| \le 4\gamma n$. Second, we will prove that in the remaining digraph, there are $m-t$ disjoint H-subdivisions, that the order of each such H-subdivision is $n_{t+1}, \ldots, n_m$, respectively.

First, similar to the proof of (i), Lemma 3.3 gives a family $\mathcal{F}$ of at most γn disjoint absorbers in D' such that for every vertex pair (u, v), it satisfies that $|\mathcal{A}_{uv} \cap \mathcal{F}| \ge \gamma^2 n$. Then applying Lemma 3.4, we can partition the family $\mathcal{F}$ into t disjoint subfamily, say $\mathcal{F}_1, \ldots, \mathcal{F}_t$, such that for each $i \in [t]$, $|V(\mathcal{F}_i)| < n_i$, and further, all 4-paths in $\mathcal{F}_i$ can be connected into a path L_i with $|V(L_i)| < n_i$. Without loss of generality, assume that $L_i = a_i \cdots b_i$ for every $i \in [t]$. For any $i \in [t]$, in the desired H-subdivision H'_i, the path, say from v^i_1 to v^i_2, can be gotten by the following strategy as below. We define $D'' = D' - \bigcup_{j=1}^{t} L_j$, and let $R_i = N^+_{D''}(v^i_1)$ and $S_i = N^-_{D''}(a_i)$ for each $i \in [t]$. Then we have that $|R_i|, |S_i| \ge \delta^0(D'') \ge (1/2-\beta)n - \gamma n \ge (1/2-2\gamma)n$, which implies that there exists an arc $x^i_1 x^i_2$ from R_i to S_i, since $D'' \subset D$ does not EC(ε). Similarly, we can prove that there is an arc $y^i_1 y^i_2$ from $N^+_{D''}(b_i)$ to $N^-_{D''}(v^i_2)$ for each $i \in [t]$. Then we get a path $v^i_1 x^i_1 x^i_2 L_i y^i_1 y^i_2 v^i_2$ for any $i \in [t]$. This can be done because $\delta^0(D'') - 4t \ge (1/2-2\gamma)n - \frac{n}{Ch} \ge (1/2-3\gamma)n > 0$. Let D^3 be the remaining digraph, and clearly $\delta^0(D^3) \ge (1/2-3\gamma)n$. Further, for the other paths in the desired H-subdivision H'_i with $i \in [t]$, we can obtain these paths of length at most 3. This is because by $|N^\sigma_{D^3}(v^i_j)| \ge \delta^0(D^3) > \delta^0(D^3) - 2hm \ge (1/2-3\gamma)n - \frac{n}{C} \ge (1/2-7\gamma/2)n$ for any $\sigma \in \{+, -\}$, $i \in [m]$ and $j \in [s]$, and $D^3 \subset D$ does not satisfy EC(ε), we get that there are at least $(\varepsilon' n)^2$ arcs from $N^\sigma_{D^3}(v^i_j)$ to $N^{-\sigma}_{D^3}(v^l_k)$ with $j, k \in [s]$ and $i, l \in [m]$ and $(U_1, U_2)_{EC(\varepsilon)} = (N^\sigma_{D^3}(v^i_j), N^{-\sigma}_{D^3}(v^l_k))$. This

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suggests that we obtain t disjoint H-subdivisions $H'_1, \ldots, H'_t$ with $|\bigcup_{i=1}^{t} V(H'_i)| \le 7\gamma n/2$, and in which, the order of $H'_1, \ldots, H'_t$ is at most $n_1, \ldots, n_t$, respectively.

Let $D^4 = D^3 - \bigcup_{i=1}^{t} H'_i$, and obviously $\delta^0(D^4) \ge (1/2 - 7\gamma/2)n$. In the following, we will prove that D^4 contains $m - t$ disjoint H-subdivisions with the order $n_{t+1}, \ldots, n_m$, respectively. Similar to the proof at the end of the previous paragraph, we can get that for each $i \in \{t+1, \ldots, m\}$, there are $h-1$ disjoint paths of length at most 3 as desired in the H-subdivision H'_i, since for any vertex v_l^k with $l \in [s], k \in [m] \setminus [t]$, we have that

$$\left|N^{\sigma}_{D^4}(v_l^k)\right| \ge \delta^0(D^4) > \delta^0(D^4) - 2hm \ge (1/2 - 7\gamma/2)n - 2n/C \ge (1/2 - 4\gamma)n.$$

Finally, in every promised H-subdivision H'_i with $i \in [m] \setminus [t]$, only one path is missing, without loss of generality, say from v_{h-1}^i to v_h^i order n'_i. Let D^5 be the remaining digraph, and then clearly, $\delta^0(D^5) \ge \delta^0(D^4) - 2hm \ge (1/2 - 4\gamma)n$ since $n \ge Chm$ and $1/C \ll \gamma$. Then we can greedily get these paths by doing the following. Let i be any index with $i \in [m] \setminus [t]$. We choose a vertex $w_3 \in N^+_{D^5}(v_{h-1}^i)$ and $w_{j+1} \in N^+_{D^5}(w_j) \setminus \{w_3, \ldots, w_j\}$ for all $j \in \{3, \ldots, n'_i\}$. Further, because $n'_{t+1} + \cdots + n'_m \le \beta n$ and $\delta^0(D^5) \ge (1/2 - 4\gamma)n$, we have that $|N^+_{D^5}(w_{n'_i-2}) \setminus \{v_{h-1}^i, w_3, \ldots, w_{n'_i-2}, v_h^i\}|, |N^-_{D'}(v_h^i) \setminus \{v_{h-1}^i, w_3, \ldots, w_{n'_i-2}, v_h^i\}| \ge (1/2 - 4\gamma)n$. Let R be the previous set and S be the subsequent set. Then this implies that there is an arc from R to S, since D does not satisfy EC(ε) and $D^5 \subseteq D$. Using a similar process as shown below, we can obtain all paths of order $n'_{t+1}, \ldots, n'_m$, of the desired H-subdivisions $H'_{t+1}, \ldots, H'_m$, respectively, due to

$$\delta^0(D^5) - (n'_{t+1} + \cdots + n'_m) \ge (1/2 - 4\gamma)n - \beta n \ge (1/2 - 5\gamma)n. \tag{3.2}$$

Hence we get $m - t$ disjoint H-subdivisions of order $n_{t+1}, \ldots, n_m$, respectively.

So we get m disjoint H-subdivisions $H'_1, \ldots, H'_m$ such that $|V(H'_i)| < n_i$ for $n_i \ge \alpha n$ and $|V(H'_i)| = n_i$ for $n_i < \alpha n$. Also, by equation (3.2), we have that $|\bigcup_{i=1}^{m} V(H'_i)| \le 5\gamma n$. This implies that (ii) is true.

(iii) In the following, we will prove that Q captures the desired absorption property. By Lemma 3.4 and the constructions of $H'_1, \ldots, H'_m$, for each $i \in [t]$ (t is defined in (*ii*). Also, maybe $t = 0$ and in the case, it is H'), the distinguished path L_i contains an absorber for every vertex pair $(u, v) \in D - Q$.

Let $P_1, \ldots, P_t$ be pairwise disjoint paths in $D - Q$, where $P_i = c_i \cdots d_i$ for each $i \in [t]$. For each $i \in [t]$, choose an absorber $z_1z_2z_3z_4$ for (c_i, d_i) contained in L_i. Since $z_2c_i, d_iz_3 \in A(D)$, we can replace the subpath $z_1z_2z_3z_4$ of L_i by $z_1z_2P_iz_3z_4$. This absorbs P_i into L_i without changing the initial and terminal of L_i. Since the paths $P_1, \ldots, P_t$ are pairwise disjoint and are disjoint from Q, all these replacements can be performed simultaneously. Hence all branch-vertices of H' or $(H'_1, \ldots, H'_m$ remain unchanged. So (iii) holds.

Hence we complete the proof of this lemma. □

Let ν and τ be real numbers with $0 < \nu \le \tau < 1$. Suppose that D is a digraph and S is a vertex subset of $V(D)$. The *ν-robust out-neighbourhood* $RN^+_{\nu,D}(S)$ *of* S is defined as the set of all vertices x in D that have at least $\nu|V(D)|$ in-neighbourhoods in S. Moreover, the digraph D is called a *robust* (ν, τ)*-outexpander* if $|RN^+_{\nu,D}(S)| \ge |S| + \nu|V(D)|$ for all $S \subseteq V(D)$ with $\tau|V(D)| < |S| < (1-\tau)|V(D)|$. Kühn, Osthus and Treglown [18] depicted that for a positive integer n_0 and $1/n_0 \ll \nu \le \tau \ll \xi < 1$, if D is a digraph on $n \ge n_0$ vertices with $\delta^0(D) \ge \xi n$ and is a robust (ν, τ)-outexpander, then D contains a Hamiltonian cycle.

In particular, if D is a digraph with $\delta^0(D) \ge (1/2 - \varepsilon')n$, where $0 < \varepsilon' \ll \varepsilon \ll 1$, and D is ε'-stable with parameter ε, then we will prove that D is a robust (ν, τ)-outexpander, and so the following path-covering lemma holds. Since a detailed proof in our paper [5] has been given, we will omit its proof here.

Lemma 3.6 (Path-Covering Lemma, [5, Lemma 3.8]). *Suppose that D is a digraph on n vertices with $\delta^0(D) \ge (1/2 - 5\gamma)n$, and is ε'-stable, where $0 < \gamma \ll \varepsilon' \ll 1$. Then D contains a Hamiltonian path.*

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3.2 Completion of Theorems 1.3 and 1.5

In this subsection, we always assume that H is any digraph with $\delta(H) \geq 1$, and D is an n-vertex digraph $\delta^0(D) \geq n/2$, and D is ε'-stable. Let $C > 0$ be a constant, and take parameters α, β and γ satisfy $0 < 1/C \ll \alpha, \beta < \gamma \ll \varepsilon' \ll 1$. We first give the proof of Theorem 1.3, and then we will prove Theorem 1.5.

Proof of Theorem 1.3. Let $\mathcal{N} = \{l_1, \ldots, l_h\}$ be any integer set with $l_1 + \cdots + l_h < n$. Without loss of generality, suppose that $l_1 \geq \cdots \geq l_t \geq \alpha n > l_{t+1} \geq \cdots \geq l_h$, and the sum of the l_i less than αn is not more than βn. By Lemma 3.5(i), we can obtain an H-subdivision in D, called H', with $|V(H')| \leq \gamma n$, and in H', the paths of length less than αn have reached the lengths as required in the desired H-subdivision. However, for the paths of length at least αn, in H', there are corresponding 'long' paths, defined as $Q_1, \ldots, Q_t$, of length no more than $l_1, \ldots, l_t$, respectively, and each of these long paths contains an absorber of any vertex pair (u, v) in $D - H'$. With the aid of Lemma 3.6, we can divide the Hamiltonian path in $D - H'$ into t disjoint subpaths, defined $R_1, \ldots, R_t$, such that for each $i \in [t]$, $R_i = c_i \cdots d_i$ and the cardinality of $V(Q_i \cup R_i)$ is $l_i + 1$. Since the path Q_i contains an absorber of the pair (c_i, d_i) for any $i \in [t]$, the path Q_i can absorb R_i to get a path with the same initial and terminal as Q_i and the length l_i as desired. Therefore, we get a spanning H-subdivision, in which, the length of paths is $l_1, \ldots, l_h$, respectively. This proves Theorem 1.3 for the non-extremal case. □

Proof of Theorem 1.5. Let $n = n_1 + \cdots + n_m$ be any integer partition of n. Without loss of generality, we assume that $n_1 \geq \cdots \geq n_t \geq \alpha n > n_{t+1} \geq \cdots \geq n_m$, and the sum of the n_i less than αn is no more than βn. With the help of Lemma 3.5(ii), we can obtain m disjoint H-subdivisions $H'_1, \ldots, H'_m$, such that for each $i \in [t]$, we have $|V(H'_i)| \leq n_i$, and for any $i \in [m] \setminus [t]$, we have that $|V(H'_i)| = n_i$, and $|\bigcup_{i=1}^m V(H'_i)| \leq 5\gamma n$. Also, for every vertex pair (u, v) of $D - \bigcup_{i=1}^m H'_i$, there exists an absorber in H'_i $(i \in [t])$ for it. Let $D^0 = D - \bigcup_{i=1}^m H'_i$, and then $\delta^0(D^0) \geq (1/2 - 5\gamma)n$. Then by Lemma 3.6, we know that D^0 contains a Hamiltonian path, and then we divide this Hamiltonian path into t disjoint paths $P_1, \ldots, P_t$, of orders $n_1 - V(H'_1), \ldots, n_t - V(H'_t)$, respectively. Without loss of generality, for any $j \in [t]$, we assume that $P_j = c_j \cdots d_j$. By Lemma 3.5(iii), we get that for every $j \in [t]$, the subdigraph H'_j contains an absorber for (c_j, d_j), and so $H'_j \cup P_j$ is an H-subdivision of order n_j with the same branch-vertices as H'_j. So far, we have obtained m disjoint H-subdivisions, of order $n_1, \ldots, n_m$, respectively. Hence, this suggests that Theorem 1.5 holds for the non-extremal case when D is ε'-stable. □

4. Extremal case: Proofs of Theorems 1.3 and 1.5

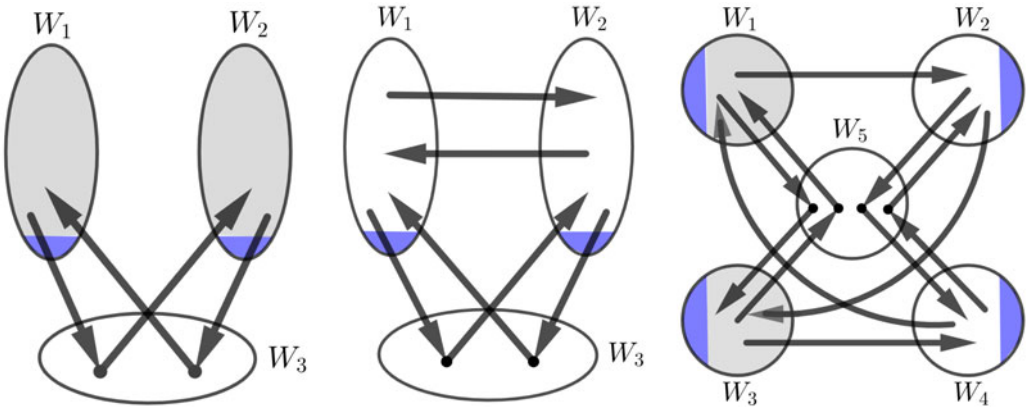
In this section, we give the unified proofs of main theorems when D is not ε'-stable. Parameters ε', ε_1 and ε are chosen with $0 < \varepsilon' \ll \varepsilon_1 \ll \varepsilon \ll 1$. For any $X, Y \subseteq V(D)$, we denote by (X, Y) the bipartite digraph with vertex set $X \cup Y$ and arc set $\{xy, yx \in A(D) | x \in X, y \in Y\}$. We define the *bidirectional neighbourhood* of a vertex x in D to be $BN(x) = \{y : xy, yx \in A(D)\}$, and the *bidirectional semi-degree* of x in D, defined as $b(x)$, is the cardinality of $BN(x)$, that is $b(x) = |BN(x)|$. Also, for a vertex subset U of D, we denote $b_U(x) = |BN(x) \cap U|$. We also say two vertex sets W_1 and W_2 to be *approximately equal*, written as $|W_1| \approx |W_2|$, if $|W_1| - |W_2| = \pm 2\varepsilon' n$.

4.1 Extremal cases and properties

We now define the following three extremal cases that will occur when D satisfies the extremal condition with parameter ε (EC(ε)).

