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Persistent massive attractors of smooth maps

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Abstract. For a smooth manifold of any dimension greater than one, we present an open set of smooth endomorphisms such that any of them has a transitive attractor with a non-empty interior. These maps are m-fold non-branched coverings, $m \geq 3$. The construction applies to any manifold of the form $S^1 \times M$, where S^1 is the standard circle and M is an arbitrary manifold.

1. *Introduction*

Consider a smooth map F of a compact manifold X (possibly with a boundary) into itself. The semigroup generated by F is a discrete-time dynamical system. In the theory of dynamical systems, we are interested in the limit behavior of the orbits. In a wide class of dynamical systems, one can find a proper subset $\Lambda \subset X$ and an open set $B \supset \Lambda$ such that the trajectories of almost every point in B accumulate (in some sense) onto Λ. Then Λ is said to be an *attractor* and B its *basin*. Various formalizations of the word 'accumulate' give rise to different (usually non-equivalent) definitions of attractors. The proper identification of an attractor is very important in applications because it is often possible to reduce the dimension or size of the system by restricting it to its attractor.

We do not consider the trivial case, $\Lambda = X$. In this situation, the notion of an attractor does not provide any additional information on the system. Now we address the following natural question suggested by Ilyashenko: if $\Lambda \neq X$, how *big* can Λ be?

There are many ways to tell what is 'big'. An incomplete list of approaches (listed from strongest to weakest) includes:

- having a non-empty interior (*T-big*, for topology);
- having a positive Lebesgue measure (*M-big*, for measure);
- having full Hausdorff dimension (*D-big*, for dimension).

Given full freedom, one can construct dynamical systems with very weird behavior. Thus, we are interested only in (counter)examples which are smooth enough and admit some sort of non-degeneracy. In this paper, we consider only the systems which are at

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least $C^{1+\alpha}$. When we speak of robustness, we assume the space of all systems is equipped with the C^1 topology.

This topic has been very hot in recent years. Abdenur, Bonatti and Díaz [**1**] conjectured that C^1-generic transitive diffeomorphisms whose non-wandering sets have non-empty interior are (globally) transitive. If this is true, no T-big attractors are possible for C^1-generic diffeomorphisms. They gave the proofs for three cases: hyperbolic diffeomorphisms, partially hyperbolic diffeomorphisms with two hyperbolic bundles, and tame diffeomorphisms. They mentioned that in the first case, the proof is folklore; in the second one, they adapted the proof of Brin [**5**]. On the other hand, Fisher [**8**] gave an interesting example of a hyperbolic set with a robustly non-empty interior. Unfortunately, this set is not transitive. Finally, a result by Gorodetski [**10**] shows that D-big transitive invariant sets robustly appear in one-parameter unfoldings of homoclinic tangencies.

However, in these papers the authors never assumed the non-wandering set to be an *attractor*. A classical result by Bowen [**4**] states that every *hyperbolic* attractor of a diffeomorphism has a zero Lebesgue measure, thus forbidding M-big hyperbolic attractors. However, in the non-hyperbolic setting, the question is still open. In [**17**], McCluskey and Manning showed that when a C^2-diffeomorphism of a 2-surface bifurcates from an Anosov to a DA (Derived from Anosov), the Hausdorff dimension of the newborn attractor is 2 (moreover, it is a continuous function of the parameter). There is also a positive result for the space of all boundary-preserving diffeomorphisms of a compact manifold *with a boundary*. Ilyashenko [**13**] showed that there exists a quasiopen set of such maps such that any map in this set has an M-big attractor.

In the realm of smooth endomorphisms, there are more positive answers. In $\dim = 1$ in the quadratic family, the famous Feigenbaum attractor with an absolutely continuous invariant measure provides an example of an M-big attractor. Also, Dobbs [**7**] showed that there exists a C^2-smooth map of the interval with a finite critical set whose Julia set (see [**6**]) has Hausdorff dimension one. This is an example of a D-big repeller.

The goal of this paper is to provide a C^1-robust example of a T-big attractor of an endomorphism. Our construction is based on skew products over expanding circle maps. Tsujii, Avila, Gouëzel [**3, 20**] and Rams [**18**] already showed that in certain concrete families of such skew products the attractor may stably carry an invariant SRB (Sinai–Ruelle–Bowen) measure which is absolutely continuous with respect to the Lebesgue measure. Thus, their attractors are M-big.

