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A sequence well distributed in the square

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Bollobás and Erdös [1] have posed the problem:

> If a is irrational, show that for $1 \leqslant i < j \leqslant p$ the number of integers t with $1 \leqslant t \leqslant p$ such that $\{(t-i)^2 a\} < d$ and $\{(t-j)^2 a\} < d$, $0 < d < 1$, is $d^2 p + o(p)$ uniformly in i, j.

We shall obtain a proof of this from a classical result in Diophantine approximation, using an estimate of Hardy and Littlewood for a certain trigonometrical sum.

Let $(x_t)_{t \in \mathbb{N}}$ be a sequence in the unit square $T = [0,1]^2$. We call the sequence (x_t) *uniformly distributed symmetrically in* T iff

$$|\{t \leqslant p : x_t \in [a,b]^2\}| = (b-a)^2 p + o(p)$$

and *well-distributed symmetrically in* T iff the sequences (x_{t+h}) are uniformly distributed symmetrically in T uniformly in h (that is, the error term $o(p)$ is independent of h) for $h \in \mathbb{Z}$. These conditions are weaker than the usual definition of uniformly or well distributed sequences mod. (1) in $\mathbb{R}^2$ (see Kuipers and Niederreiter [4], chapter I·6).

It is sufficient to show that the sequences

$$y_t = (\{(t-i)^2 a\}, \{(t-j)^2 a\})$$

with $i \neq j$ are well distributed symmetrically in T uniformly in $i-j$: this will be Theorem 6.

For fixed i, j with $i \neq j$, (y_t) is well distributed mod. (1) in $\mathbb{R}^2$, and hence well distributed symmetrically in T, by a result of Weyl ([5], Satz 14): see also [4], theorem I·6·4.

Proposition 1. *Let $f_1, \ldots, f_n$ be polynomials with the property that no non-trivial integer linear combination is constant mod.* (1). *Then the sequence $(f_1(t), \ldots, f_n(t))$ is uniformly distributed mod.* (1). ∎

It is easy to show that (y_t) satisfies the conditions of the Proposition, so in this note we establish uniformity.

We first obtain a sufficient condition for any sequence (x_t) to be well distributed symmetrically in T.

Proposition 2. *(x_t) is well distributed symmetrically in T if for all continuous symmetric functions $f: T \to \mathbb{C}$ we have*

$$\frac{1}{p} \sum_{t=1}^{p} f(x_{t+h}) \to \int_T f \quad \text{uniformly in } h. \tag{1}$$

Proof. Assume (x_t) satisfies condition (1). Let $S = [a,b]^2$ and I_S be the characteristic

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function of S: put $d = b - a$. Choose $\delta > 0$. I_S is symmetric, so we can choose continuous symmetric functions $f_1 < I_S < f_2$ such that $\int_T f_2 - f_1 < \delta$. Now

$$\begin{aligned} d^2 - \delta &\leqslant \int_T f_2 - \delta \leqslant \int_T f_1 = \lim \frac{1}{p} \Sigma f_1(x_{t+h}) \\ &\leqslant \liminf \frac{1}{p} \Sigma I_S(x_{t+h}) \leqslant \limsup \frac{1}{p} \Sigma I_S(x_{t+h}) \\ &\leqslant \lim \frac{1}{p} \Sigma f_2(x_{t+h}) = \int_T f_2 \leqslant \int_T f_1 + \delta \\ &\leqslant d^2 + \delta \end{aligned}$$

uniformly in h, using condition (1) twice. So

$$\frac{1}{p} \sum_{t=1}^{p} I_S(x_{t+h}) \to d^2$$

uniformly in h. ∎

We next find a condition equivalent to (1) in terms of trigonometric sums (a 'Weyl criterion', cf. [2], theorem I·5·2). For any map $f: T \to \mathbb{C}$, define

$$f^{\mathrm{sym}}(x, y) = \tfrac{1}{2}(f(x, y) + f(y, x)).$$

Clearly f is symmetric iff $f = f^{\mathrm{sym}}$. Further define $e(z) = \exp(2\pi i z)$ and, for $r, s \in \mathbb{Z}$, let $e_{r,s}(x, y) = e(rx + sy)$, $f_{r,s} = e_{r,s}^{\mathrm{sym}}$. By the Stone–Weierstrass Theorem, the linear span of the functions $e_{r,s}$ is dense in the space $C(T, \mathbb{C})$ of continuous functions from T to $\mathbb{C}$ with the supremum norm and hence the linear span of the $f_{r,s}$ is dense in the space $C^{\mathrm{sym}}(T, \mathbb{C})$ of symmetric functions in $C(T, \mathbb{C})$.