(1) Extremal Case 1 with parameter ε(EC1(ε)): We can partition $V(D)$ into three disjoint vertex sets W_1, W_2 and W_3, such that for every $i \in [2]$, $|W_i| = (1/2 - \varepsilon)n \pm \sqrt{10\varepsilon}|W_i|$, and $|W_3| \leq 2\sqrt{10\varepsilon} \cdot \max\{|W_1|, |W_2|\}$, and the following properties hold (see Fig. 1(*a*)).

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(a) The extremal case 1 (**EC1**). (b) The extremal case 2 (**EC2**). (c) The extremal case 3 (**EC3**).

Figure 1. The extremal cases 1-3. Note that in this figure, a thick arrow pointing between two vertex sets indicates that the induced digraph by them is ε-almost one-way complete, and the grey shading indicates that the digraph induced by this vertex set is ε-almost complete. Also, the blue shaded areas indicate the exceptional vertices in the corresponding vertex sets.

(1.1) For each $i \in [2]$, there exists a subset W_i' in W_i with $|W_i'| \leq 10\sqrt{\varepsilon}|W_i|$ such that for every vertex w in $W_i \setminus W_i'$, $\delta^0_{W_i}(w) \geq (1-10\sqrt{\varepsilon})|W_i|$, and for every vertex w in W_i', $\delta^0_{W_i}(w) \geq \varepsilon^{1/3}|W_i|/2$. We also say that $D[W_i]$ is *ε-almost complete*, for each $i \in [2]$.

(1.2) For any vertex $w \in W_3$, we have that either $d^+_{W_1}(w), d^-_{W_2}(w) > (1/2-\varepsilon^{1/3})n$, or $d^-_{W_1}(w), d^+_{W_2}(w) > (1/2-\varepsilon^{1/3})n$, and $\delta^0_{W_i}(w) \leq \varepsilon^{1/3}n/2$ for each $i \in [2]$.

(2) Extremal Case 2 with parameter ε(EC2(ε)): We can partition $V(D)$ into three disjoint vertex sets W_1, W_2 and W_3 such that for every $i \in [2]$, $|W_i| = (1/2-\varepsilon)n \pm \sqrt{10\varepsilon}|W_i|$, and $|W_3| \leq 2\sqrt{10\varepsilon} \cdot \max\{|W_1|, |W_2|\}$, and the following hold (see Fig. 1(*b*)).

(2.1) For every $i \in [2]$, apart from at most $10\sqrt{10\varepsilon}|W_i|$ exceptional vertices, all vertices in W_i have bidirectional semi-degrees at least $(1-10\sqrt{\varepsilon})|W_{3-i}|$ in W_{3-i}, and the semi-degrees of these exceptional vertices are at least $\varepsilon^{1/3}|W_{3-i}|/8$ in W_{3-i}. In this case, we also call (W_1, W_2) to be a *ε-almost complete bipartite digraph*.

(2.2) For every vertex $w \in W_3$, we have that either $d^+_{W_1}(w), d^-_{W_2}(w) > (1/2-\varepsilon^{1/3})n$, or $d^-_{W_1}(w), d^+_{W_2}(w) > (1/2-\varepsilon^{1/3})n$, and $\delta^0_{W_i}(w) \leq 2\varepsilon^{1/3}n$ for each $i \in [2]$.

(3) Extremal Case 3 with parameters ε and ε_1(EC3(ε, ε_1)): We can partition $V(D)$ into five disjoint vertex sets W_1, W_2, W_3, W_4 and W_5 in D with $|W_5| \leq 4\sqrt{10\varepsilon}n$, $\varepsilon^{1/3}n - \sqrt{10\varepsilon}n < |W_1|, |W_3| < (1/2-3\varepsilon_1/4)n + \sqrt{10\varepsilon}n$ and $\varepsilon^{1/3}n - \sqrt{10\varepsilon}n < |W_2|, |W_4| < (1/2-\varepsilon_1/4)n + \sqrt{10\varepsilon}n$ such that $|W_1| \approx |W_3|$ and $|W_2| \approx |W_4|$. Furthermore, the following statements hold (see Fig. 1(*c*)).

(3.1) For each $i \in \{1, 3\}$, apart from at most $10\sqrt{\varepsilon}|W_i|$ exceptional vertices, all vertices have the semi-degrees in W_i at least $(1-10\sqrt{\varepsilon})|W_i|$, and the semi-degrees of these exceptional vertices in W_i are at least $\varepsilon^{1/3}|W_i|/2$.

(3.2) For $\{i, j\} = \{2, 4\}$, in W_i, apart from at most $10\sqrt{10\varepsilon}|W_i|$ exceptional vertices, all vertices in W_i have bidirectional semi-degrees at least $(1-10\sqrt{\varepsilon})|W_j|$ in W_j, and the semi-degrees of these exceptional vertices are at least $\varepsilon^{1/3}|W_j|/8$ in W_j. For the vertex set W_j, we can get a similar conclusion by swapping all i and j above, respectively.

(3.3) Without loss of generality, for any $i \in [2]$, we assume that $|W_i| \geq |W_{2+i}|$. Then for every vertex w in W_5 and any pair $\{j_1, j_2\} \in \{\{1, 3\}, \{2, 4\}\}$, it satisfies either $d^+_{W_{j_1}}(w), d^-_{W_{j_2}}(w) \geq (1-\varepsilon^{1/3})|W_{j_1}|$ or $d^-_{W_{j_1}}(w), d^+_{W_{j_2}}(w) \geq (1-\varepsilon^{1/3})|W_{j_1}|$, and $\delta^0_{W_j}(w) \leq \varepsilon^{1/3}n/2$ for any $j \in \{1, 3\}$, and $\delta^0_{W_j}(w) \leq 2\varepsilon^{1/3}n$ with $j \in \{2, 4\}$.

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(3.4) For $i \in [4]$, we have that $e^+(W_i, W_{i+1}) \geq |W_i| \cdot |W_{i+1}| - \varepsilon' n^2/2$, where the subscript of W_{i+1} takes the remainder of modulo 4 for $i = 4$. In particular, in this case we also say that (W_i, W_{i+1}) is ε'-*almost one-way complete* corresponding to the vertex sets W_i and W_{i+1}.

Based on the extremal condition (EC(ε)) and the definitions of EC1(ε), EC2(ε) and EC3($\varepsilon, \varepsilon_1$), we can use traditional structural analysis methods to effectively demonstrate the following conclusion. Since its proof strategy essentially follows from that of Lemma 3.14 in [5], it is deferred to Appendix A.

Lemma 4.1. *Suppose that* $0 < \varepsilon' \ll \varepsilon_1 \ll \varepsilon \ll 1$. *Let* D *be an* n*-vertex digraph with* $\delta^0(D) \geq n/2$. *If* D *satisfies the extremal condition (EC*(ε')*), then* D *must belong to either EC1*(ε)*, EC2*(ε)*, or EC3*($\varepsilon, \varepsilon_1$).

Furthermore, we establish the following lemmas, which demonstrate how to properly handle the exceptional vertices in each extremal configuration of D and will be frequently applied in the proofs of the main theorems. A set $\mathcal{P}$ is said to *pass through* a vertex u if there exists an element in $\mathcal{P}$ that contains u. If this holds for every vertex in a set U, then $\mathcal{P}$ is said to pass through U. We say a path P is of the form $W_1 W_2 \cdots W_k$ if $P = w_1 w_2 \cdots w_k$, where $w_i \in W_i$ for each $i \in [k]$.

Lemma 4.2. *Let* D *be an* n*-vertex digraph with* $\delta^0(D) \geq n/2$. *If* D *belongs to EC1*(ε)*, then there is a set* $\mathcal{P}$ *of disjoint* W_1*-paths and* W_2*-paths passing through* W_3. *Moreover, each path in* $\mathcal{P}$ *has length at most* 5 *and contains at most two vertices from* W_3.

Proof. We define $W_{3,1}$ (resp., $W_{3,2}$) to be the set of vertices w in W_3 satisfying that $d^-_{W_1}(w), d^+_{W_2}(w) > (1/2 - \varepsilon^{1/3})n$ (resp., $d^+_{W_1}(w), d^-_{W_2}(w) > (1/2 - \varepsilon^{1/3})n$). If $|W_{3,1}| = |W_{3,2}|$, then for any $u \in W_{3,1}$ and any $v \in W_{3,2}$, by the definitions of $W_{3,1}$ and $W_{3,2}$, we have that

$$|N^-_{W_1}(u) \cap N^+_{W_1}(v)| \geq (1 - 2\varepsilon^{1/3})n - |W_1| > n/3 \quad \text{and}$$
$$|N^+_{W_2}(u) \cap N^-_{W_2}(v)| \geq (1 - 2\varepsilon^{1/3})n - |W_2| > n/3.$$

This implies that there exist many (more than $|W_{3,1}|$) disjoint paths P with the form $W_2 v W_1 u W_2$, and many (more than $|W_{3,1}|$) disjoint paths P' of the form $W_1 u W_2 v W_1$. Greedily, we can get that a family $\mathcal{P}$ of disjoint paths that cover all vertices of W_3 and each path in $\mathcal{P}$ is a W_1-path or a W_2-path, as requested.

So in the following, we consider the case of $|W_{3,1}| \neq |W_{3,2}|$. Without loss of generality, suppose that $|W_{3,1}| > |W_{3,2}|$ and let $r = |W_{3,1}| - |W_{3,2}|$. We denote by M_1 (resp., M_2) to be a set of disjoint arcs from $W_{3,1}$ to W_1 (resp., from W_2 to $W_{3,1}$), with M_1 and M_2 being disjoint from each other. We further choose each M_i ($i \in [2]$) to be maximal. Set $M = M_1 \cup M_2$, and for each $i \in [2]$, define $W'_i = W_i \setminus V(M)$ and $W'_{3,1} = W_{3,1} \setminus V(M)$. We now consider two subcases. □

Case 1: $|M| \geq r$. In this case, since for any $w \in W_{3,1}$, we have $d^-_{W_1}(w), d^+_{W_2}(w) > (1/2 - \varepsilon^{1/3})n$, there are r disjoint W_1-paths and W_2-paths of the forms $W_1 M_1$ and $M_2 W_2$. Clearly, $|W_{3,1}| - r = |W_{3,2}|$. Then for all remaining vertices u in $W_{3,1}$ and v in $W_{3,2}$, similar to the previous paragraph and by the definitions of $W_{3,1}$ and $W_{3,2}$, there are $|W_{3,2}|$ disjoint W_2-paths (or W_1-paths) of form $W'_2 v W'_1 u W'_2$ (or $W'_1 u W'_2 v W'_1$), covering all remaining vertices in $W_{3,1}$ and all vertices in $W_{3,2}$, as desired.

Case 2: $|M| < r$. Here, since M_1 and M_2 are maximum and $n = |W'_1| + |W'_2| + |W'_{3,1}| + |W_{3,2}| + |V(M)|$, for any vertex $u \in W'_1$, we have that

$$\begin{aligned} d^-_{W'_2}(u) &\geq \delta^0(D) - (|W'_1| + |W_{3,2}| + d^-_M(u)) \\ &\geq \frac{|W'_2| - |W'_1| + |W'_{3,1}| - |W_{3,2}| + |V(M)|}{2} - d^-_M(u). \end{aligned} \tag{4.1}$$

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Symmetrically, for any vertex v in W_2', we obtain that

$$d^+_{W_1'}(v) \geq \delta^0(D) - (|W_2'| + |W_{3,2}| + d^+_M(v))$$
$$\geq \frac{|W_1'| - |W_2'| + |W_{3,1}'| - |W_{3,2}| + |V(M)|}{2} - d^+_M(v). \tag{4.2}$$

From equations (4.1) and (4.2), for each pair of vertices $u \in W_1'$, $v \in W_2'$, we have that

$$|N^-_{W_2'}(u) \cup N^+_{W_1'}(v)| \geq |W_{3,1}'| - |W_{3,2}| + |V(M)| - (d^-_M(u) + d^+_M(v)). \tag{4.3}$$

We first state the following conclusion:

Claim 4.3. *For an arc xy in M, the following holds.*

(i) If there exists a pair (u, v) with $u \in W_1'$, $v \in W_2'$ such that $d^-_{xy}(u) + d^+_{xy}(v) = 3$, then the value of $|W_{3,1}'| - |W_{3,2}|$ can be reduced by 1.

(ii) If there exist two distinct pairs (u_1, v_1), (u_2, v_2) with $u_1, u_2 \in W_1'$, $v_1, v_2 \in W_2'$ (possibly $u_1 = u_2$ or $v_1 = v_2$) such that for each $i \in [2]$, $d^-_{xy}(u_i) + d^+_{xy}(v_i) = 4$, then the value of $|W_{3,1}'| - |W_{3,2}|$ can be reduced by 2.

Proof. (i) First, consider $xy \in M_1$, that is, xy is an arc from $W_{3,1}$ to W_1. If $vx \in A(D)$, then by the definition of $W_{3,1}'$, for any $w \in W_{3,1}'$, we have that $d^-_{W_1'}(w), d^+_{W_2'}(w) > (1 - 3\varepsilon^{1/3})n/2$, and for any vertex $v' \in W_2'$, since $D[W_2]$ is ε-almost complete, we have that that $\delta^0_{W_2'}(v') \geq (1 - 10\sqrt{\varepsilon})|W_2'|$. This suggests that there is a vertex v' in W_2' satisfying $wv', v'v \in A(D)$. So there is a W_1'-path with the form $W_1' wv'vxy$, This path passes through one more vertex from remaining of $|W_{3,1}'|$ than the original arc xy alone, effectively reducing $|W_{3,1}'| - |W_{3,2}|$ by 1.

Otherwise, $vx \notin A(D)$. Then from $d^-_{xy}(u) + d^+_{xy}(v) = 3$, we have $yu, xu, vy \in A(D)$. We then replace xy with xu and simultaneously have a W_1-path of the form $W_1' wv'vy$, where $w \in W_{3,1}'$, $v' \in N^+_{W_2'}(w) \cap N^-_{W_2'}(v)$. Note such a v' exists because $d^+_{W_2'}(w) \geq (1 - 3\varepsilon^{1/3})|W_2'|$ and $\delta^0_{W_2'}(v) \geq (1 - 10\sqrt{\varepsilon})|W_2'|$. In this case, the value of $|W_{3,1}'| - |W_{3,2}|$ is also reduced by 1.

The case for $xy \in M_2$, that is, $x \in W_2, y \in W_{3,1}$, is handled similarly. If $yu \in A(D)$, a W_1-path of the form $W_1' wv'xyu$ exists, where $w \in W_{3,1}'$, $v' \in W_2'$. Otherwise, we have $vx, vy, xu \in A(D)$. We replace xy with vy and gain an additional W_1-path of the form $W_1' wv'xyu$, where $w \in W_{3,1}'$, $v' \in N^+_{W_2'}(w) \cap N^-_{W_2'}(x)$. In all scenarios, the value of $|W_{3,1}'| - |W_{3,2}|$ decreases by 1.