In Tsujii's example, the fiber maps are uniformly contracting, which ensures the existence of an attractor. Viana [**21**] used another mechanism to create an attractor, namely unimodal maps in fibers. The result is an M-big attractor exhibiting all positive Lyapunov exponents almost everywhere. As a result of this expansion property, Viana's attractor is also T-big, as shown in [**2**]. Moreover, it is C^3-robust.

In our main theorem, Theorem 2.4, we also construct an example of a T-big attractor. However, the argument is more in Tsujii's spirit, thus showing that a single expanding direction is already enough to create robust T-big attractors. Moreover, our example is robust in the space of C^1-smooth endomorphisms. The skew products from [**3, 18, 20**] can be included in this space as subsets of infinite codimension.

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Our proof is based on the theory of partially hyperbolic perturbations started by Hirsch, Pugh and Shub [**11**] and continued by Gorodetski [**9**], and Ilyashenko and Negut [**16**]. First, we construct a certain skew product over an expanding circle map. Then we prove that every skew product which is close enough to it has a massive attractor (Theorem 2.9). Finally, we show that the same is true for endomorphisms which are not necessarily skew products (Theorem 2.4).

2. *Main theorems*

Let X be some smooth manifold and let a smooth map $F : X \to X$ define a discrete-time dynamical system. In this paper, we use the term 'region' to describe the closure of a non-empty connected open set in X. First, we recall some classical definitions related to attractors. Denote by int A the interior of any set A.

Definition 2.1. A compact set $D \subset X$ is called *trapping* if $F(D) \subset \operatorname{int} D$. Then the closed F-invariant set

$$A_{\max} := \bigcap_{n=0}^{\infty} F^n(D)$$

is said to be a *maximal attractor* for F.

This definition gives the description of an attractor from a geometrical point of view. For the statistical description, we recall the following definition.

Definition 2.2. An F-invariant measure μ is called *Sinai–Ruelle–Bowen (SRB)* if there exists a measurable set $E \subset X$, $\operatorname{Leb}(E) > 0$, such that for any test function $\psi \in C(X)$, and any $x \in E$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(F^i(x)) = \int_X \psi \, d\mu.$$

The set $E = E(\mu)$ is called the *basin* of μ.

Definition 2.3. Let $A_{\max} \neq X$ be a maximal attractor for a certain trapping region $D \supset A_{\max}$. We say that $A_{\max}$ is *massive* if:

(1) $A_{\max}$ is a region;

(2) $A_{\max}$ is the support of an invariant ergodic SRB measure μ such that $E(\mu) \supset D$.

In particular, any massive attractor is T-big and transitive.

Now let M be any compact Riemannian manifold, and fix any $m \in \mathbb{N}$, $m \geq 3$. The existence theorems in this paper will be proven in a constructive way. The dimension of M and the number m are the main parameters of the construction.

Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the standard circle and let $X := S^1 \times M$. Denote by $C^1(X)$ the space of all C^1-smooth m-to-1 coverings of X by itself, with C^1-topology.

THEOREM 2.4. *There exists a non-empty open set $\mathcal{O} \subset C^1(X)$ such that any $\mathcal{F} \in \mathcal{O}$ has a massive attractor $A_{\max}(\mathcal{F})$.*

Remark 2.5. In our construction, any covering $\mathcal{F}$ is a factor of an invertible dynamical system $\hat{\mathcal{F}}$. Any such $\hat{\mathcal{F}}$ has an invariant hyperbolic set $\Lambda(\hat{\mathcal{F}})$ and $A_{\max}(\mathcal{F})$ is the projection

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of this set. In particular, this means μ itself is hyperbolic: its Lyapunov exponents are non-zero.

Remark 2.6. Because the attractor is hyperbolic, and by Ruelle's classical result [**19**], $\mu(\mathcal{F})$ depends on $\mathcal{F}$ in a differentiable way.

Remark 2.7. It is essential in our construction that $m \geq 2$. For technical reasons, we assume $m \geq 3$, but we believe all theorems are true for $m = 2$ as well.

Remark 2.8. Most of the construction is localized within a small ball $\hat{A} \subset M$, so the global properties of M are pretty much irrelevant to us. The compactness assumption about M is used solely to apply the results of [**16**].