PROPOSITION 3. (1) *holds for the sequences* (x_{t+h}) *uniformly in* h *iff for* r, s *not both zero,*

$$\frac{1}{p} \sum_{t=1}^{p} f_{r,s}(x_{t+h}) \to 0 \quad \textit{uniformly in } h. \tag{2}$$

Proof. Since $\int_T f_{r,s} = 0$ in this case, (1) immediately gives (2).

Assume, then, that (2) holds and let $f \in C^{\mathrm{sym}}(T, \mathbb{C})$, $\delta > 0$; choose a_k, r_k, s_k such that

$$\left| f - \sum_{k=1}^{n} a_k f_{r_k, s_k} \right| < \delta. \tag{3}$$

Then

$$\left| \frac{1}{p} \sum_{1}^{p} f(x_{t+h}) - \int_T f \right| \leqslant X + Y + Z,$$

where

$$\begin{aligned} X &= \left| \frac{1}{p} \sum_{1}^{p} f(x_{t+h}) - \frac{1}{p} \sum_{1}^{p} \sum_{1}^{n} a_k f_{r_k, s_k}(x_{t+h}) \right|, \\ Y &= \left| \frac{1}{p} \sum_{1}^{p} \sum_{1}^{n} a_k f_{r_k, s_k}(x_{t+h}) - \int_T \sum_{1}^{n} a_k f_{r_k, s_k}(x_{t+h}) \right|, \\ Z &= \left| \int_T \sum_{1}^{n} a_k f_{r_k, s_k}(x_{t+h}) - \int_T f \right|. \end{aligned}$$

Now $X, Z < \delta$ by assumption and $Y < \delta$ for sufficiently large p, independent of h, by (2): since $f_{0,0} = 1$, this term makes no contribution to Y. The result follows. ∎

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We now establish a slightly strengthened version of an estimate of Hardy and Littlewood ([2], theorem 2·14). Define

$$s_n^2(x,y) = \sum_{r\leqslant n} \exp(\pi i x(r-\tfrac{1}{2})^2)\cos(\pi(2r-1)y),$$

$$s_n^3(x,y) = \sum_{r\leqslant n} \exp(\pi i x r^2)\cos(2\pi r y),$$

$$s_n^4(x,y) = \sum_{r\leqslant n} (-1)^r \exp(\pi i x r^2)\cos(2\pi r y),$$

and, for $b = 2$, 3 or 4, put

$$s^b_{m,n}(x,y) = s^b_{m+n}(x,y) - s^b_m(x,y).$$

Define b' by $2' = 4$, $3' = 3$, $4' = 2$. We have the 'approximate functional equation for the theta-function'.

PROPOSITION 4. $s_n^b = \sqrt{(i/x)}\exp(-\pi i y^2/x)\, s^{b'}_{nx}(-1/x, y/x) + O(1)/\sqrt{x}$ *where* $O(1)$ *denotes a function of* n, x, y *bounded by an absolute constant* A.

Proof. This is Hardy and Littlewood [2], theorem 2·128: see also [3], theorem A. |

COROLLARY 4·1. $s^b_{m,n} = \sqrt{(i/x)}\exp(-\pi i y^2/x)\, s^{b'}_{mx,nx}(-1/x, y/x) + O(1)/\sqrt{x}$ *where* $O(1)$ *denotes a function of* n, x, y *bounded by the absolute constant* $2A$. |

THEOREM 5. *Let* x *be irrational. Then*

$$\frac{1}{n} s^b_{m,n}(x,y) \to 0$$

uniformly in m *and* y.

Proof. We follow the proof of theorem 2·14 of [2]. Suppose x is expressed as a continued fraction,

$$x = 1/(a_1 + 1/(a_2 + 1/(a_3 + \ldots)))$$

and put $x_0 = x$, $x_{r-1} = 1/(a_r + x_r)$; $y_0 = y$, $y_{r+1} = \{y_r/x_r\}$; $p_r = x_0 \ldots x_{r-1}$. Let w denote an unspecified complex number of unit modulus. Then, by Corollary 4·1,

$$\begin{aligned} s^b_{m,n}(x,y) &= \sqrt{(w/x)}\, s^{b'}_{mx,nx}(-1/x, y/x) + O(1)/\sqrt{x}, \\ &= \sqrt{(w/x)}\, s^{b'}_{mx,nx}(-a_1 - x_1, y_1) + O(1)/\sqrt{x}, \\ &= \sqrt{(w/x)}\, s^{b'}_{mx,nx}(-x_1, y_1) + O(1)/\sqrt{x}, \\ &= \sqrt{(w/x)}\, s^{b'}_{mx,nx}(x_1, y_1) + O(1)/\sqrt{x}, \end{aligned}$$

since $s^b_n(-x,y) = s^b_n(x,y)$. Repeating,

$$s^b_{m,n}(x,y) = w s^{b(r)}_{mp_r,np_r}(\pm x_r, y_r)/\sqrt{p_r} + O(1)/\sqrt{x_0} + O(1)/\sqrt{(x_0x_1)} + \ldots + O(1)/\sqrt{p_r}.$$