(ii) We prove only the case $xy \in M_1$; the proof for $xy \in M_2$ is analogous except for the choice of intermediate vertices. Since $d^-_{xy}(u_i) + d^+_{xy}(v_i) = 4$ for each $i \in [2]$, we obtain two disjoint W_1-paths of the forms $W_1' w_1 v' v_2 x u_1$ and $W_1' w_2 v'' v_1 y$, where $w_1, w_2 \in W_{3,1}'$, $v', v'' \in W_2'$. Taking these two paths reduces $|W_{3,1}'| - |W_{3,2}|$ by 2. □

For an arc $xy \in M$, a pair of vertices $(u, v) \in W_1' \times W_2'$ is called *xy-feasible* if either $d^-_{xy}(u) + d^+_{xy}(v) \leq 3$, or $d^-_{xy}(u) + d^+_{xy}(v) = 4$ and there exists another distinct pair $(u', v') \in W_1' \times W_2'$ such that $d^-_{xy}(u') + d^+_{xy}(v') = 4$ (in which case, (u', v') is called a *companion pair* for (u, v)). Note that all companion pairs are pairwise disjoint. Furthermore, (u, v) is called *M-feasible* if it is xy-feasible for all $xy \in M$. We say M is *feasible* if all but at most $\sqrt{10\varepsilon}n^2$ pairs in (W_1', W_2') is M-feasible.

Claim 4.4. *M is feasible.*

Proof. Fix an arc $xy \in M$. We first show that at most one pair $(u, v) \in W_1' \times W_2'$ is not xy-feasible. Indeed, if there were two distinct such pairs (u_1, v_1) and (u_2, v_2), then necessarily, for each $i \in [2]$, we would have $d^-_{xy}(u_i) + d^+_{xy}(v_i) = 4$. Thus each of these two pairs would be a companion pair

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for the other one, a contradiction. Therefore, for each fixed arc $xy \in M$, at most one pair is not xy-feasible. Hence the number of pairs which are not M-feasible is atmost $|M|$. Since $|M| < r \leq 2\sqrt{(}10\varepsilon n$, this number is at most $\sqrt{10\varepsilon}n^2$ for sufficiently large n. Hence all but at most $\sqrt{10\varepsilon}n^2$ pairs in $W_1' \times W_2'$ are M-feasible, which verifies the feasibility of M.

It follows from equation (4.3) and the proof of Claim 4.4 that for $i \in [2]$, there are subsets $U_i \subseteq W_i'$ with $|U_i| \leq 2\sqrt{10\varepsilon}n$ such that all pairs $(u, v) \in (W_1' \setminus U_1(\times (W_2' \setminus U_2)$ are M-feasible and satisfy the degree condition $|N^-_{W_2'}(u) \cup N^+_{W_1'}(v)| \geq |W'_{3,1}| - |W_{3,2}| + |V(M)| - (d^-_M(u) + d^+_M(v)) := s$. Moreover, it follows from the pigeonhole principle that at least $|W_1' \setminus U_1| \cdot |W_2' \setminus U_2|/4$ of these pairs share the same value of $|V(M)| - (d^-_M(u) + d^+_M(v))$. By König-Hall's theorem (on arcs from W_2' to W_1'), this guarantees s disjoint arcs vu from W_2' to W_1'. For s distinct $w \in W'_{3,1}$, the definition of $W'_{3,1}$ gives $d^-_{W_1'}(w), d^+_{W_2'}(w) > d^+_{W_2'}(w) > (1/2 - 2\varepsilon^{1/3})n$. Together with the property of W_2', this implies

$$\left|N^-_{W_2'}(v) \cap N^+_{W_2'}(w)\right| \geq (1 - 10\sqrt{\varepsilon})|W_2'| + (1/2 - 2\varepsilon^{1/3})n - |W_2'| > n/3.$$

Thus, we obtain s disjoint W_1'-paths of the forms $W_1' w W_2' v u$, each utilising a fresh arc $vu \in (W_1', W_2')$ and a new vertex $w \in W'_{3,1}$. The remaining $r - s$ vertices of $W_{3,1}$ are passed through by disjoint paths of the forms $W_1' M_1$ and $M_2 W_2'$ using arcs in M. Finally, for each $i \in [2]$, define $W^0_{3,i}$ as the remainder of $W_{3,i}$ after deleting the vertices on these paths. Clearly, $|W^0_{3,1}| = |W^0_{3,2}|$. This case thereby reduces to Case 1 discussed earlier, allowing us to find disjoint W_1-paths or W_2-paths that pass through $W^0_{3,1} \cup W^0_{3,2}$.

Hence, this completes the proof of Lemma 4.2. □

Lemma 4.5. *Let D be an n-vertex digraph with $\delta^0(D) \geq n/2$. If D belongs to EC2(ε), then there exists a set $\mathcal{P}$ of disjoint paths that passes through the vertex set W_3 and satisfies $|W_1 \setminus V(\mathcal{P})| = |W_2 \setminus V(\mathcal{P})|$.*

Proof. We use $W_{3,1}$ (resp., $W_{3,2}$) to define the set of vertices w in W_3 that satisfy $d^-_{W_1}(w), d^+_{W_2}(w) > (1/2 - \varepsilon^{1/3})n$ (resp., $d^+_{W_1}(w), d^-_{W_2}(w) > (1/2 - \varepsilon^{1/3})n$). Clearly, we can construct two families $\mathcal{P}_1$ and $\mathcal{P}_2$ of disjoint paths with the following properties: every path in $\mathcal{P}_1$ has the alternating form $W_1 W_{3,1} W_2 \cdots W_1 W_{3,1} W_2$ and $\mathcal{P}_1$ covers all vertices of $W_{3,1}$; similarly, every path in $\mathcal{P}_2$ follows the form $W_2 W_{3,2} W_1 \cdots W_2 W_{3,2} W_1$, and $\mathcal{P}_2$ covers all vertices of $W_{3,2}$.

In the following, we may assume $|W_1| \neq |W_2|$. Otherwise, taking $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2\}$ immediately proves the lemma. Without loss of generality, suppose $|W_1| > |W_2|$ and set $r = |W_1| - |W_2|$. We further define the set of disjoint arcs from $W_{3,1}$ to W_1 (resp., from W_1 to $W_{3,2}$) as M_1 (resp., M_2), and M_1 and M_2 are also disjoint. Choose each M_i ($i \in [2]$) to be maximal, and set $M = M_1 \cup M_2$. If $|M| = r$, then, according to the properties of $W_{3,1}$ and $W_{3,2}$, we can find r disjoint 3-paths of the forms $W_1 M_1$ and $M_2 W_1$. These 3-paths can be directly incorporated into the construction of $\mathcal{P}_1$ and $\mathcal{P}_2$, described above. Hence, the lemma holds with $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2\}$.

Now suppose that $|M| < r$. Define $W_i' = W_i \setminus V(M)$ for $i \in \{1, 3\}$. Combining the semi-degree condition $\delta^0(D) \geq n/2$ and $n = |W_1'| + |V(M)| + |W_2| + |W_3'|$, we obtain that for any vertex $u \in W_1'$,

$$\begin{aligned} d_{W_1'}(u) &\geq 2\delta^0(D) - (|W_3'| + 2|W_2| + d_M(u)) \\ &\geq |W_1'| - |W_2| + |V(M)| - d_M(u). \end{aligned} \tag{4.4}$$

Claim 4.6. *For an arc $xy \in M$, the following holds.*

(i) If there exists a vertex $u \in W_1'$ with $d_{xy}(u) = 3$, then the value of $|W_1'|$ can be effectively reduced by 1.

(ii) If there exists a pair of vertices $u_1, u_2 \in W_1'$ with $d_{xy}(u_i) = 4$ for each $i \in [2]$, then the value of $|W_1'|$ can be effectively reduced by 2.

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Proof. We only give the proof for (i) and (ii) when $xy \in M_1$, the case $xy \in M_2$ is analogous.

(i) If $yu \in A(D)$, replacing xy with the path xyu reduces the value of $|W_1'|$ by 1. Otherwise, $d_{xy}(u) = 3$ suggests that $uy, ux, xu \in A(D)$. In this case, replace xy with xuy to achieve the same effect.

(ii) Since $d_{xy}(u_i) = 4$ for each $i \in [2]$, replacing xy with the path xu_1yu_2 reduces the value of $|W_1'|$ by 2. □

Similar to the proof of Lemma 4.2, for an arc $xy \in M$, we call a vertex $u \in W_1'$ *xy-feasible* if either $d_{xy}(u) \leq 3$, or $d_{xy}(u) = 4$ and there exists a distinct vertex $u' \in W_1'$ (called a *companion* of u) such that $d_{xy}(u') = 4$. Note that all companions are required to be distinct from each other. We also call $u \in W_1'$ to be *M-feasible* if it is xy-feasible for any $xy \in M$. Furthermore, we say M is *feasible* if all but at most $2\sqrt{10\varepsilon}n$ vertices in W_1' is M-feasible.

Claim 4.7. *M is feasible.*

Proof. If for every vertex $u \in W_1'$ and every arc $xy \in M$, we have $d_{xy}(u) \leq 3$, or $d_{xy}(u) = 4$ with approximately $2\sqrt{10\varepsilon}n$ available companions of u, then M is clearly feasible. Hence, we assume that there exists a vertex $u_1 \in W_1'$ and an arc $x_1y_1 \in M$ such that $d_{x_1y_1}(u_1) = 4$, but u_1 has no available companion (note that u_1 may have had companions, but if those partners are already assigned to other vertices, they cannot be reused for u_1). For each fixed arc $xy \in M$, pair up the vertices $u \in W_1'$ satisfying $d_{xy}(u) = 4$ arbitrarily. Then for this fixed xy, at most one such vertex is left without acompanion, and hence at most one vertex is not xy-feasible. Therefore, by taking the union over all arcs $xy \in M$, at most $|M|$ vertices are not M-feasible. Since $|M| < r \leq 2\sqrt{10\varepsilon}n$, all but at most $2\sqrt{10\varepsilon}n$ vertices in W_1' are M-feasible. This yields that M is feasible. □

Recall from equation (4.4) that $d_{W_1'}(u) \geq |W_1'| - |W_2| + |V(M)| - d_M(u) := s$. Applying the pigeonhole principle, at least $(|W_1'| - 2\sqrt{10\varepsilon}n)/4$ of these vertices $u \in W_1'$ share the same value of $|V(M)| - d_M(u)$. This yields an additional set M_3 of s disjoint arcs in $D[W_1']$. Since (W_1, W_2) is ε-almost bipartite complete, Claim 4.7 allows us to combine the arcs in $M \cup M_3$ with some vertices from $W_1' \cup W_2 \cup W_3'$ to obtain a set $\mathcal{P}_3$ of disjoint (W_1, W_2)-paths such that $|V(W_1) \setminus V(\mathcal{P}_3)| = |V(W_2) \setminus V(\mathcal{P}_3)|$. For the remaining vertices in W_3, we can, as we did at the beginning of the proof, construct two disjoint sets of (W_1, W_2)-paths $\mathcal{P}_1$ and $\mathcal{P}_2$, that together cover all of these remaining vertices of W_3. Finally, let $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$ as requested. Hence this completes the proof of Lemma 4.5. □

Lemma 4.8. *Suppose that $0 < \varepsilon' \ll \varepsilon_1 \ll \varepsilon \ll 1$. Let D be an n-vertex digraph with $\delta^0(D) \geq n/2$. If D belongs to EC3($\varepsilon, \varepsilon_1$), then there is a set $\mathcal{P}$ of disjoint W_1-paths, W_3-paths, (W_2, W_4)-paths and (W_4, W_2)-paths such that $|W_2 \setminus V(\mathcal{P})| = |W_4 \setminus V(\mathcal{P})|$, and $\mathcal{P}$ passes through the vertex set W_5.*

Proof. We prove this lemma by the following two claims, the first of which states that there is a set $\mathcal{P}'$ of disjoint paths with length at most 4, such that for any $w \in W_5$ satisfying that $d^{\sigma}_{W_1}(w), d^{-\sigma}_{W_3}(w) \geq (1 - \varepsilon^{1/3})|W_1|$ for some $\sigma \in \{+, -\}$, we have $\mathcal{P}'$ through w; and the second of which shows that there exists a set $\mathcal{P}''$ of disjoint paths such that it passes through any $w \in W_5$ satisfying that $d^{\sigma}_{W_2}(w), d^{-\sigma}_{W_4}(w) \geq (1 - \varepsilon^{1/3})|W_2|$. Thus, taking $\mathcal{P} = \mathcal{P}' \cup \mathcal{P}''$ completes the proof of the lemma.

Claim 4.9. *For any vertex $w \in W_5$, if for some $\sigma \in \{+, -\}$ we have $d^{\sigma}_{W_1}(w), d^{-\sigma}_{W_3}(w) \geq (1 - \varepsilon^{1/3})|W_1|$, then for any $i \in \{1, 3\}$, there is a W_i-path passing through w.*

Proof. First, consider any vertex $w \in W_5$ such that $d^{-}_{W_1}(w), d^{+}_{W_3}(w) > (1 - 2\varepsilon^{1/3})|W_1|$. Since (W_3, W_4) is ε-almost one-way complete, we get that $e^{+}(N^{+}_{W_3}(w), W_4) \geq |N^{+}_{W_3}(w)| \cdot |W_4| - \varepsilon' n^2/2$. Consequently, there exist at least $|N^{+}_{W_3}(w)| - \varepsilon' n \geq (1 - 3\varepsilon^{1/3})|W_1|$ vertices in W_3 with the following property: for each such vertex u, we have $d^{+}_{W_4}(u) \geq |W_4|/2 \geq \varepsilon^{1/3} n/4$. Next, using the fact

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that (W_4, W_1) is also ε-almost one-way complete, we obtain $e^+(N^+_{W_4}(u), W_1) \geq |N^+_{W_4}(u)| \cdot |W_1| - \varepsilon' n^2/2$. The above inequalities imply the existence of a path of the form $wW_3W_4W_1$. On the other hand, since $d^-_{W_1}(w) > (1-2\varepsilon^{1/3})|W_1|$, there is an arc from some vertex in W_1 to w. Combining these, we obtain a W_1-path of length at most 4 that passes through w.

By symmetry, we can also find a W_3-path of length at most 4 passing through w. Indeed, from the ε-almost one-way completeness of (W_4, W_1), we have $e^-(W_1, W_4) \geq |W_1| \cdot |W_4| - \varepsilon' n^2/2$. Together with $|N^-_{W_1}(w)| \geq (1-\varepsilon^{1/3})|W_1|$, we get that there exists a vertex set $U \subset W_4$ with $|U| \geq |W_4| - \varepsilon' n$ such that for any vertex $u \in U$, we have $|N^+(u) \cap N^-_{W_1}(w)| \geq |N^-_{W_1}(w)|/2$. Further, since (W_3, W_4) is ε-almost one-way complete, we also have $e^-(U, W_3) \geq |U| \cdot |W_3| - \varepsilon' n^2/2$. These facts guarantee a path of the form $W_3W_4W_1w$. Together with the arc from w to W_3 (which follows from $d^+_{W_3}(w) \geq (1-2\varepsilon^{1/3})|W_1|$), we obtain the desired W_3-path passing through w.