To establish Theorem 2.4, first we prove it for a special class of skew products. Let $h : \varphi \mapsto m\varphi \mod 1$ be the standard linear expanding map of S^1. Denote by $\mathcal{C}(M)$ the space of all skew products over h with the fiber M, i.e., the maps of the form

$$F : (\varphi, x) \mapsto (h\varphi, f_\varphi(x)), \quad \varphi \in S^1, x \in M. \tag{2.1}$$

Here, f_φ is a C^1-diffeomorphism onto its image, f_φ is C^0 in φ. The metric on $\mathcal{C}(M)$ is defined as

$$\operatorname{dist}(F, G) := \sup_\varphi \operatorname{dist}_{C^1}(f_\varphi^{\pm 1}, g_\varphi^{\pm 1}).$$

THEOREM 2.9. *There exists a non-empty open set in $\mathcal{C}(M)$ such that any skew product from this set has a massive attractor.*

Remark 2.10. Theorem 2.9 can be generalized in a straightforward way to the case where h is any uniformly hyperbolic m-to-1 endomorphism of a smooth manifold, with $m \geq 3$.

In §§3–8, we first prove Theorem 2.9, and then in §9, we apply Gorodetski–Ilyashenko–Negut techniques to obtain Theorem 2.4.

3. *Geometric construction*

In this section, we establish two results which are not explicitly related to any dynamics, and might have other geometrical applications. We recently found a preprint by Homburg [**12**] in which he independently gives an argument equivalent to §3.1. However, our Theorem 3.1 is stronger, and thus we give a complete proof here.

For regions A and B, we will write $A \Supset B$ if $\operatorname{int} A \supset B$. Note that this relation commutes with finite unions and intersections.

THEOREM 3.1. *For any $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists a region $A \subset \mathbb{R}^n$ and a C^∞-smooth arc of affine maps E_t, $t \in [0, 1]$, such that:*

(1) *for any $t \in [0, 1]$, E_t is contracting;*
(2) *$E_0 A \cup E_1 A \Supset A$;*
(3) *for any $t \in [0, 1]$, E_t is ε-close to the identity in* $\operatorname{Aff}(n)$.

The proof is divided into two steps. The first step is to construct the region and the affine maps satisfying (1) and (2). The second step is to modify both the region and the maps to make them satisfy (3).

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3.1. *One box fits into its two smaller images.* Fix some $0 < \lambda < 1$ close enough to 1. Let $B = [-r_1, r_1] \times [-r_2, r_2] \times \cdots \times [-r_n, r_n] \subset \mathbb{R}^n$ be a standard rectangular box, and let the numbers $r_i > 0$ satisfy the inequalities

$$\lambda r_i > r_{i+1}, \quad 1 \le i < n, \quad \text{and} \quad \lambda r_n > \tfrac{1}{2} r_1. \tag{3.1}$$

Let $R \in \mathrm{SO}(n)$, $R : (x_1, \ldots, x_n) \mapsto ((-1)^{n+1} x_n, x_1, \ldots, x_{n-1})$; let $\Lambda = \lambda \cdot \mathrm{Id}$. Note that $\Lambda R B$ is the standard rectangular box with sides $\lambda r_n, \lambda r_1, \ldots, \lambda r_{n-1}$.

As a result of (3.1), there exist two translations T_0, T_1 of the space $\mathbb{R}^n$ such that

$$\Lambda T_0 R B \cup \Lambda T_1 R B \ni B. \tag{3.2}$$

Indeed, maps of the form $(x_1 \pm s, x_2, \ldots, x_n)$ suffice.

Let $T_t := tT_0 + (1-t)T_1$, $t \in [0, 1]$. Now the affine maps $G_t := \Lambda T_t R$ are contracting with a rate arbitrarily close to identity, and $G_0 B \cup G_1 B \ni B$. But the rotation R is fixed and far from identity, so the statement (3) cannot be achieved with these maps.

3.2. *Making the maps close to identity.* Now we construct new affine maps E_t, $t \in [0, 1]$, and a new region A. The idea is to find E_t satisfying $E_t^k = G_t$ for $k \in \mathbb{N}$ big enough. Denote by $\mathbb{R}^+$ the space $\mathbb{R}$ as an additive group and denote by $\mathrm{Iso}(n)$ the group of isometries of $\mathbb{R}^n$. The following proposition is a simple exercise in linear algebra.