Replacing the first term by the trivial estimate np_r,

$$|s^b_{m,n}(x,y)| < n\sqrt{p_r} + A[1/\sqrt{p_1} + \ldots + 1/\sqrt{p_r}].$$

But $$x_{r-1} = 1/(a_r + x_r) \leqslant 1/(1 + x_r),$$

so $$x_{r-1}x_r \leqslant x_r/(1+x_r) \leqslant \tfrac{1}{2}.$$

The second term in the estimate is then

$$\begin{aligned} A[1/\sqrt{p_1} + \ldots + 1/\sqrt{p_r}] &= A[\sqrt{(x_1 \ldots x_{r-1})} + \ldots + 1]/\sqrt{p_r} \\ &< A[1 + 1 + \sqrt{\tfrac{1}{2}} + \sqrt{\tfrac{1}{2}} + \ldots]/\sqrt{p_r} \\ &= AK/\sqrt{p_r}, \end{aligned}$$

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where $$K = 2\sum_0^\infty (\sqrt{\tfrac{1}{2}})^k = 2\sqrt{2}/(\sqrt{2}-1).$$

So $$|s^b_{m,n}(x,y)| < n\sqrt{p_r} + AK/\sqrt{p_r}.$$

Now $p_r < 2^{2-r/2} \to 0$ as $r \to \infty$. Choose $r = r(n)$ such that $np_{r+1} < 1 < np_r$. Then $p_r \to 0$ as $n \to \infty$ independently of m, y, so the first term is $o(n)$ uniformly in m, y. Further, $1/\sqrt{p_r} \leqslant \sqrt{n}$, so that the second term is $O(n)$ uniformly in m, y. The result follows. |

We can now solve the problem.

THEOREM 6. *Let a be irrational. The sequences $x_t = (\{a(t-i)^2\}, \{a(t-j)^2\})$ are well distributed symmetrically in T uniformly in $j-i$.*

Proof. By Proposition 3, it is sufficient to show that, for r, s not both zero,

$$\frac{1}{p}\sum_1^p f_{r,s}(x_t) \to 0$$

uniformly in i, j with $i \neq j$. But this sum is

$$\begin{aligned}
&\tfrac{1}{2}\sum_1^p e(a(r(t-i)^2 + s(t-j)^2)) + e(a(s(t-i)^2 + r(t-j)^2)) \\
&\quad = \sum e(a(r+s)\,((t-i)^2 + (t-j)^2)/2)\cos(\pi a(r-s)\,((t-i)^2 - (t-j)^2)) \\
&\quad = \sum e\left(a(r+s)\left(u^2 + \left(\frac{i-j}{2}\right)^2\right)\right)\cos(2\pi a(r-s)\,(j-i)\,u),
\end{aligned}$$

where $u = t - (i+j)/2$. So the sum, up to a factor of unit modulus, is just

$$s^3_{q,p}((r+s)a,\ (r-s)(j-i)a) \quad \text{if } i+j \text{ is even, } i+j = 2q,$$

and $$s^2_{q,p}\ ((r+s)a,\ (r-s)(j-1)a) \quad \text{if } i+j \text{ is odd, } i+j = 2q+1.$$

By Theorem 4 each is $o(p)$ uniformly in i, j. The result follows. |

REFERENCES

[1] B. BOLLOBÁS and P. ERDÖS. An extremal problem of graphs with diameter 2. *Mathematics Magazine* **48** (1975), 281–283.

[2] G. H. HARDY and J. E. LITTLEWOOD. Some problems of Diophantine approximation. II. The trigonometrical series associated with the elliptic θ-functions. *Acta Mathematica* **37** (1914), 193–238.

[3] G. H. HARDY and J. E. LITTLEWOOD. Some problems of Diophantine approximation: an additional note on the trigonometrical series associated with the elliptic theta-functions. *Acta Mathematica* **47** (1925), 189–198.

[4] L. KUIPERS and H. NIEDERREITER. *Uniform Distribution of Sequences* (Wiley, 1974).

[5] H. WEYL. Uber die Gleichverteilung von Zahlen mod. Eins. *Math. Ann.* **77** (1916), 313–352.

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