Finally, for any vertex w satisfying $d^+_{W_1}(w), d^-_{W_3}(w) \geq (1-2\varepsilon^{1/3})|W_1|$, we can similarly prove that there is a W_1-path (W_3-path) of length at most 4 through it. Since the proof strategy of this case essentially follows from that in the preceding two paragraphs, we omit the repetitive details. This completes the proof of the claim. □

We partition W_5 into $W_{5,1} \cup W_{5,3}$, assigning vertices satisfying both alternatives arbitraily, such that for every vertex $w \in W_{5,1}$ satisfying $d^-_{W_1}(w), d^+_{W_3}(w) > (1-\varepsilon^{1/3})|W_1|$ and every vertex $w \in W_{5,3}$ satisfies $d^+_{W_1}(w), d^-_{W_3}(w) > (1-\varepsilon^{1/3})|W_1|$. If $|W_{5,1}| = |W_{5,3}|$, then as established in the proof of Lemma 4.2, we can find a set $\mathcal{P}'$ of disjoint W_1-paths with the form $W_1W_{5,1}W_3W_{5,3}W_1$, and passing through $W_{5,1} \cup W_{5,3}$. So in the following, we consider the case of $|W_{5,1}| \neq |W_{5,3}|$. Without loss of generality, suppose that $|W_{5,1}| > |W_{5,3}|$. By repeatedly applying Claim 4.9 to $|W_{5,1}| - |W_{5,3}|$ vertices $w \in W_{5,1}$, we obtain $|W_{5,1}| - |W_{5,3}|$ disjoint W_1-paths of the form $W_1wW_3W_4W_1$. This can be done because the number of arcs between the relevant vertex sets significantly exceeds the cardinality of W_5. During this process, we use at most $10\varepsilon^{1/2}|W_2|$ from W_2 and at most $10\varepsilon^{1/2}|W_4|$ vertices from W_4. For the remaining in $W_{5,1}$ and all vertices in $W_{5,3}$, we can, as before, find disjoint W_1-paths of the form $W_1W_{5,1}W_3W_{5,3}W_1$. Let $\mathcal{P}'$ be the set of all these disjoint W_1-paths. By construction, $\mathcal{P}'$ passes through $W_{5,1} \cup W_{5,3}$. For convenience, set $W'_i = W_i \setminus V(\mathcal{P}')$ for each $i \in [5]$. Clearly, $|W'_i| \geq \varepsilon^{1/3} n/4$. Moreover, for any $w \in W'_5$, we have either $d^-_{W_2}(w), d^+_{W_4}(w) \geq (1-\varepsilon^{1/3})|W_2|$ or $d^-_{W_4}(w), d^+_{W_2}(w) \geq (1-\varepsilon^{1/3})|W_2|$.

Claim 4.10. *There is a set $\mathcal{P}''$ of disjoint W_1-paths, W_3-paths, (W_2, W_4)-paths and (W_4, W_2)-paths, satisfying that $|W'_2 \setminus V(\mathcal{P}'')| = |W'_4 \setminus V(\mathcal{P}'')|$, and $\mathcal{P}''$ passes through W'_5.*

Proof. We denote by $W'_{5,1}$ (resp., $W'_{5,2}$) to be the set of vertices $w \in W'_5$ satisfying that $d^-_{W_2}(w), d^+_{W_4}(w) \geq (1-\varepsilon^{1/3})|W_2|$ (resp., $d^-_{W_4}(w), d^+_{W_2}(w) \geq (1-\varepsilon^{1/3})|W_2|$). Clearly, since (W_2, W_4) is an ε-almost complete bipartite digraph, we can construct two families of disjoint paths $\mathcal{P}_1$ and $\mathcal{P}_2$ with the following properties: each path in $\mathcal{P}_1$ has the form $W'_2W'_{5,1}W'_4 \cdots W'_2W'_{5,1}W'_4$, and $\mathcal{P}_1$ passes through $W'_{5,1}$; and each path in $\mathcal{P}_2$ has the form $W'_4W'_{5,2}W'_2 \cdots W'_4W'_{5,2}W'_2$, and and $\mathcal{P}_2$ passes through $W'_{5,2}$. If $|W'_2| = |W'_4|$, then we take $\mathcal{P}'' = \{\mathcal{P}_1, \mathcal{P}_2\}$ as required.

Therefore, in what follows we assume $|W'_2| \neq |W'_4|$. First we consider the case when $|W_2| \geq |W_4|$ and $|W'_2| \geq |W'_4|$. We take a family $\mathcal{P}_0$ (maybe, $\mathcal{P}_0 = \mathcal{P}'$ and not necessarily disjoint from $\mathcal{P}'$) of disjoint W_1-paths, W_3-paths, (W_2, W_4)-paths and (W_4, W_2)-paths such that the positive difference $r := |W'_2 \setminus V(\mathcal{P}_0)| - |W'_4 \setminus V(\mathcal{P}_0)|$ is minimised. Let $W''_2 := W'_2 \setminus V(\mathcal{P}_0)$, and then take an arbitrary vertex $u \in W''_2$. Then first, we have $d^+_{W_1}(u) = 0$ and $d^-_{W_3}(u) = 0$. If not, we could extend $\mathcal{P}_0$ by a suitable W_1-path and a W_3-path, respectively, containing u, which would reduce r by 1, contradicting the minimality of $\mathcal{P}_0$. Second, for any vertex $w \in W'_5$ satisfying $d^-_{W_2}(w), d^+_{W_4}(w) \geq (1-\varepsilon^{1/3})|W_2|$ (resp., $d^+_{W_2}(w), d^-_{W_4}(w) \geq (1-\varepsilon^{1/3})|W_2|$), the minimality of $\mathcal{P}_0$ implies $wu \notin A(D)$ (resp., $uw \notin A(D)$). Otherwise, a path adjustment similar to the proof of Lemma 4.5 would reduce r by 1. Third, for each path P of the form $W_1w_1W_3w_2W_1$ where $w_1 \in W_{5,1}$ and $w_2 \in W_{5,3}$,

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we have $d_{\{w_1\}}(u) = 0$ and $d_{\{w_2\}}(u) \le 2$. Since, otherwise, we can replace P with the path of the form $W_1uw_1W_3w_2W_1$ and $W_1w_1uW_3w_2W_1$, respectively, a contradiction to the minimality of r. Fourthly, each path P of the form $W_1u'W_3wW_1$ where $u' \in W_2$ and $w \in W_{5,3}$, we have $d_{\{u'\}}(u) = 0$ and $d_{\{w\}}(u) \le 2$. Otherwise, we can replace P with the path of the form $W_1W_2W_2W_3wW_1$, a contradiction to the minimality of r again.

Consequently, from the four properties above, we can conclude that $d_{\mathcal{P}_0}(u) \le |V(\mathcal{P}_0)|$. Together with the lower bound of $\delta^0(D)$, we have $d_{W_2''}(u) \ge |W_2''| - |W_4''| = r$. This implies that there exist r pairwise disjoint arcs in $D[W_2'']$. Recall that for any vertex $w \in W_5'$, it satisfies $d^{\sigma}_{W_2'}(w), d^{-\sigma}_{W_4'}(w) > (1 - 2\varepsilon^{1/3})|W_2'|$ for some $\sigma \in \{+, -\}$. Since (W_2'', W_4'') is an ε-almost complete bipartite digraph, similar to the proof of Lemma 4.5, there exist disjoint (W_2, W_4)-paths that cover these r arcs and all vertices in W_5'. Let $\mathcal{P}''$ be the union of these paths and $\mathcal{P}_0$, then we have $|W_2'' \setminus V(\mathcal{P}'')| = |W_4'' \setminus V(\mathcal{P}'')|$.

The proof for the remaining cases is analogous to that of the case presented above. Therefore, we omit it to avoid redundancy. □

Hence by the two claims above, this completes the proof of this lemma. □

Before giving the proofs of Theorems 1.3 and 1.5, we also need the following conclusions that can help us to iteratively find disjoint H-subdivisions of arbitrary orders in a dense digraph with sufficiently large semi-degree.

Proposition 4.11. *Suppose that C is a positive integer and η is any real number with $0 < 1/C \ll \eta \ll 1$. For any integer partition $a = a_1 + \cdots + a_k$ with $a \ge Ck$ and $a_i \ge 2$ for each i, let $T = (A, B)$ be a balanced bipartite digraph of order $2a$. For any $\sigma \in \{+, -\}$, if T satisfies that for any vertex $u \in A$ and any $v \in B$, $d^{\sigma}_B(u) \ge (1 - \eta)a$ and $d^{\sigma}_A(v) \ge (1 - \eta)a$, then for any vertex set $U \subseteq V(T)$ with $U \cap A = \{x_1^0, \ldots, x_k^0\}$ and $U \cap B = \{y_1^0, \ldots, y_k^0\}$, T contains k disjoint paths $P_1, \ldots, P_k$ such that for each $i \in [k]$, the initial and terminal of P_i is x_i^0 and y_i^0, respectively, and $|V(P_i) \cap A| = |V(P_i) \cap B| = a_i$.*

Proof. For convenience, for any $i \in [k]$, let $r_i = a_i - 1$. For each $i \in [k]$, we choose $r_i + 1$ vertices $x_i^0, x_i^1, \ldots, x_i^{r_i}$ from A with the last vertex $x_i^{r_i} \in N_A^-(y_i^0)$ such that all these vertices are distinct and their union is A. We construct an auxiliary bipartite graph $Q = (\tilde{A}, B')$ such that $\tilde{A} = \bigcup_{i=1}^{k}\{(x_i^0, x_i^1), (x_i^1, x_i^2), \ldots, (x_i^{r_i-1}, x_i^{r_i})\}$ and $B' = B \setminus U$, where each 'vertex' (x_i^j, x_i^{j+1}) in $\tilde{A}$ connects with all the vertices in $N_T^+(x_i^j) \cap N_T^-(x_i^{j+1})$. Obviously, any perfect matching in Q that saturates $\tilde{A}$ corresponds to an embedding of disjoint paths $P_1, \ldots, P_k$ in T as required. We claim that such a perfect matching exists. In fact, $|\tilde{A}'| = |B'| = a - k$ and $d_Q(z) \ge 2(1 - \eta)|B'| - |B'| \ge (1 - 2\eta)|B'|$ for $z \in \tilde{A}$. Additionally, we deduce that $d_Q(u) \ge (1 - 2\eta)|\tilde{A}|$ for any vertex $u \in B'$. Therefore, the degrees of the vertices in Q are all at least $(1 - 2\eta)(a - k)$. By the König-Hall's theorem, we conclude that Q contains a perfect matching, which completes the proof of this proposition. □

Recall that for a digraph H, we have $m_H = \min_{H=(U_1,U_2)} \big||U_1| + e(U_2) - (|U_2| + e(U_1))\big|$. For a bipartition (U_1, U_2) attaining this minimum, we further set $m_H' = |V(H)| + e(U_2) + e(U_1)$, Using the proposition above, we obtain the following result.

Lemma 4.12. *Let H be any digraph with h arcs and $\delta(H) \ge 1$. Assume that $0 < 1/C \ll \xi \ll 1$ and let $a = a_1 + \cdots + a_k$ satisfy $a \ge Chk$ and $a_i \ge (3h + |V(H)| - m_H')/2$ for each i. Let T be a digraph of order $km_H' + 2a$ with bipartition $V(T) = A \cup B$, where $|A| - |B| = t$ with $t = 0$ if $k \equiv 0 \pmod 2$ and $t = m_H$ if $k \equiv 1 \pmod 2$. If for any $\sigma \in \{+, -\}$, every $u \in A$ and $v \in B$, we have that $d^{\sigma}_B(u), d^{\sigma}_A(v) \ge (1 - \xi)a$, then the following hold.*

(i) If $k = 1$, then T contains a spanning H-subdivision.

(ii) T admits a perfect H-subdivision tiling in which the i-th subdivision has order $m_H' + 2a_i$, for $i \in [k]$.

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Proof. Without loss of generality, assume that the bipartition of H achieving m_H is (U_1, U_2) and, after swapping U_1 and U_2 if necessary, assume that $m_H = |U_1| + e(U_2) - (|U_2| + e(U_1)) \geq 0$. Then, by definition, $m'_H = |U_1| + e(U_2) + (|U_2| + e(U_1))$. Write $U_1 = \{x_1, \ldots, x_r\}$, $U_2 = \{y_1, \ldots, y_s\}$. We first prove statement (i), and then statement (ii).

(i) In this case, we have $k = 1$. Consequently, $|A| = |B| + m_H$. We select the image set of U_1 from A and the image set of U_2 from B. For simplicity, we still denote each image set by U_i for $i \in [2]$, with $U_1 = \{x_1, \ldots, x_r\}$, $U_2 = \{y_1, \ldots, y_s\}$. Below we first use steps (i1) and (i2) to associate each arc in H with two short subpaths; we then apply Proposition 4.11 to concatenate these subpaths pairwise into one long path.

Step (i1): arcs between U_1 and U_2. Consider a pair $(x_i, y_j) \in U_1 \times U_2$. If $x_i y_j \in A(H)$, then, using the lower bounds on the semi-degrees of vertices in A and B, we can find two disjoint paths $P^1_{i,j} = x_i v_{i,j} x^1_{i,j}$ and $Q^1_{i,j} = y^1_{i,j} u_{i,j} y_j$ with $x^1_{i,j}, u_{i,j} \in A \setminus U_1$ and $v_{i,j}, y^1_{i,j} \in B \setminus U_2$. If $y_j x_i \in A(H)$, then similarly, we find two disjoint paths $P^2_{i,j} = y^2_{i,j} x_i$ and $Q^2_{i,j} = y_j x^2_{i,j}$ with $x^2_{i,j} \in A \setminus U_1$ and $y^2_{i,j} \in B \setminus U_2$.

Step (i2): arcs inside U_1 and U_2. Consider a pair $(x_i, x_j) \in U_1 \times U_1$. If $x_i x_j \in A(H)$, then using the abundance of arcs between A and B again, we obtain disjoint paths $R^1_{i,j} = x_i v^2_{i,j} x^3_{i,j}$ and $R^2_{i,j} = y^3_{i,j} x_j$, where $v^2_{i,j}, y^3_{i,j} \in B \setminus U_2$ and $x^3_{i,j} \in A \setminus U_1$. Similarly, consider a pair $(y_i, y_j) \in U_2 \times U_2$. We find two disjoint paths $S^1_{i,j} = y_i x^4_{i,j}$ and $S^2_{i,j} = y^4_{i,j} u^3_{i,j} y_j$ with $x^4_{i,j}, u^3_{i,j} \in A \setminus U_1$ and $y^4_{i,j} \in B \setminus U_2$.

Note that in this process, we allow for some paths to be single vertices, that is, a path is then considered a vertex in some U_i for $i \in [2]$. Furthermore, all vertices outside $U_1 \cup U_2$ that appear in these paths are chosen distinctly. Also, at each choice we avoid previously used vertices. Because the total number of new vertices employed is at most $3h$ and every vertex still has at least $(1 - \xi)a - 3h \geq (1 - 3\xi/2)a$ (since $h \leq a/(Ck)$ and $1/C \ll \xi$) available neighbours, the greedy selection succeeds.

After performing the corresponding step for every arc of H, we obtain a family of internally disjoint paths that realise the connections prescribed by H. The total number of vertices consumed is $2e^+(U_1, U_2) + e^-(U_1, U_2) + e(U_1) + 2e(U_2)$ from A and $2e^+(U_1, U_2) + e^-(U_1, U_2) + 2e(U_1) + e(U_2)$ from B. Let A_0 (respectively, B_0) denote the subset of A (respectively, B) obtained by deleting all vertices that lie on any of the paths constructed above. Clearly, since $|U_1| + e(U_2) - (|U_2| + e(U_1)) = m_H$, we have $|A_0| = |B_0|$.

Recall that $r = |U_1|, s = |U_2|$. Sequentially, for every $\sigma \in \{+, -\}$, every $u \in A_0 \cup \bigcup_{i \in [r], j \in [s], l \in [4]} \{x^l_{i,j}\}$ and $v \in B_0 \cup \bigcup_{i \in [r], j \in [s], l \in [4]} \{y^l_{i,j}\}$ (where we allow that some sets $\{x_{i_0,j_0}\}$ and $\{y_{i_1,j_2}\}$ may be empty for certain indices $i_0, i_1 \in [r], j_0, j_1 \in [s]$), we have that $d^\sigma_{B'}(u), d^\sigma_{A'}(v) \geq (1 - \xi)a - 3h - |V(H)|$. Using the facts that $|V(H)| \leq 2h$, $h \leq \frac{a}{Ck}$ and $0 < 1/C \ll \xi \ll 1$, we obtain

$$d^\sigma_{B'}(u), d^\sigma_{A'}(v) \geq (1 - \xi)a - 3h - 2h \geq (1 - \xi - 5/(Ck))a \geq (1 - \xi - 5\xi/k)a \geq (1 - 2\xi)a.$$

To complete a spanning H-subdivision, it now suffices to find, in almost complete bipartite digraph (A_0, B_0), a collection of disjoint paths $P_1, \ldots, P_h$, such that for each i, the initial of P_i is some $x^l_{i,j}$ and its terminal vertex is the corresponding $y^l_{i,j}$ (where $l \in [4]$). This is guaranteed by Proposition 4.11 with 2ξ playing the role of η. Hence, we have shown that T contains a spanning H-subdivision, proving the statement (i).