PROPOSITION 3.2. *For $t \in [0, 1]$, there exist Lie group homomorphisms $P_t : \mathbb{R}^+ \to \mathrm{Aff}(n)$ such that $P_t(1) = G_t$ and for all $x \in \mathbb{R}$, $P_t(x) = \lambda^x \cdot U_t(x)$, $U_t(x) \in \mathrm{Iso}(n)$.*

As a result of the homomorphism property, for any $t \in [0, 1]$, $\lim_{k \to +\infty} P_t(1/k) = \mathrm{Id}$. Therefore, by choosing $k \in \mathbb{N}$ big enough, we can make $P_t(1/k)$ arbitrarily close to identity. Now let $E_t := P_t(1/k)$. Note that

$$E_t^k = \left(P_t\left(\frac{1}{k}\right) \right)^k = P_t\left(k \cdot \frac{1}{k}\right) = P_t(1) = G_t.$$

Also note that E_t is a contraction with the uniform rate $\lambda^{1/k} < 1$. By construction, the map $t \mapsto E_t$ is C^∞-smooth.

Now, for $i = 0, 1$ we can find regions $B_i^{(0)}, \ldots, B_i^{(k)}$ with the following properties.

(1) $B_i^{(0)} = B$.

(2) $E_i B_i^{(j)} \ni B_i^{(j+1)}$.

(3) $B_0^{(k)} \cup B_1^{(k)} \ni B$.

We do this inductively: take $E_i B$, shrink it a little, denote the result by $B_i^{(1)}$, then repeat $k-1$ times. This procedure works because of the gap in (3.2).

Let $A_i := \bigcup_{j=0}^{k-1} B_i^{(j)}$, $A := A_0 \cup A_1$.

PROPOSITION 3.3. $E_0 A \cup E_1 A \ni A$.

Proof.

$$E_0 A \cup E_1 A \supset E_0 A_0 \cup E_1 A_1 = E_0\left(\bigcup_{j=0}^{k-1} B_0^{(j)}\right) \cup E_1\left(\bigcup_{j=0}^{k-1} B_1^{(j)}\right)$$
$$\ni \bigcup_{j=1}^{k-1} B_0^{(j)} \cup \bigcup_{j=1}^{k-1} B_1^{(j)} \cup B = A. \qquad \square$$

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Thus, Theorem 3.1 is also proven.

3.3. *Nonlinear version.*

THEOREM 3.4. *For any manifold M, there exists a region $\hat{A}$ such that for any $\varepsilon > 0$, there exists a C^∞-smooth arc of diffeomorphisms $\tilde{f}_t : [0, 1] \times M \to M$ such that:*

(1) *for all $t \in [0, 1]$, $\tilde{f}_t(\hat{A}) \Subset \hat{A}$;*
(2) *for all $t \in [0, 1]$ for all $x \in \hat{A}$, $\|D\tilde{f}_t(x)\| < 1$;*
(3) *for all $t \in [0, 1]$, $\operatorname{dist}_{C^1}(\tilde{f}_t, \mathrm{Id}) < \varepsilon$;*
(4) *there exists a region $A \Subset \hat{A}$ such that $\tilde{f}_0(A) \cup \tilde{f}_1(A) \Supset A$.*

Proof. Take a Morse function H on M with the global minimum p. Consider the gradient flow of H. There exists a small $\tau > 0$ such that the time-τ shift S^τ along the flow is ε-close to the identity map Id.

In a neighborhood of p, the gradient flow of H can be linearized to a diagonal form. Now we work in these linear coordinates. Let $\hat{A}$ be a diffeomorphic ball with the center at p such that $S^\tau(\hat{A}) \Subset \hat{A}$. Such a ball can be obtained using the Lyapunov function method.

Take $\varepsilon_1 > 0$ such that being ε_1-close to Id in the linear coordinates guarantees ε-closeness to Id in the original coordinates. Take some smaller ball $\check{A} \Subset \hat{A}$. Apply Theorem 3.1 in $\check{A}$ to construct a region $A \subset \check{A}$ and the affine maps $E_t : \check{A} \to \check{A}$, $t \in [0, 1]$ that are ε_1-close to Id and $E_0(A) \cup E_1(A) \Supset A$.

Now glue together S^τ and E_t. Namely, we can find maps $\tilde{f}_t$ such that:

(1) $\tilde{f}_t\big|_{M \setminus \hat{A}} = S^\tau$;
(2) $\tilde{f}_t\big|_{\check{A}} = E_t$.

This is possible because, by construction, the maps E_t are isotopic to S^τ in $\check{A}$. By choosing $\check{A}$ small enough, we can ensure all $\tilde{f}_t$ are close to the identity. □

4. *The skew product*

To prove Theorem 2.9, we provide a single skew product $F_0 \in \mathcal{C}(M)$, and show that every $F \in \mathcal{C}(M)$ which is close enough to F_0 has a massive attractor.

Let $L_0, L_1 \subset S^1$ be disjoint closed arcs, with the length of each arc equal to $1/m$. Then, for $i = 0, 1$, we have $h(L_i) = S^1$. Fix a small $\varepsilon > 0$ and apply Theorem 3.4 to get the C^∞-arc of diffeomorphisms $\tilde{f}_t : [0, 1] \times M \to M$.

Now consider a C^∞ map $l : S^1 \to [0, 1]$ such that

$$l|_{L_0} = 0, \quad l|_{L_1} = 1.$$

Let $f_\varphi := \tilde{f}_{l(\varphi)}$ in (2.1). This map (2.1) is the desired C^∞ smooth skew product F_0. Actually, we only need it to be C^2 to apply the results from [**16**]. By construction, for any small enough ε, both F_0 and any $F \in \mathcal{C}(M)$ close enough to F_0 satisfy the *modified dominated splitting condition* [**16**, Definition 2], which we rewrite in the form

$$\max\left(\frac{1}{m} + \left\|\frac{\partial f_\varphi^{\pm 1}}{\partial \varphi}\right\|_{C^0}, \left\|\frac{\partial f_\varphi^{\pm 1}}{\partial x}\right\|_{C^0}\right) = L < m. \tag{4.1}$$

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5. *The lift to a solenoid*

5.1. *Solenoid skew products.* For any $F \in \mathcal{C}(M)$, we consider its solenoid extension $\hat{F}$. Here, we follow [**14**, p. 459]. Recall the definition of the solenoid map. For any $R \geq 2$, $\alpha \ll 1$, let

$$D := \{z \in \mathbb{C} \mid |z| \leq R\}, \quad B := S^1 \times D,$$

and consider the solenoid map $H : B \to B$:

$$H : (\varphi, z) \mapsto (h\varphi, e^{2\pi i\varphi} + \alpha z), \quad \varphi \in S^1, z \in D.$$

For proper R, α (depending only on m), H is a 1–1 map onto its image. We fix these values of R, α and will not need them again. The maximal attractor of H,

$$\Lambda := \bigcap_{n \geq 0} H^n B,$$

is called a *Smale–Williams solenoid*. This set is invariant, locally maximal, and hyperbolic. The restriction $H|_\Lambda$ is invertible.

Given $F : S^1 \times M \to S^1 \times M$, we define $\hat{F} : B \times M \to B \times M$ by the following formula:

$$\hat{F} : (b, x) \mapsto (Hb, f_\varphi(x)), \quad b := (\varphi, z) \in B.$$

$\hat{F}$ is a skew product over H with fiber maps not depending on z. Thus, it can also be viewed as a skew product over h, with the fibers $D \times M$. Denote by π the projection along z, $\pi : (\varphi, z, x) \mapsto (\varphi, x)$. Then $F \circ \pi = \pi \circ \hat{F}$.

Obviously, $\hat{F}$ has $\Lambda \times M$ as an invariant set. For $f_\varphi^{\pm 1}$ uniformly C^1-close to identity, $\Lambda \times M$ is partially hyperbolic with $\dim E^s = 2$, $\dim E^u = 1$, $\dim E^c = \dim M$.