(ii) We select $\lceil k/2 \rceil$ image sets of U_1 and $\lfloor k/2 \rfloor$ image sets of U_2 from A, and correspondingly $\lceil k/2 \rceil$ image sets of U_2 and $\lfloor k/2 \rfloor$ image sets of U_1 from B. This yields a total of k matched pairs. Label the first $\lceil k/2 \rceil$ pairs as $(U_{1,1}, U_{2,1}), \ldots, (U_{1,\lceil k/2 \rceil}, U_{2,\lceil k/2 \rceil})$, and the remaining $\lfloor k/2 \rfloor$ pairs as $(U_{2,\lceil k/2 \rceil + 1}, U_{1,\lceil k/2 \rceil + 1}), \ldots, (U_{2,k}, U_{1,k})$.

For each pair $(U_{1,i}, U_{2,i})$ with $i \in [\lceil k/2 \rceil]$ and each pair $(U_{2,j}, U_{1,j})$ with $j \in \{\lceil k/2 \rceil + 1, \ldots, k\}$, we carry out steps (i1) and (i2) from part (i) analogously. This process constructs, for each index i, a set of $2h$ paths $P^1_{i,1}, P^2_{i,1} \ldots, P^1_{i,h}, P^2_{i,h}$ whose intersection is contained in $U_{1,i} \cup U_{2,i}$, and for each index j, a set of $2h$ paths $Q^1_{j,1}, Q^2_{j,1} \ldots, Q^1_{j,h}, Q^2_{j,h}$ whose intersection lies in $U_{2,j} \cup U_{1,j}$. These

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constructions satisfy the balance properties that $|A \cap \bigcup_{l\in[h]} V(P^1_{i,l} \cup P^2_{i,l})| - |B \cap \bigcup_{l\in[h]} V(P^1_{i,l} \cup P^2_{i,l})| = m_H$ and $|A \cap \bigcup_{l\in[h]} V(Q^1_{j,l} \cup Q^2_{j,l})| - |B \cap \bigcup_{l\in[h]} V(Q^1_{j,l} \cup Q^2_{j,l})| = -m_H$.

Let A_0 and B_0 denote the subsets of A and B, respectively, obtained by deleting all vertices appearing in any of the paths constructed above. Since each construction for index $i \in [k]$ contributes a net imbalance of either m_H or $-m_H$, summing over all k indices and taking into account the definition of t, we obtain $|A_0| = |B_0|$. Moreover, for any $\sigma \in \{+, -\}$, every vertex $u \in A_0$ and $v \in B_0$ satisfies

$$d^\sigma_{B_0}(u), d^\sigma_{A_0}(v) \geq (1-\xi)a - \xi a/10 - 5hk - 2k \geq (1-\xi)a - 6a/C \geq (1-2\xi)a.$$

This follows because $|V(H)| \leq 2h$, $h \leq a/(Ck)$ and the inequality $0 < 1/C \ll \xi \ll 1$.

Finally, we will apply Proposition 4.11 with 2ξ playing the role of η. This yields a collection of disjoint (A_0, B_0)-paths such that: **(ii1)** for each $i \in [\lceil k/2 \rceil]$ and $l \in [h]$, there is a path of specified length from the terminal of $P^1_{i,l}$ to the initial of $P^2_{i,l}$; **(ii2)** for $j \in \{\lceil k/2 \rceil + 1, \ldots, k\}$ and $l \in [h]$, there is a path of specified length from the terminal of $Q^1_{j,l}$ to the initial of $Q^2_{j,l}$.

This completes the construction of a perfect H-subdivision tiling in T where the subdivisions have orders $m'_H + 2a_1, \ldots, m'_H + 2a_k$, respectively, thereby proving (ii). □

4.2 Proof of Theorem 1.3

Let $C \geq C_0$ be an integer and choose parameters ε', ε_1 and ε satisfying the hierarchy $1/C \ll \varepsilon' \ll \varepsilon_1 \ll \varepsilon \ll 1$. In this section, we present the proof of Theorem 1.3 for the case where D is not ε'-stable.

Proof. Let H be a digraph with h arcs and minimum degree $\delta(H) \geq 1$, and let D be a digraph on $n \geq Ch$ vertices with $\delta^0(D) \geq \frac{n+h}{2} - 1$, as described in Theorem 1.3. If $h = 1$, which corresponds to the case where H consists of a single arc, then a spanning H-subdivision in D is exactly a Hamiltonian path. In this scenario, the minimum semi-degree satisfies $\delta^0(D) \geq \frac{n+h}{2} - 1 = \frac{n-1}{2}$. By Ghouila-Houri's theorem [9], this minimum semi-degree condition ensures that D contains a Hamiltonian path. Therefore, in what follows, we assume $h \geq 2$. By Lemma 4.1, the digraph D belongs to EC1(ε), EC2(ε), or EC3($\varepsilon, \varepsilon_1$).

Case a.1: *D* belongs to EC1(ε). In this case, the vertex set $V(D)$ can be partitioned into three disjoint sets W_1, W_2 and W_3, such that for each $i \in [2]$, we have $|W_i| = (1/2 - \varepsilon)n \pm \sqrt{10\varepsilon}|W_i|$, and each induced subdigraph $D[W_i]$ is ε-almost complete. Since $\delta^0(D) \geq (n+h)/2 - 1$, for any vertex $u \in W_1$ and any vertex $v \in W_2$, the following holds.

$$\begin{cases} uv \in A(D) & \text{and} \quad |N^+(u) \cap N^-(v)| \geq 2\delta^0(D) - (n-1) \geq h-1, \text{or} \\ uv \notin A(D) & \text{and} \quad |N^+(u) \cap N^-(v)| \geq 2\delta^0(D) - (n-2) \geq h. \end{cases}$$

Symmetrically, we also have that either

$$\begin{cases} vu \in A(D) & \text{and} \quad |N^-(u) \cap N^+(v)| \geq 2\delta^0(D) - (n-1) \geq h-1, \text{or} \\ vu \notin A(D) & \text{and} \quad |N^-(u) \cap N^+(v)| \geq 2\delta^0(D) - (n-2) \geq h. \end{cases}$$

This implies the existence of two disjoint minimal paths $P_1 = u_1 \cdots v_1$ and $P_2 = v'_1 \cdots u'_1$, each of length at most 2, such that P_1 is a (W_1, W_2)-path and P_2 is a (W_2, W_1)-path. Let $W'_3 = W_3 \setminus V(P_1 \cup P_2)$. Furthermore, by Lemma 4.2, there exist two disjoint path sets $\mathcal{P}_1$ and $\mathcal{P}_2$, such that $\mathcal{P}_1 \cup \mathcal{P}_2$ covers all vertices of W'_3, and for each $j \in [2]$, $\mathcal{P}_j$ is a set of disjoint W'_i-paths of length at most 5. Note that it is possible that P_1 and P_2 are subpaths of some path(s) in $\mathcal{P}_1 \cup \mathcal{P}_2$. Utilising property (1.1) of EC1(ε) and Proposition 4.11, after first dealing with a bounded number

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of exceptional vertices with not too big semi-degrees in W_2, we obtain a Hamiltonian path P in $D[W_2 \cup V(\mathcal{P}_2)]$ that connects v_1 to v_1'.

Furthermore, the property (1.1) of EC1(ε) ensures that except for a set R of W_1, every vertex w in $W_1 \setminus R$ satisfies $\delta^0_{W_1}(w) \geq (1-10\sqrt{\varepsilon})|W_1|$ and any $u \in R$ has $\delta^0_{W_1}(u) \geq \varepsilon^{1/3}|W_1|/2$. Then, evidently, we can use some vertices from $W_1 \setminus (R \cup \{u_1, u_2\})$ to connect all vertices of R into a single path, denoted $P' = u_2' \cdots u_2$. Moreover, there exists a vertex $u \in W_1 \setminus (V(P') \cup \{u_1, u_2\})$ such that $P_2 \circ u \circ P' := P_0$ form a path. Set $W_1' := W_1 \setminus V(P_1 \cup P_0)$ and $P' := P_1 \circ P \circ P_0$.

Without loss of generality, let xy be an arc in H. We then let u_1 serve as the embedding of x, and u_2 as the embedding of y. Let $H^- = H - xy$. Partition $W_1' \cup \{u_1, u_2\}$ into two subsets W_1^1 and W_1^2 such that $|W_1^1| - |W_1^2| = m_{H^-}$. Finally, applying the proof of Lemma 4.12(i) with the images of x and y fixed as u_1 and u_2, respectively, we can find a spanning H^--subdivision H' in $D[W_1^1 \cup W_1^2]$. Possible isolated vertices of H^- cause no difficulty. Clearly, the union $H' \cup P'$ then forms the required spanning H-subdivision, completing the proof of Theorem 1.3 for the Extremal Case 1 with parameter ε (EC1(ε)).

Case a.2: D belongs to EC2(ε). In this case, the vertex set $V(D)$ can be partitioned into three disjoint sets W_1, W_2 and W_3 such that for each $i \in [2]$, we have $|W_i| = (1/2 - \varepsilon)n \pm \sqrt{10\varepsilon}|W_i|$, and the pair (W_1, W_2) forms an ε-almost complete bipartite digraph. Without loss of generality, assume $|W_1| \geq |W_2|$. Choose a bipartitions (U_1, U_2) of H attaining m_H, and swap the two parts if necessary, so that $t := |U_1| + e(U_2) - (|U_2| + e(U_1)) \geq 0$. By Lemma 2.1, we have $t \leq h/2 - 1$.

We now construct a collection $\mathcal{P}$ of disjoint paths that passes W_3. The construction splits into two cases based on the balance between W_1 and W_2:

(a.2.1) If $|W_1| - t \geq |W_2|$, then by the proof of Lemma 4.5, there exists a collection $\mathcal{P}$ of disjoint paths such that $|W_1 \setminus V(\mathcal{P})| = |W_2 \setminus V(\mathcal{P})| + t$ and passes through W_3.

(a.2.2) Otherwise, we have $|W_1| - t < |W_2|$. Since $t \leq h/2 - 1$, a procedure analogous to the proof of Lemma 4.5 also yields a collection $\mathcal{P}$ of disjoint paths such that $|W_1 \setminus V(\mathcal{P})| = |W_2 \setminus V(\mathcal{P})| + t$ and passes through W_3.

Furthermore, for $j \in [2]$ and $\sigma \in \{+, -\}$, any vertices $u, v \in W_j$ satisfy $|N^{\sigma}_{W_{3-j}}(u) \cap N^{-\sigma}_{W_{3-j}}(v)| \geq (1-10\sqrt{\varepsilon})|W_{3-j}| + \frac{\varepsilon^{1/3}|W_{3-j}|}{8} - |W_{3-j}| > \frac{\varepsilon^{1/3}|W_{3-j}|}{10} > |W_3|$. This inequality guarantees robust connectivity between the sets, which ensures the existence of a specific path P of the form $u_1 P_1^1 v_1 P_1^2 u_2 P_1^3 u_3 v_2 P_1^4 v_3 \cdots$, where for each i, $u_i \in W_1$, $v_i \in W_2$, and the path $P_i^j \in \mathcal{P}$ are of the following types: P_i^1 is a (W_2, W_1)-path, P_i^2 is a (W_1, W_2)-path, P_i^3 is a W_1-path and P_i^4 is a W_2-path. This path P also satisfies $|W_1 \setminus V(P)| = |W_2 \setminus V(P)| + t$.

We then embed U_1 into $W_1 \setminus V(P)$ and U_2 into $W_2 \setminus V(P)$. For simplicity, we still denote these embedded sets by U_1 and U_2, respectively. Finally, applying the proof of Lemma 4.12(i) to the remaining ε-almost complete bipartite digraph $(W_1 \setminus V(P), W_2 \setminus V(P))$ yields the desired spanning H-subdivision. In this construction, the path P can naturally serve as a subpath of one of the branch paths in the subdivision.

Case a.3: D belongs to EC3($\varepsilon, \varepsilon_1$). In this case, $V(D) = \bigcup_{i=1}^{5} W_i$ with $|W_i| \approx |W_{i+2}|$ for $i \in [2]$ satisfying the following properties: for each $i \in \{1, 3\}$, $D[W_i]$ is ε-almost complete; the pair (W_2, W_4) is ε-almost bipartite complete; and for any $j \in [4]$, the pair (W_j, W_{j+1}) is ε-almost one-way complete (where the subscript of W_{j+1} taken the remainder of modulo 4 when $j = 4$).

Subcase a.3.1: $|W_1| \geq |W_2|$.

By Lemma 4.8, there exists a set $\mathcal{P}$ of disjoint W_1-paths, W_3-paths, (W_2, W_4)-paths and (W_4, W_2)-paths, such that $|W_2 \setminus V(\mathcal{P})| = |W_4 \setminus V(\mathcal{P})|$ and $\mathcal{P}$ passes through W_5. For convenience, we denote by $\mathcal{P}_i$ the set of all W_i-paths in $\mathcal{P}$ for $i \in \{1, 3\}$, by $\mathcal{P}_2^1$ the set of all (W_2, W_4)-paths, and by $\mathcal{P}_2^2$ the set of all (W_4, W_2)-paths. For each $i \in [4]$, let $W_i' = W_i \setminus V(\mathcal{P})$.

Step a.3.1 Constructing a W_1-path passing through $W_3' \cup V(\mathcal{P}_3)$.

Since $D[W_3]$ is ε-almost complete, we can argue similarly to Case a.1 to obtain a Hamiltonian path $P = a \cdots b$ in $D[W_3' \cup V(\mathcal{P}_3)]$. Utilising the property that (W_j, W_{j+1}) is ε-almost one-way

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complete for any $j \in [4]$, we can find two disjoint 3-paths x_1y_1a and by_2x_2 with $x_1, x_2 \in W_1'$, $y_1 \in W_2'$ and $y_2 \in W_4'$. This implies the existence of a W_1-path $P_1 = x_1 \cdots x_2$ passing through $W_3' \cup V(\mathcal{P}_3)$.

Step a.3.2 Constructing a W_1-path passing through $W_2' \cup W_4' \cup V(\mathcal{P}_2^1 \cup \mathcal{P}_2^2)$.

Since (W_2', W_4') is ε-almost complete balanced bipartite, we can find a Hamiltonian (W_2, W_4)-path P' in $D[W_2' \cup W_4' \cup V(\mathcal{P}_2^1 \cup \mathcal{P}_2^2)]$ of the form

$$P_1^1 u_1 v_1 P_2^1 \cdots u_{k-1} v_{k-1} P_k^1 u_{k+1} P_1^2 v_{k+1} u_{k+2} P_2^2 \cdots v_{k+s-1} u_{k+s} P_s^2,$$

where for any integer i, $P_i^1 \in \mathcal{P}_2^1$ and $P_i^2 \in \mathcal{P}_2^2$, with $u_i \in W_2'$ and $v_i \in W_4'$. Furthermore, as (W_1, W_2) and (W_4, W_1) are ε-almost one-way complete, there exist vertices $x_1', x_2' \in W_1'$ such that there is an arc from x_1' to the initial of P' and an arc from the terminal of P' to x_2'. Consequently, we obtain a W_1-path $P_2 = x_1' \cdots x_2'$ passing through $W_2' \cup W_4' \cup V(\mathcal{P}_2^1 \cup \mathcal{P}_2^2)$.