5.2. *Generic endomorphisms of* $S^1 \times M$. The same construction works for any $\mathcal{F} \in C^1(X)$. For $(z, \varphi, x) \in B \times M$, let

$$\hat{\mathcal{F}}(z, \varphi, x) = (e^{2\pi i\varphi} + \alpha z, \mathcal{F}(\varphi, x)).$$

Note that for C^1-close $\mathcal{F}$, $\mathcal{G}$, we get C^1-close extensions $\hat{\mathcal{F}}$, $\hat{\mathcal{G}}$.

6. *Sink skew products and their attractors*

Let $q : B \to B$ be a dynamical system and $F : B \times M \to B \times M$ be a skew product over q.

Definition 6.1. A skew product F is called a *sink skew product* if there exists a *sink region* $S \subset M$ diffeomorphic to a closed ball D^n such that for all $b \in B$ the fiber map f_b has the following properties.

(1) f_b is a Morse–Smale diffeomorphism.
(2) $\exists!\, x \in \mathrm{Fix}(f_b)$ within int S and this x is an attracting fixed point.
(3) f_b brings S strictly into itself and is contracting on S uniformly in b.
(4) All inverse maps f_b^{-1} are expanding on S uniformly in b.
(5) The maps f_b and f_b^{-1} depend continuously on b in the C^0 topology.

Apparently, as the region $B \times S$ is trapping (property 3), it admits a maximal attractor $A_{\max} = \bigcap_{n \geq 0} F^n(B \times S)$. It is a closed invariant set: $F^{-1}(A_{\max}) = A_{\max}$. Now assume q is invertible. In this case, $A_{\max}$ has a really simple form.

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LEMMA 6.2. *Let F be a sink skew product over an invertible $q : B \to B$. Let P be an ergodic invariant measure on B. Consider $A_{\max} := \bigcap_{n=0}^{\infty} F^n(B \times S)$. Then:*

(1) *$A_{\max}$ is the graph Γ of a continuous function $\gamma =: B \to S$. The projection $j|_\Gamma : \Gamma \to B$ is a bijection. Under this bijection, $F|_\Gamma$ is conjugated to the dynamics in the base;*
(2) *$A_{\min} = \Gamma$;*
(3) *there exists an ergodic SRB measure μ_∞ in $B \times S$. Its basin contains $B \times S$. This measure is supported on Γ and is precisely the pull-back of the measure P under the bijection $j|_\Gamma : \Gamma \to B$. Therefore,* $\operatorname{supp} \mu_\infty = \Gamma$.

Remark 6.3. Statement (2) means that all reasonable definitions of an attractor coincide with Γ; see [**15**, (12)]. In particular, $A_{\min} = A_{\text{stat}} = A_M = A_{\max}$.

Lemma 6.2 is rather elementary, and appears to be folklore. The proof can be found, for instance, in [**15**, §§2.3–2.5]. Though [**15**] state their results only for $B = \Sigma\dagger$, the same argument works perfectly for any invertible $q : B \to B$ in the base. Also, formally speaking, they deal not with sink skew products, but with their specific subset, so-called North–South skew products, but the same proofs extend easily to sink skew products.

By construction, any F close enough to F_0 and any $\hat{F}$ close enough to $\hat{F}_0$ are sink skew products with $S = \hat{A}$. Therefore, Lemma 6.2 applies to $\hat{F}$.

7. *Projection of the attractor is dense within a strip*

Note that the projection $\mu = \pi_* \mu_\infty$ of the SRB measure μ_∞ for $\hat{F}$ is an SRB measure for F; $\operatorname{supp} \mu = \pi\Gamma$. Take $A \Subset S$ from Theorem 3.4.

PROPOSITION 7.1. *For any $F \in \mathcal{C}(M)$ which is C^1-close enough to F_0, we have*

$$S^1 \times A \subset \pi\Gamma. \tag{7.1}$$

Let $\frac{1}{2} < \lambda < 1$ be the uniform upper rate of contraction of the fiber maps in the sink S:

$$\lambda = \sup_{\varphi \in S^1, x \in S} \|Df_\varphi(x)\|.$$

We also denote by S_φ and A_φ the regions S and A in the fiber over φ:

$$S_\varphi := \{\varphi\} \times S, \quad A_\varphi := \{\varphi\} \times A,$$

and let $\Gamma_\varphi := S_\varphi \cap \pi A_{\max}$. We write $|A| := \operatorname{diam} A$.

The proof will be carried out by induction, i.e., we will show that for any $\varphi \in S^1$ and any $n \in \mathbb{N}$, there exists a finite covering of A_φ by balls of radius $\lambda^{n-1} \cdot |A|$ whose centers lie in the set Γ_φ. As n could be taken arbitrarily large, this will prove the density of Γ_φ in A_φ for any $\varphi \in S^1$ and thus the density of $\pi A_{\max}$ in $S^1 \times A$. Finally, the fact that $\pi A_{\max}$ has to be a closed set finishes the proof of the proposition.

Base: $n = 1$. Any point from the non-empty set Γ_φ suffices as the center of a single ball of radius $|A|$, which clearly contains A_φ.

† Σ is the bi-infinite analog of Σ^+. The shift $\sigma : \Sigma \to \Sigma$ is invertible.

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Step. Assume that for some $n \in \mathbb{N}$ and for any $\varphi \in S^1$, there exists a finite covering of A_φ by balls of radius $\lambda^{n-1} \cdot |A|$ with centers in the set Γ_φ. Denote by O_φ the set of all centers of the balls.

Take any $\varphi' \in S^1$. There exist two preimages of φ', φ'_0 and φ'_1, within L_0 and L_1 respectively:

$$h\varphi'_i = \varphi', \quad \varphi'_i \in L_i.$$

Because of the invariance of $A_{\max}$ and $\pi A_{\max}$, we have

$$\Gamma_{\varphi'} = \bigcup_{\varphi \in h^{-1}\varphi'} f_\varphi(\Gamma_\varphi) \supset f_{\varphi'_0}(\Gamma_{\varphi'_0}) \cup f_{\varphi'_1}(\Gamma_{\varphi'_1}).$$

We know that every f_φ contracts S_φ with a uniform upper rate of $\lambda < 1$. This observation, combined with the induction assumption, gives us that balls of radius $\lambda^n \cdot |A|$ with centers in $f_\varphi(O_\varphi)$ constitute a covering of the regions $f_\varphi(A_\varphi) \subset S_{\varphi'}$, $\varphi \in h^{-1}\varphi'$. Now note that the maps $f_{\varphi'_i}$ are C^1-close to the maps f_i originally generated by Theorem 3.4, and, therefore,

$$A \subset f_{\varphi'_0}(A) \cup f_{\varphi'_1}(A). \tag{7.2}$$

Thus, the balls of radius $\lambda^n \cdot |A|$ with centers in $f_{\varphi'_0}(O_{\varphi'_0}) \cup f_{\varphi'_1}(O_{\varphi'_1})$ cover the whole region A.

The induction step is complete, and Proposition 7.1 is proven.

8. *From a strip to the whole attractor*

In this section, we finish the proof of Theorem 2.9. From §7, we know that

$$S^1 \times A \subset \operatorname{supp} \mu.$$

As μ is ergodic, its support is a transitive invariant set. Therefore,

$$\operatorname{supp} \mu = \overline{\bigcup_{n=-\infty}^{+\infty}} F^n(S^1 \times A).$$

For any finite $N \in \mathbb{N}$, the set $\bigcup_{n=-N}^{N} F^n(S^1 \times A)$ is the closure of a non-empty connected open set. Thus, this is true for the whole $\operatorname{supp} \mu$. What remains to be proven is that there exists an open $U \supset \operatorname{supp} \mu$ such that

$$\operatorname{supp} \mu = \bigcap_{n=0}^{+\infty} F^n(U).$$

In the solenoid extension $\hat{F}$, the set Γ is a hyperbolic attractor with a uniform rate of contraction along the fibers. For a small $\varepsilon > 0$, let

$$\hat{U}_\varepsilon := \{(z, \varphi, x) \mid \operatorname{dist}_M(x, \gamma(z, \varphi)) < \varepsilon\}.$$

This set is open, and $U := \pi \hat{U}_\varepsilon$ is an open neighborhood of $\operatorname{supp} \mu$. In the solenoid extension,

$$\Gamma = \bigcap_{n=0}^{+\infty} \hat{F}^n(\hat{U}_\varepsilon).$$

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Thus,

$$\operatorname{supp}\mu = \pi\Gamma = \bigcap_{n=0}^{+\infty} \pi\hat{F}^n(\hat{U}_\varepsilon) = \bigcap_{n=0}^{+\infty} F^n(U).$$

The proof of Theorem 2.9 is complete.