Step a.3.3: Constructing paths within W_1.

Since $D[W_1]$ is also ε-almost complete, there exists a W_1-path $P_3 = x_1'' \cdots x_2''$ passing through $V(\mathcal{P}_1)$ and exceptional vertices of W_1'. Finally, in the ε-almost complete digraph $D[W_1' \setminus (\{x_1, x_2'\} \cup V(P_3))]$, we can find two disjoint paths $P_4 = x_2 \cdots x_1'$ and $P_5 = x_2' \cdots x_1''$. Let $P := P_1 \circ P_4 \circ P_2 \circ P_5 \circ P_3$.

Without loss of generality, let xy be an arc in H. We then let x_1 serve as the embedding of x, and x_2'' as the embedding of y. Let $H^- = H - xy$. Partition $(W_1 \setminus V(P)) \cup \{x_1, x_2''\}$ into two subsets W_1^1 and W_1^2 such that $|W_1^1| - |W_1^2| = m_{H^-}$. Finally, applying the proof of Lemma 4.12(i) with the images of x and y fixed x_1 and x_2'', respectively, we can find a spanning H^--subdivision H' in $D[W_1^1 \cup W_1^2]$. Possible isolated vertices of H^- cause no difficulty. Clearly, the union $H' \cup P$ then forms the required spanning H-subdivision.

Subcase a.3.2: $|W_1| < |W_2|$.

Let (U_1, U_2) be a bipartition of H satisfying the conclusion of Lemma 2.1 and define $t := |U_1| + e(U_2) - (|U_2| + e(U_1)) \geq 0$. Following an approach analogous to that in Case a.2, and by an argument similar to that of Lemma 4.8, there exists a set $\mathcal{P}$ of disjoint W_1-paths, W_3-paths, (W_2, W_4)-paths and (W_4, W_2)-paths, such that $|W_2 \setminus V(\mathcal{P})| = |W_4 \setminus V(\mathcal{P})| + t$ and $\mathcal{P}$ passes through W_5. We use the same notation $\mathcal{P}_i, \mathcal{P}_2^1, \mathcal{P}_2^2$, as in Subcase a.3.1, and define $W_i' = W_i \setminus V(\mathcal{P})$ for $i \in [4]$.

Moreover, since each $D[W_i]$ (for $i \in \{1, 3\}$) is ε-almost complete, there is a Hamiltonian path $Q_i = x_i \cdots y_i$ of $D[W_i' \cup V(\mathcal{P}_i)]$. By the ε-almost one-way completeness of (W_i, W_{i+1}) for $i \in [4]$, where the indice is taken modulo 4 when $i = 4$, we can find vertices $u_1, u_2 \in W_2'$ and $v_1, v_2 \in W_4'$ such that $\{v_1x_1, y_1u_1, u_2x_3, y_3v_2\} \subset A(D)$. This yields two disjoint paths: $Q_1' = v_1P_1u_1$ passing through $W_1' \cup V(\mathcal{P}_1)$, and $Q_3' = u_2P_3v_2$ passing through $W_3' \cup V(\mathcal{P}_3)$.

We now embed U_1 into $W_2' \setminus \{u_1, u_2\}$ and U_2 into $W_4' \setminus \{v_1, v_2\}$ (retaining the notation U_1 and U_2 for the embedded sets, respectively). Finally, applying Lemma 4.12(i) to the remaining ε-almost complete bipartite digraph between these two sets produces the desired spanning H-subdivision. In this construction, the paths from $\mathcal{P}_2^1$ and $\mathcal{P}_2^2$, as well as the paths Q_1' and Q_3' are incorporated as subpaths of the corresponding branch paths.

Hence, this completes the proof of Theorem 1.3. □

4.3 Proof of Theorem 1.5

Proof. Let H be a digraph with h arcs and minimum degree $\delta(H) \geq 1$. Let C_0 be the constant from Theorem 1.5. For any integer m and any integer partition $n = n_1 + \cdots + n_m$, let D be a digraph on n vertices satisfying $\delta^0(D) \geq \frac{n+m+h}{2} - 1$. By increasing C_0 if necessary, we may assume that $n \geq C_0hm$. Fix a hierarchy of parameters $1/C_0 \ll \varepsilon' \ll \varepsilon_1 \ll \varepsilon \ll 1$. We prove Theorem 1.5 under the assumption that D is not ε'-stable. If $h = 1$, that is, $H = xy$, then m disjoint

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H-subdivisions correspond simply to m disjoint paths. Since in this case, $\delta^0(D) \geq \frac{n+m+h}{2} - 1 = \frac{n+m-1}{2} \geq \frac{n-1}{2}$, by Ghouila-Houri's theorem [9], D contains a Hamiltonian path. We can then partition this Hamiltonian path consecutively into m disjoint segments whose orders are respectively $n_1, \ldots, n_m$. Therefore, in what follows we assume $h \geq 2$. We also assume $m \geq 2$, since otherwise the statement reduces to Theorem 1.3, which has already been proved.

By Lemma 4.1, every such digraph D either belongs to EC1(ε) or EC2(ε) or EC3($\varepsilon, \varepsilon_1$). We therefore consider the following cases.

Case b.1: D **belongs to EC1**(ε). In this case, $V(D) = W_1 \cup W_2 \cup W_3$ such that for each $i \in [2]$, $D[W_i]$ is ε-almost complete. By Lemma 4.2, there exists a set $\mathcal{P}$ of disjoint W_1-paths and W_2-paths, each of length at most 5, and passing through W_3. For $i \in [2]$, let $\mathcal{P}_i \subseteq \mathcal{P}$ be the set of W_i-paths, and define $W_i' = W_i \setminus V(\mathcal{P})$. We now distinguish two subcases.

First, suppose that there exists a subscript $i_0 \in [m]$ such that $|W_1 \cup V(\mathcal{P}_1)| = n_1 + \cdots + n_{i_0}$ (or, symmetrically, $|W_2 \cup V(\mathcal{P}_2)| = n_{i_0+1} + \cdots + n_m$). Since each W_i' (for $i \in [2]$) is ε-almost complete, we can partition it into two subsets W_i^1 and W_i^2, and simultaneously partition each $\mathcal{P}_i$ into two parts $\mathcal{P}_i^1$ and $\mathcal{P}_i^2$ such that $|W_i^1| + |V(\mathcal{P}_i^1)| = |W_i^2| + |V(\mathcal{P}_i^2)| + t_i$, where $t_i = 0$ if the number of H-subdivisions placed in this part is even, and $t_i = m_H$ otherwise. Clearly, each induced subdigraph $D[W_i^j]$ $(i, j \in [2])$ remains ε-almost complete, and the pair (W_i^1, W_i^2) forms an ε-almost complete bipartite digraph. We then proceed as follows: first, we incorporate the exceptional vertices of W_i' that have lower but still sufficient semi-degree. Afterwards, we apply Lemma 4.12(ii) to (W_i^1, W_i^2) to obtain disjoint H-subdivisions whose orders meet the required conditions. Throughout this process, the paths in $\mathcal{P}_i$ can be used directly as segments of the branch paths of these H-subdivisions.

Second, we assume that no such index i_0 exists. Given the minimum semi-degree condition $\delta^0(D) \geq \frac{n+m+h}{2} - 1$, it follows that for any vertices $u \in W_1$ and $v \in W_2$, we have that either

$$\begin{cases} uv \in A(D) & \text{and} \quad |N^+(u) \cap N^-(v)| \geq 2\delta^0(D) - (n-1) \geq m + h - 1, \text{or} \\ uv \notin A(D) & \text{and} \quad |N^+(u) \cap N^-(v)| \geq 2\delta^0(D) - (n-2) \geq m + h, \end{cases}$$

Similarly, we also get that either

$$\begin{cases} vu \in A(D) & \text{and} \quad |N^-(u) \cap N^+(v)| \geq 2\delta^0(D) - (n-1) \geq m + h - 1, \text{or} \\ vu \notin A(D) & \text{and} \quad |N^-(u) \cap N^+(v)| \geq 2\delta^0(D) - (n-2) \geq m + h, \end{cases}$$

Consequently, there exist two disjoint minimal paths P_1 and P_2, where P_1 is directed from W_1 to W_2, and P_2 from W_2 to W_1. For convenience, let $W_i' = W_i \setminus V(P_1 \cup P_2)$ for $i \in [2]$. Now, by an argument analogous to the proof of Lemma 4.2 (noting that during the process, P_1 and P_2 can always be suitably adjusted to remain disjoint from the final required arc set), we obtain a collection $\mathcal{P}'$ of disjoint W_1'-paths and W_2'-paths, and passing through $W_3 \setminus V(P_1 \cup P_2)$. For $i \in [2]$, let $\mathcal{P}_i' \subseteq \mathcal{P}'$ denote the subset of W_i'-paths.

Let $i_0 \in [m]$ be the largest index satisfying $n_1 + \cdots + n_{i_0} \leq |W_1 \cup V(\mathcal{P}_1')|$, and define $r = |W_1' \cup V(\mathcal{P}_1')| - (n_1 + \cdots + n_{i_0})$. Recall that in the digraph $D[W_1']$, except for $10\sqrt{\varepsilon}|W_1'|$ exceptional vertices, every vertex w satisfies $\delta^0_{W_1'}(w) \geq (1 - 10\sqrt{\varepsilon})|W_1'|$, and these exceptional vertices have semi-degree at least $\varepsilon^{1/3}|W_1'|/2$ within W_1'. Using these properties, we can find an r-vertex path P_0 whose internal vertices lie in $D[W_1']$ connecting the terminal of P_2 to the initial of P_1. Define W_2-path $P := P_2 \circ P_0 \circ P_1$.

We now proceed similarly to the previous case. Since each $W_i' \setminus V(P)$ (for $i \in [2]$) is ε-almost complete, we can partition it into two subsets W_i^1 and W_i^2. Simultaneously, we partition each $\mathcal{P}_i'$ into two subcollections $\mathcal{P}_i^1$ and $\mathcal{P}_i^2$ such that $|W_1^1| + |V(\mathcal{P}_i^1)| = |W_1^2| + |V(\mathcal{P}_i^2)| + t_1$, and

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$|W_2^1| + |V(\mathcal{P}_i^1)| = |W_1^2| + |V(\mathcal{P}_i^2)| + |V(P)| + t_2$, where $t_i = 0$ or m_H according to the parity of the number of H-subdivisions placed in the corresponding part. Clearly, each induced subdigraph $D[W_i^j]$ $(i, j \in [2])$ remains ε-almost complete, and each pair (W_i^1, W_i^2) forms an ε-almost complete bipartite digraph. Then within the subdigraph $D[W_1' \cup V(\mathcal{P}_1') \setminus V(P)]$, we obtain disjoint H-subdivisions of orders $n_1, \ldots, n_{i_0}$, by first using the exceptional vertices of W_1', and then applying Lemma 4.12(ii) to (W_1^1, W_1^2). Throughout this construction, paths from $\mathcal{P}_1$ can be used directly as segments of these H-subdivisions. Similarly, within the subdigraph $D[W_2' \cup V(\mathcal{P}_2') \cup V(P)]$, we obtain disjoint H-subdivisions of orders $n_{i_0+1}, \ldots, n_m$, by first using exceptional vertices of W_2' and then applying Lemma 4.12(ii) to (W_2^1, W_2^2). Here, the path P and the paths from $\mathcal{P}_2'$ can be directly used to construct some paths of these H-subdivisions.

Case b.2: D belongs to EC2(ε). In this case, we have that $V(D) = W_1 \cup W_2 \cup W_3$ such that for each $i \in [2]$, $|W_i| = (1/2 - \varepsilon)n \pm \sqrt{10\varepsilon}|W_i|$, and (W_1, W_2) forms an ε-almost complete bipartite digraph. By the minimum semi-degree condition of D, for any $u \in W_1$ and $v \in W_2$, we have

$$|N^+(u) \cap N^-(v)| \geq 2\delta^0(D) - (n-1) \geq m + h - 1. \tag{4.5}$$

Choose a bipartition (U_1, U_2) of H attaining m_H, and swap the two parts if necessary, so that $t := |U_1| + e(U_2) - (|U_2| + e(U_1)) \geq 0$. By Lemma 2.1, we have $t \leq h/2 - 1$. We proceed according to the parity of m and t.

Subcase b.2.1: m is even, and t is odd.

For all even n_i, we use equation (4.5) to obtain a collection $\mathcal{P}$ of l disjoint 3-paths from W_1 to W_2, where $l = \sum_{i=1}^{m} (1 - (\lceil \frac{n_i}{2} \rceil - \lfloor \frac{n_i}{2} \rfloor))$. Next, following an argument similar to the proof of Lemma 4.5 (note that in this setting the degree calculated in equation (4.4) decreases by at most $2m$, and the remaining degree is at least $n - m + h - 2$, which still meets the condition of Lemma 4.5), we obtain a family $\mathcal{P}'$ of disjoint paths covering all remaining vertices of W_3 and satisfying $|W_1 \setminus V(\mathcal{P} \cup \mathcal{P}')| = |W_2 \setminus V(\mathcal{P} \cup \mathcal{P}')|$. Finally, applying the method of Lemma 4.12(ii) to the ε-almost complete bipartite digraph $(W_1 \setminus V(\mathcal{P} \cup \mathcal{P}'), W_2 \setminus V(\mathcal{P} \cup \mathcal{P}'))$, yields the required set of disjoint H-subdivisions. In this construction, each H-subdivision of even order uses exactly one path from $\mathcal{P}$, while paths from $\mathcal{P}'$ may be incorporated as subpaths in the largest subdivision.

Subcase b.2.2: Both m and t are even.

The argument differs from Subcase b.2.1 only in the initial step: here we apply equation (4.5) to obtain a collection $\mathcal{P}$ of l disjoint 3-paths, where $l = \sum_{i=1}^{m} (\lceil \frac{n_i}{2} \rceil - \lfloor \frac{n_i}{2} \rfloor)$. Each resulting H-subdivision of odd order then uses exactly one path from $\mathcal{P}$. The remaining steps are identical to Subcase b.2.1.

Subcase b.2.3: Both m and t are odd.

As in Subcase b.2.1, we use equation (4.5) to obtain a collection $\mathcal{P}$ of l disjoint 3-paths with $l = \sum_{i=1}^{m} (1 - (\lceil \frac{n_i}{2} \rceil - \lfloor \frac{n_i}{2} \rfloor))$. Then, similarly to the proof of Lemma 4.5, we find a family $\mathcal{P}'$ of disjoint paths covering all remaining vertices of W_3 such that $|W_1 \setminus V(\mathcal{P} \cup \mathcal{P}')| = |W_2 \setminus V(\mathcal{P} \cup \mathcal{P}')| + t$. Applying the proof of Lemma 4.12(ii) to the ε-almost complete bipartite digraph $(W_1 \setminus V(\mathcal{P} \cup \mathcal{P}'), W_2 \setminus V(\mathcal{P} \cup \mathcal{P}'))$, gives the required disjoint H-subdivisions, with each even-order subdivision using exactly one path from $\mathcal{P}$, and paths from $\mathcal{P}'$ usable in the largest subdivision.

Subcase b.2.4: m is odd and t is even.