9. *Perturbation in the space of all smooth endomorphisms*

In this section, we prove our main theorem, Theorem 2.4. Consider any map $\mathcal{F} \in C^1(X)$ which is C^1-close to F_0. We want to show that $\mathcal{F}$ has a massive attractor.

Note that though F_0 is a skew product, $\mathcal{F}$ is usually not. The straight vertical foliation is no longer invariant, so the skew product structure is lost. However, for small enough perturbations of F_0, we are able to rectify the perturbed map and recover the structure. Here, the partially hyperbolic techniques from [**9, 11, 16**] come into play.

First of all, we consider the solenoidal extensions of F_0 and $\mathcal{F}$. Denote them by $\hat{F}_0$ and $\hat{\mathcal{F}}$. Their fiber maps are independent of z. According to §5, they are C^1-close. In the rest of this section, we use the notations of §5.

We now invoke the following theorem from [**14**, Theorem 5] (which is in fact a modified version of theorems [**16**, Theorems 2, 1]).

THEOREM 9.1. *Suppose that the fiber maps of $\hat{F}_0$ satisfy the modified dominated splitting condition (4.1). Then there exists a $\rho > 0$ such that any map $\mathcal{G}$ that is ρ-close to $\hat{F}_0$ in C^1,*

$$\operatorname{dist}_{C^1}(\hat{F}_0^{\pm 1}, \mathcal{G}^{\pm 1}) \le \rho,$$

has the following properties.

(a) *There exists a semiconjugacy $p : S^1 \times D \times M \to S^1$ such that the diagram*

$$\begin{array}{ccc} S^1 \times D \times M & \xrightarrow{\mathcal{G}} & S^1 \times D \times M \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ S^1 & \xrightarrow{h} & S^1 \end{array}$$

commutes. Moreover, the map

$$\hat{\mathcal{H}} : S^1 \times D \times M \to S^1 \times D \times M, \quad \hat{\mathcal{H}}(\varphi, z, x) := (p(\varphi, z, x), z, x)$$

is a homeomorphism.

(b) *The fibers of p,*

$$\tilde{N}_\varphi := p^{-1}(\varphi),$$

are the graphs of smooth maps

$$\tilde{\beta}_\varphi : D \times M \to S^1.$$

The maps $\tilde{\beta}_\varphi$ are continuous in φ.

Actually, the maps $\tilde{\beta}_\varphi$ are even Hölder continuous in φ, but we do not use this fact. The following statement is proven in [**14**, §5.2].

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THEOREM 9.2. *In the setting of Theorem 9.1, suppose $\mathcal{G} = \hat{\mathcal{F}}$ is a solenoidal extension of $\mathcal{F}$ that is ρ-close to F_0. Then the maps $p(\varphi, z, x)$ and $\tilde{\beta}_\varphi(z, x)$ are independent of z.*

It follows from Theorem 9.1 that we can rectify the perturbed skew product $\mathcal{G}$. Let $\hat{F} = \hat{\mathcal{H}} \circ \hat{\mathcal{F}} \circ \hat{\mathcal{H}}^{-1}$. Statement (a) of Theorem 9.1 implies that $\hat{F}$ is a solenoidal skew product.

By Theorem 9.2, the map $\mathcal{H} : S^1 \times M \to S^1 \times M$ such that $\mathcal{H}(\varphi, x) = (p(\varphi, x), x)$ is a homeomorphism. Thus, $\hat{F}$ is the solenoidal extension of a circle skew product $F := \mathcal{H} \circ \mathcal{F} \circ \mathcal{H}^{-1}$.

Ilyashenko and Negut [**16**] also provide an important addition which we apply here.

THEOREM 9.3. *The fiber maps f_b of the skew product F are C^1-close to those of the skew product F_0, in the following sense:*

$$\operatorname{dist}(f_b^{\pm 1}, f_{(0),b}^{\pm 1})_{C^1} \le O(\rho).$$

This is statement (12), proved in [**16**, §6.2].

Thus, F is C^1-close to F_0. By Theorem 2.9, F has a massive attractor. Now note that the property to have a massive attractor is preserved by a homeomorphic conjugacy. Theorem 2.4 is proven.

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