This case mirrors Subcase b.2.3: we apply equation (4.5) to obtain a collection $\mathcal{P}$ of l disjoint 3-paths with $l = \sum_{i=1}^{m} (\lceil \frac{n_i}{2} \rceil - \lfloor \frac{n_i}{2} \rfloor)$. Each H-subdivision of odd order then uses exactly one path from $\mathcal{P}$. The remaining steps are identical to Subcase b.2.3.

Having established Subcases b.2.1 through b.2.4, we conclude that Theorem 1.5 holds when D belongs to EC2(ε).

Case b.3: D is EC3($\varepsilon, \varepsilon_1$). In this case, the vertex set admits a partition $V(D) = \bigcup_{i=1}^{5} W_i$ with $|W_i| \approx |W_{i+2}|$ for $i \in [2]$ and the following properties hold. For $i \in \{1, 3\}$, the induced subdigraph $D[W_i]$ is ε-almost complete; The pair (W_2, W_4) forms an ε-almost complete bipartite digraph; For

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$j \in [4]$ (where the index of W_{j+1} is taken modulo 4 when $j = 4$), the pair (W_j, W_{j+1}) is ε-almost one-way complete.

By Lemma 4.8, there is a collection $\mathcal{P}$ of disjoint paths, specifically, W_1-paths, W_3-paths, (W_2, W_4)-paths and (W_4, W_2)-paths, that passes through W_5 and satisfies $|W_2 \setminus V(\mathcal{P})| = |W_4 \setminus V(\mathcal{P})|$. Denote by $\mathcal{P}_1$, $\mathcal{P}_2^1$, and $\mathcal{P}_2^2$, $\mathcal{P}_3$ the subsets of $\mathcal{P}$ consisting of W_1-paths, (W_2, W_4)-paths, (W_4, W_2)-paths, and W_3-paths, respectively. Since (W_1, W_2) and (W_4, W_1) are ε-almost one-way complete, we can proceed as in Claim 4.9 of Lemma 4.8: for each path $P \in \mathcal{P}_2^1$, we can choose a pair (u, v) of distinct vertices in $W_1 \setminus V(\mathcal{P}_1)$ such that the concatenation $P' := u \circ P \circ v$ is a W_1-path. Let $\mathcal{P}_1'$ be the set of these extended paths. Similarly, since (W_2, W_3) and (W_3, W_4) are ε-almost one-way complete, for each path $Q \in \mathcal{P}_2^2$, we can select a pair (u', v') in $W_3 \setminus V(\mathcal{P}_3)$ such that $Q' := u' \circ Q \circ v'$ is a W_3-path. Denote the collection of these paths by $\mathcal{P}_3'$. Note that all paths in $\mathcal{P}_1' \cup \mathcal{P}_3'$ are disjoint. After these extensions we still have $|W_2 \setminus V(\mathcal{P}_1' \cup \mathcal{P}_3')| = |W_4 \setminus V(\mathcal{P}_1' \cup \mathcal{P}_3')|$. For $i \in [4]$, define $W_i' = W_i \setminus V(\mathcal{P}_1 \cup \mathcal{P}_1' \cup \mathcal{P}_3' \cup \mathcal{P}_3)$. Clearly, (W_2', W_4') is an ε-almost complete balanced bipartite digraph. From the definition of the class EC3$(\varepsilon, \varepsilon_1)$, we have that for $i \in [4]$, $|W_i| > \varepsilon^{1/3}n - \sqrt{10\varepsilon}n$, and $|W_5| \le 4\sqrt{10\varepsilon}n$. Consequently, for $i \in [4]$,

$$|W_i'| \ge \varepsilon^{1/3}n - 2\sqrt{10\varepsilon}n - 3|W_5| \ge \varepsilon^{1/3}n/2.$$

Recall that $m_H = \min_{U_1 \cup U_2 = V(H)} \big||U_1| + e(U_2) - (|U_2| + e(U_1))\big|$, where the minimum is taken over all bipartitions (U_1, U_2) of the digraph H, and that $n = n_1 + \cdots + n_m$. Suppose first that there exists a balanced subdigraph (W_2^0, W_4^0) of (W_2', W_4'), together with a subset $I \subseteq [m]$, such that $|W_1'| + |W_2^0| + |W_4^0| + |V(\mathcal{P}_1 \cup \mathcal{P}_1')| = \sum_{i \in I} n_i$. (If this holds, symmetry gives $|W_3'| + |W_2' \setminus W_2^0| + |W_4' \setminus W_4^0| + |V(\mathcal{P}_3' \cup \mathcal{P}_3)| = \sum_{i \in [m] \setminus I} n_i$). Then we directly proceed to step b.3. If no such pair (W_2^0, W_4^0) exists, then without loss of generality, we can select a balanced subdigraph (W_2'', W_4'') of (W_2', W_4') and a subset $I_0 \subseteq [m]$ such that $s := |W_1'| + |W_2''| + |W_4''| + |V(\mathcal{P}_1 \cup \mathcal{P}_1')| - \sum_{i \in I_0} n_i \ge 3$ and $s \le \max_{i \in [m] \setminus I_0} n_i - m_H$. Since each pair (W_i', W_{i+1}') is ε-almost one-way complete and $D[W_1']$ is ε-almost complete, we can find a W_3'-path P_0 that uses exactly $s - 2$ vertices of W_1', one vertex of W_2'', and one vertex of W_4''. For consistency, we now update the sets as follows: $I := I_0$, $W_1' := W_1' \setminus V(P_0)$, $W_2^0 := W_2'' \setminus V(P_0)$, $W_4^0 := W_4'' \setminus V(P_0)$, and $W_3' := W_3' \setminus V(P_0)$, $\mathcal{P}_3' := \mathcal{P}_3' \cup P_0$. Hence for each case, we always have

$$|W_1'| + |W_2^0| + |W_4^0| + |V(\mathcal{P}_1 \cup \mathcal{P}_1')| = \sum_{i \in I} n_i. \tag{4.6}$$

$$|W_3'| + |W_2' \setminus W_2^0| + |W_4' \setminus W_4^0| + |V(\mathcal{P}_3' \cup \mathcal{P}_3)| = \sum_{i \in [m] \setminus I} n_i. \tag{4.7}$$

Clearly, the pairs (W_2^0, W_4^0) and $(W_2' \setminus W_2^0, W_4' \setminus W_4^0)$ are ε-almost complete balanced bipartite, and (W_1', W_2^0), (W_4^0, W_1'), $(W_2' \setminus W_2^0, W_3')$ and $(W_3', W_4' \setminus W_4^0)$ are ε-almost one-way complete. Also, for each $i \in \{1, 3\}$, the induced subdigraph $D[W_i']$ is ε-almost complete. More precisely, for each i, except for a set R_i of W_i', every vertex w in $W_i' \setminus R_i$ satisfies $\delta^0_{W_i'}(w) \ge (1 - 20\sqrt{\varepsilon})|W_i'|$ and any $u \in R$ has $\delta^0_{W_i'}(u) \ge \varepsilon^{1/3}|W_i'|/10$. Without loss of generality, (by reordering indices if needed) let $I = 1, \ldots, |I|$. Finally, we proceed to Step b.3.

Step b.3. Using the properties above, we can find $|I|$ disjoint W_1'-paths $P_1, \ldots, P_{|I|}$, where each $P_j = u_j \cdots v_j$ has order $|V(P_j)| := n_j' \le n_j - m_H$. Similarly, we can find $m - |I|$ disjoint W_3'-paths $P_{|I|+1}, \ldots, P_m$, where each $P_j = u_j \cdots v_j$ has order $|V(P_j)| := n_j' \le n_j - m_H$. Let $\widetilde{W}_1$ and $\widetilde{W}_3$ denote the vertices of W_1' and W_3' respectively, not used in these paths.

Now, let xy be an arbitrary arc of H and set $H^- = H - xy$. For each $j \in [m]$, we will use u_j as the embedding of vertex x, and v_j as the embedding of vertex y. Partition $\widetilde{W}_1 \cup \bigcup_{j \in I}\{u_j, v_j\}$ into

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two subsets W_1^1 and W_1^2 such that $|W_1^1| - |W_1^2| = m_{H^-}$. Similarly, partition $\widetilde{W}_3 \cup \bigcup_{j\in[m]\setminus I}\{u_j, v_j\}$ into two subsets W_3^1 and W_3^2 such that $|W_3^1| - |W_3^2| = m_{H^-}$.

Finally, we apply the proof of Lemma 4.12(ii) to (W_1^1, W_1^2) with the images of x and y fixed as u_j and v_j, respectively, and with the residual order $n_j - |V(P_j)| + 2$ for each $j \in I$. This yields $|I|$ disjoint H^--subdivisions $H_1^-, \ldots, H_{|I|}^-$ within $D[W_1' \cup W_2^0 \cup W_4^0 \cup V(\mathcal{P}_1 \cup \mathcal{P}_1')]$. For each $i \in I$, the union $H_i^- \cup P_i$ then forms a required H-subdivision of order n_i. Applying the proof of Lemma 4.12(ii) to (W_3^1, W_3^2) with the images of x and y fixed as u_j and v_j, respectively, and with the residual order $n_j - |V(P_j)| + 2$ for the H^--subdivision corresponding to each $j \in [m] \setminus I$, similarly yields $m - |I|$ disjoint H^--subdivisions $H_{|I|+1}^-, \ldots, H_m^-$ within $D[W_3' \cup (W_2' \setminus W_2^0) \cup (W_4' \setminus W_4^0) \cup V(\mathcal{P}_3 \cup \mathcal{P}_3')]$. For each $j \in [m] \setminus I$, the union $H_j^- \cup P_j$ forms a required H-subdivision of order n_j.

Hence, Theorem 1.5 holds when D belongs to EC3$(\varepsilon, \varepsilon_1)$. Consequently, this completes the proof of Theorem 1.5. □

5. Concluding remarks

Recall that for a digraph H and any bipartition (U_1, U_2) of its vertex set, we have $m_H = \min_{U_1\cup U_2 = V(H)} |(|U_1| + e(U_2)) - (|U_2| + e(U_1))|$. Following the same line of proof as in our paper, the minimum semi-degree conditions in Theorems 1.3 and 1.5 can in fact be strengthened to $\delta^0(D) \geq \frac{n}{2} + m_H$ and $\delta^0(D) \geq \frac{n+m}{2} + m_H$, respectively. Since the argument carries over almost verbatim, we leave the verification to the interested reader. We also note that in some cases, the parameter m_H provides a better bound than $h/2 - 1$. For instance, if H is a balanced bipartite digraph with at least three arcs, then we have $m_H = 0$, which typically yields a tighter estimate.

For a graph H with edge labels $e_1, \ldots, e_{|E(H)|}$ and an integer set $\mathcal{N} = \{n_1, \ldots, n_{|E(H)|}\}$, an $(\mathcal{N}H)$-*subdivision* is obtained from H by subdividing each edge e_i into a path of the length n_i.

Coll, Magnant and Nowbandegani [4] studied spanning H-subdivision with prescribed path lengths under a degree sum condition. Specifically, they showed that for a graph H with h edges, $\delta(H) \geq 1$, and vertices labelled $v_1, \ldots, v_{|V(H)|}$, there exists $n_0 > 0$ such that the following holds: for any set of integers $\mathcal{N} = \{n_1, \ldots, n_h\}$ with $n_i \geq n_0$ for all i, and any n-vertex graph G with $\sigma_2(G) \geq n + 2h - 1$, if $x_1, \ldots, x_{|V(H)|}$ are distinct vertices in $V(G)$, then G contains a spanning $(\mathcal{N}H)$-subdivision in which each vertex x_j corresponds to v_j. This result improves earlier work of Chizmar et al. [6].

It is natural to ask analogous questions for digraphs. We therefore propose the following.

Conjecture 5.1. *Let H be a digraph with h arcs and $\delta(H) \geq 1$. There is a constant n_0 and a real number C_H, depending only on H such that the following holds. If D is an n-vertex digraph satisfying $d_D^+(u) + d_D^-(v) \geq n + C_H$ for every pair of vertices $u, v \in V(D)$ with $uv \notin A(D)$, then for any set of integers $\mathcal{N} = \{n_1, \ldots, n_h\}$ with $n_i \geq n_0$ for all i, the digraph D contains a spanning $(\mathcal{N}H)$-subdivision.*

Problem 5.2. *In the context of Conjecture 5.1, given an integer set $\mathcal{N} = \{n_1, \ldots, n_k\}$, is it always possible to find a perfect H-subdivision tiling in D, where the orders of the H-subdivisions are $n_1, \ldots, n_k$, respectively?*

Acknowledgements

The authors would like to thank the referees for their helpful comments and suggestions. Zhilan Wang and Jin Yan are supported by National Natural Science Foundation of China (No. 12571373, 12071260), and Yangyang Cheng is supported by a PhD studentship of ERC Advanced Grant 883810.

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Appendix A. Proof of Lemma 4.1

We use structural analysis to prove Lemma 4.1.

Proof of Lemma 4.1. Since D satisfies EC(ε'), there exist two (not necessarily disjoint) vertex sets U_1 and U_2 with $|U_i| \geq (1/2 - \varepsilon')n$ for every $i \in [2]$, and $e^+(U_1, U_2) \leq (\varepsilon' n)^2$. We consider the case by case based on the cardinality of $U_1 \cap U_2$. For convenience, let $U_0 := U_1 \cap U_2$.

Case A.1: $|U_0| \leq \varepsilon_1 n$. In this case, we will prove that D belongs to EC1(ε). First, we define $W_i = U_i \setminus U_0$ for each $i \in [2]$, and $R = V(D) \setminus (W_1 \cup W_2)$. Clearly W_1 and W_2 are disjoint and $e^+(W_1, W_2) \leq e^+(U_1, U_2) \leq (\varepsilon' n)^2$. Additionally, for every $i \in [2]$, since $\varepsilon' \ll \varepsilon_1 \ll \varepsilon$ and $|U_1 \cap U_2| = |U_0| \leq \varepsilon_1 n$, we have that $|W_i| = |U_i \setminus (U_1 \cap U_2)| \geq (1/2 - \varepsilon' - \varepsilon_1)n \geq (1/2 - \varepsilon)n$. Further, together with $\delta^0(D) \geq n/2$, $|W_1| \leq (1/2 + \varepsilon)n$, $|V(D) \setminus (W_1 \cup W_2)| \leq 2\varepsilon n$,

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$e^+(W_1, \overline{W_1}) = e^+(W_1, W_2) + e^+(W_1, D \setminus (W_1 \cup W_2))$, and $\varepsilon' \ll \varepsilon_1 \ll \varepsilon$, we can deduce that

$$e(W_1) \geq \sum_{u \in W_1} d^+(u) - e^+(W_1, \overline{W_1}) \geq |W_1| \cdot n/2 - (\varepsilon' n)^2 - (1/2 + \varepsilon)n \cdot 2\varepsilon n$$
$$\geq |W_1|^2 - 2\varepsilon n^2. \tag{A.1}$$

Following the same calculation as in equation (A.1), we can sum in-degrees of vertices in W_2 to obtain that $e(W_2)| \geq |W_2|^2 - 2\varepsilon n^2$.

Further, for $i \in [2]$, let E_i be the set of vertices u in W_i such that $d^\sigma_{W_i}(u) \leq \varepsilon^{1/3}|W_i|$ for some $\sigma \in \{+, -\}$. Then we have that $|E_i| \leq \sqrt{10\varepsilon}|W_i|$. Further, if there exists a vertex x in $E_1 \cup R$ (resp., a vertex y in $E_2 \cup R$) such that for each $\sigma \in \{+, -\}$, $d^\sigma_{W_2}(x) > \varepsilon^{1/3}|W_2|$ (resp., $d^\sigma_{W_1}(y) > \varepsilon^{1/3}|W_1|$), then we put x (resp., y) into the vertex set W_2 (resp., W_1) and update the vertex sets W_1 and W_2. Repeat the above operation until there are no such vertices x and y. Let $W_i := W_i \setminus E_i$ for each $i \in [2]$ and $W_3 = V(D) \setminus (W_1 \cup W_2)$. Obviously, $|W_i| = (1/2 - \varepsilon)n \pm \sqrt{10\varepsilon}|W_i|$ for $i \in [2]$ and $|W_3| \leq 2\sqrt{10\varepsilon} \cdot \max\{|W_1|, |W_2|\}$, and by using the lower bound of $\delta^0(D)$, we can get that W_1, W_2 and W_3 satisfy the properties (1.1) and (1.2) of EC1(ε).

Case A.2: $|U_0| \geq (1/2 - \varepsilon_1)n$. For this case, we will prove that D belongs to EC2(ε). First, let $W_1 = U_0$, $W_2 = V(D) \setminus U_0$. Then we have that $|W_2| \geq (1/2 - \varepsilon')n$ since $V(D) = W_1 \cup W_2$ and $|W_1| = |U_0| \leq |U_i| \leq |V(D) \setminus U_{3-i}| \leq (1/2 + \varepsilon')n$, where $i \in [2]$. Further, it is evident that $e(W_1) = e(U_0) \leq e(U_1, U_2) \leq (\varepsilon' n)^2$. Combining with $\delta^0(D) \geq n/2$ and $|W_2| \leq (1/2 + \varepsilon_1)n$ due to $|V(D) \setminus W_2| = |W_1| = |U_0| \geq (1/2 - \varepsilon_1)n$, and $\varepsilon' \ll \varepsilon_1 \ll \varepsilon$, we can conclude that

$$e^+(W_1, W_2) \geq |W_1| \cdot n/2 - (\varepsilon' n)^2 = |W_1| \cdot (1/2 + \varepsilon_1)n - |W_1| \cdot \varepsilon_1 n - (\varepsilon' n)^2$$
$$\geq |W_1| \cdot |W_2| - (\varepsilon_1 + (\varepsilon')^2)n^2$$
$$\geq |W_1| \cdot |W_2| - \varepsilon n^2 \tag{A.2}$$

Similar to equation (A.2), by calculating the sum of in-degrees of vertices in W_1, we can also obtain that $e^+(W_2, W_1) \geq |W_1| \cdot |W_2| - \varepsilon n^2$.

Continually, we denote by E_1 (resp., E_2) be the set of vertices u in W_1 (resp., W_2) such that $b_{W_2}(u) \leq \varepsilon^{1/3}|W_2|$ (resp., $b_{W_1}(u) \leq \varepsilon^{1/3}|W_1|$). Then we have that $|E_i| \leq \sqrt{10\varepsilon}|W_i|$. Further, if there exists a vertex x in E_1 (resp., a vertex y in E_2) such that $b_{W_2}(x) > \varepsilon^{1/3}|W_2|$ (resp., $b_{W_1}(y) > \varepsilon^{1/3}|W_1|$), then we put x (resp., y) into the vertex set W_2 (resp., W_1) and update the vertex sets W_1 and W_2. Repeat the above operation until there are no such vertices x and y. Finally, let $W_i := W_i \setminus E_i$ for any $i \in [2]$ and $W_3 = V(D) \setminus (W_1 \cup W_2)$. Obviously, $|W_i| = (1/2 - \varepsilon)n \pm \sqrt{10\varepsilon}|W_i|$ for $i \in [2]$ and $|W_3| \leq 2\sqrt{10\varepsilon} \cdot \max\{|W_1|, |W_2|\}$, and we see, using the lower bound of $\delta^0(D)$, that W_1, W_2 and W_3 satisfy the properties (2.1) and (2.2) of EC2(ε).

Case A.3: $\varepsilon_1 n < |U_0| < (1/2 - \varepsilon_1)n$. In this case, we will declare that D belongs to EC3($\varepsilon, \varepsilon_1$). Let $W_1 = U_1 \setminus U_0$, $W_2 = V(D) \setminus (U_1 \cup U_2)$, $W_3 = U_2 \setminus U_0$ and $W_4 = U_0$. We first estimate the cardinalities of vertex sets W_1-W_4. The following conclusion is true.

Claim A.1. *For each $j \in \{1, 3\}$, $(1/2 - \varepsilon'/2)n \leq |W_j| + |W_2| \leq (1/2 + \varepsilon')n$.*

Proof. On the one hand, since $e(W_4) + e^+(W_4, W_3) = e(U_0) + e^+(U_0, U_2 \setminus U_0) \leq e^+(U_1, U_2) \leq (\varepsilon' n)^2$, by calculating the out-degrees of vertices in W_4, we have that

$$n/2 \cdot |W_4| \leq \sum_{w \in W_4} d^+(w) = e^+(W_4, W_1) + e(W_4) + e^+(W_4, W_3) + e^+(W_4, W_2)$$
$$\leq e^+(W_4, W_1) + (\varepsilon' n)^2 + e^+(W_4, W_2)$$
$$\leq |W_4| \cdot |W_1| + (\varepsilon' n)^2 + |W_4| \cdot |W_2|.$$

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Since $|W_4| = |U_0| > \varepsilon_1 n$ and $\varepsilon' \ll \varepsilon_1$, this implies that $(1/2 - \varepsilon'/2)n \le |W_1| + |W_2|$. Similarly, by calculating in-degrees of vertices of W_4, we also obtain that $(1/2 - \varepsilon'/2)n \le |W_2| + |W_3|$.

On the other hand, due to $|U_i| \ge (1/2 - \varepsilon')n$ for each $i \in [2]$, it can be deduced that $|W_1| + |W_2| = |V(D) \setminus U_2| \le (1/2 + \varepsilon')n$ and $|W_2| + |W_3| = |V(D) \setminus U_1| \le (1/2 + \varepsilon')n$. Hence, the claim holds. □

In the following, we will first prove that $|W_1|$ is approximately equal to $|W_3|$, and similarly, $|W_2|$ is approximately equal to $|W_4|$.

Claim A.2. *$-3\varepsilon' n/2 \le |W_1| - |W_3| \le 3\varepsilon' n/2$ and $-\varepsilon' n \le |W_2| - |W_4| \le 2\varepsilon' n$.*

Proof. By Claim A.1, we get that for each $j \in \{1, 3\}$,

$$(1 - \varepsilon')n/2 - |W_2| \le |W_j| \le (1/2 + \varepsilon')n - |W_2|.$$

Hence, we have that

$$-3\varepsilon' n/2 \le |W_1| - |W_3| \le 3\varepsilon' n/2. \tag{A.3}$$

Also, by Claim A.1 again, we have $(1/2 - \varepsilon'/2)n - |W_3| \le |W_2| \le (1/2 + \varepsilon')n - |W_3|$, and

$$(1/2 - \varepsilon'/2)n + |W_3| \le |W_1| + |W_2| + |W_3| \le (1/2 + \varepsilon')n + |W_3|.$$

Together with $n = |W_1| + |W_2| + |W_3| + |W_4|$, this suggests that

$$(1/2 - \varepsilon')n - |W_3| \le |W_4| \le (1 + \varepsilon')n/2 - |W_3|.$$

Hence, we obtain that

$$-\varepsilon' n \le |W_2| - |W_4| \le 2\varepsilon' n. \tag{A.4}$$

Inequalities equations (A.3) and (A.4) imply that $|W_1| \approx |W_3|$ and $|W_2| \approx |W_4|$. □

We then estimate the cardinality of the vertex set W_i for each $i \in [4]$. The following claim holds.

Claim A.3. *We declare that the statements holds as follows.*
(i) $\varepsilon_1 n/2 < |W_j| < (1/2 - 3\varepsilon_1/4)n$ *for* $j \in \{1, 3\}$.
(ii) $\varepsilon_1 n/2 < |W_i| < (1/2 - \varepsilon_1/4)n$, *for each* $i \in \{2, 4\}$.

Proof. Since $|W_4| = |U_0|$ and $\varepsilon_1 n/2 < |U_0| < (1/2 - \varepsilon_1)n$, clearly,

$$\varepsilon_1 n < |W_4| < (1/2 - \varepsilon_1/4)n. \tag{A.5}$$

In the following, we will estimate the upper and lower bounds of $|W_j|$ for $j \in \{1, 3\}$. Since $|U_1|, |U_2| \ge (1/2 - \varepsilon')n$ and $\varepsilon' \ll \varepsilon_1$, we have $|W_j| \ge (1/2 - \varepsilon')n - |W_4| > (1/2 - \varepsilon')n - (1/2 - \varepsilon_1)n \ge \varepsilon_1 n/2$. Also by Claim A.1, $V(D) = W_1 \cup W_2 \cup W_3 \cup W_4$ and $|W_4| = |U_0| > \varepsilon_1 n$, we can deduce that

$$\begin{aligned} |W_j| &= |V(D)| - (|W_{j+2}| + |W_2|) - |W_4| \\ &< n - (1/2 - \varepsilon'/2)n - \varepsilon_1 n \le (1/2 - 3\varepsilon_1/4)n, \end{aligned} \tag{A.6}$$

where the subscript of W_{j+2} takes the remainder of modulo 4. So we conclude that

$$\varepsilon_1 n/2 < |W_j| < (1/2 - 3\varepsilon_1/4)n \text{ for } j \in \{1, 3\}. \tag{A.7}$$

Next, we will estimate the upper and lower bounds of $|W_2|$. On the one hand, due to $|U_1| \ge (1/2 - \varepsilon')n$, $|W_2| = |V(D)| - (|W_1| + |W_3| + |W_4|) = |V(D)| - |U_1 \cup U_2|$, and $\varepsilon' \ll \varepsilon_1$, we have that

$$|W_2| = n - (|U_1| + |U_2 \setminus U_0|) < n - ((1/2 - \varepsilon')n + \varepsilon_1 n/2) < (1/2 - \varepsilon_1/4)n.$$

On the other hand, by Claim A.1 again and equation (A.6), we obtain that

$$|W_2| \ge (1/2 - \varepsilon'/2)n - |W_1| > (1/2 - \varepsilon'/2)n - (1/2 - 3\varepsilon_1/4)n \ge \varepsilon_1 n/2.$$

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Together with equation (A.5), we can conclude that

$$\varepsilon_1 n/2 < |W_i| < (1/2 - \varepsilon_1/4)n, \text{ for each } i \in \{2, 4\}. \tag{A.8}$$

Therefore, we have successfully proven this claim. □

It is worth noting that if there exists an $i \in \{1, 3\}$ such that $|W_i| < \varepsilon^{1/3}n$, then we treat the vertices in $W_1 \cup W_3$ as exceptional vertices and set $(W_2, W_4) := (W_1, W_2)_{EC2(\varepsilon')}$. In this case, by appropriately choosing new parameters, the situation can be reduced to the extremal case EC2(ε). Similarly, if there exists a $j \in \{2, 4\}$ such that $|W_j| < \varepsilon^{1/3}n$, then we treat the vertices in $W_2 \cup W_4$ as exceptional vertices and set $(W_1, W_3) := (W_1, W_2)_{EC1(\varepsilon)}$. Again, with suitable adjustment of parameters, this can be reduced to the extremal case EC1(ε). Therefore, for all $i \in [4]$, we have $|W_i| \geq \varepsilon^{1/3}n$.

In what follows, we will prove that properties (3.1), (3.2), (3.3), and (3.4) of EC3($\varepsilon, \varepsilon_1$) hold. First, it yields from $e^+(U_1, U_2) \leq (\varepsilon' n)^2$ that, for each $j \in \{1, 4\}$, $e^+(W_j, W_3 \cup W_4) \leq e^+(U_1, U_2) \leq (\varepsilon' n)^2$. By Claim A.1, $\delta^0(D) \geq n/2$ and $\varepsilon' \ll \varepsilon$, this implies that

$$\begin{aligned} e^+(W_j, W_1 \cup W_2) &\geq |W_j| \cdot n/2 - (\varepsilon' n)^2 = |W_j| \cdot (1/2 + \varepsilon')n - \varepsilon' n|W_j| - (\varepsilon' n)^2 \\ &\geq |W_j| \cdot (|W_1| + |W_2|) - \varepsilon' n^2/2 \end{aligned} \tag{A.9}$$

Second, since for each $j \in \{3, 4\}$, $e^+(W_1 \cup W_4, W_j) \leq e^+(U_1, U_2) \leq (\varepsilon' n)^2$, and by Claim A.1, we get that

$$\begin{aligned} e^+(W_2 \cup W_3, W_j) &\geq n/2 \cdot |W_j| - (\varepsilon' n)^2 = (1/2 + \varepsilon')n \cdot |W_j| - \varepsilon' n|W_3| - (\varepsilon' n)^2 \\ &\geq (|W_2| + |W_3|) \cdot |W_j| - \varepsilon' n^2/2. \end{aligned} \tag{A.10}$$

Inequalities equations (A.9) and (A.10) imply that the property (3.4) of **EC3** holds.

Further, for each $i \in \{1, 3\}$, we define E_i to be the set of vertices u in W_i with $d^{\sigma}_{W_i}(u) \leq \varepsilon^{1/3}|W_i|$ for some $\sigma \in \{+, -\}$. Clearly, $|E_i| \leq \sqrt{10\varepsilon}|W_i|$. Further, if there exists a vertex x in $E_1 \cup W_2 \cup W_4$ (resp., a vertex y in $(E_3 \cup W_2 \cup W_4)$ such that for each $\sigma \in \{+, -\}$, $d^{\sigma}_{W_3}(x) > \varepsilon^{1/3}|W_3|)$ (resp., $d^{\sigma}_{W_1}(y) > \varepsilon^{1/3}|W_1|$), then we put x (resp., y) into the vertex set W_3 (resp., W_1) and update the vertex sets W_1 and W_3. Repeat the above operation until no such vertices x and y exist. Then let $W_i := W_i \setminus E_i$ for each $i \in \{1, 3\}$.

Symmetrically, recall that $D[W_2 \cup W_4]$ is an ε-almost complete bipartite digraph. For any $j \in \{2, 4\}$, let E_j be the set of those vertices u in W_j such that $b_{W_{j+2}}(u) \leq \varepsilon^{1/3}|W_{j+2}|$, where the subscript of W_{j+2} takes the remainder of modulo for $j = 4$. Clearly, $|E_j \cap W_j| \leq \sqrt{10\varepsilon}|W_j|$ for every $j \in \{2, 4\}$. If there exists a vertex x in $E_2 \cup (V(D) \setminus (W_1 \cup W_3))$ (resp., a vertex y in $E_4 \cup (V(D) \setminus (W_1 \cup W_3)))$ such that $b_{W_4}(x) > \varepsilon^{1/3}|W_4|$ (resp., $b_{W_2}(y) > \varepsilon^{1/3}|W_2|$), then we put x (resp., y) into the vertex set W_4 (resp., W_2) and update the sets W_2 and W_4. We repeat this operation until no such vertices x and y exist. Then let $W_2 := W_2 \setminus E_2$ and $W_4 := W_4 \setminus E_4$. Finally set $W_5 = \bigcup_{i=1}^{4} E_i$, and then together with $\delta^0(D) \geq n/2$, the properties (3.1), (3.2), and (3.3) hold.

Hence the proof of Lemma 4.1 is completed. □

Cite this article: Wang Z, Cheng Y, and Yan J (2026). Spanning H-subdivisions and perfect H-subdivision tilings in dense digraphs. *Combinatorics, Probability and Computing*. https://doi.org/10.1017/S0963548326100509

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